Wilson-loop One-point Functions in ABJM Theory

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Based on Yunfeng Jiang, **JW** and Peihe Yang, [2306.05773]
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- So we can use weakly coupled gravity/string theory to compute quantities in strongly coupled gauge theory in the large N limit.
- The quantities includes amplitudes, correlation functions of local operators, vacuum expectation values of loop operators, entanglement entropy...

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- The non-perturbative tools in the field theory side of gauge/gravity correspondence include integrability, supersymmetric localization, bootstrap...
- Integrability makes people be able to compute many quantities in the large N limit, even beyond the BPS sectors.

Integrability in AdS_5/CFT_4

- Minahan and Zarembo (02) found that the planar one-loop anomalous dimension matrix in the $SU(2)$ sector of $\mathcal{N}=4$ SYM is essentially the Hamiltonian of Heisenberg XXX spin chain. This Hamiltonian is integrable!

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- Integrability is an important non-pertubative tool in AdS_5/CFT_4 correspondence.

Integrability in AdS_4/CFT_3

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- The integrable structure was also found in this AdS_4/CFT_3 correspondence. [Minahan, Zarembo, 08][Bak, Rey, 08][Gromov, Vieira, 08]
- Almost every aspect of integrability in this case is more complicated and difficult.

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- In ABJM theory, IBS also appears in similar three-point functions [Yang, Jiang, Komatsu, **JW**, 21] and domain wall one-point functions [Kristjansen, Vu, Zarembo, 21].
- One aim of this talk is to show that IBS also appears in some BPS Wilson-loop one-point functions in ABJM theory.

Heisenberg XXX spin chain

- The Hilbert space of a closed XXX spin chain,

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- We consider the Hamiltonian

$$H = J \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z \right) , \tag{2}$$

with periodic boundary condition,

$$S_{L+1}^{\alpha} = S_1^{\alpha}, \ \alpha = x, y, z. \tag{3}$$

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- Here $U = T(0) = Q_1$ is a shift operator.

IBS for XXX chain

- The definition of IBS [Piroli, Pozsgay, Vernier, 17] for XXX chain is that the state $|B\rangle$ satisfying

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- This is equivalent to

$$T(u)|B\rangle = T(-u)|B\rangle\,. \tag{6}$$

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- Eigenstates of integrable Hamiltonian can be labelled by Bethe roots, solutions to certain Bethe ansatz equations (BAEs).
- A selection rule for the overlap of an integrable boundary state and a Bethe state: the overlap is nonzero only when the Bethe roots satisfy certain pairing conditions.
- When this selection rule is satisfied, the overlap can often be expressed as a product of super-Gaudin-determinant and a prefactor. Great simplification!

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- This theory should be low energy effective theory of N M2-branes putting at the tip of $\mathbf{C}^4/\mathbf{Z}_k$.

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- Two limits:
  't Hooft limit (planar limit): $N, k \to \infty$, $\lambda \equiv \frac{N}{k}$ fixed;
  M-theory limit: $N \to \infty$, k fixed.

Holographic dual

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- When $N \gg k^5$, this theory is dual to M-theory on $AdS_4 \times S^7/Z_k$.
- When $k \ll N \ll k^5$, a better description is in terms of IIA superstring theory on $AdS_4 \times \mathbf{CP}^3$.

Bosonic $1/6$-BPS circular WLs

- We consider the Wilson loops (WLs) along $x^\mu = (R\cos\tau, R\sin\tau, 0), \tau \in [0, 2\pi]$.

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- The construction is the following,

$$W^B_{1/6} = \mathrm{Tr}\mathcal{P}\ \exp\left(-i\oint d\tau \mathcal{A}^B_{1/6}(\tau)\right)\ , \tag{7}$$

$$\hat{W}^B_{1/6} = \mathrm{Tr}\mathcal{P}\ \exp\left(-i\oint d\tau \hat{\mathcal{A}}^B_{1/6}(\tau)\right)\ , \tag{8}$$

$$\mathcal{A}^B_{1/6} = A_\mu \dot{x}^\mu + \frac{2\pi}{k} R_I{}^J Y^I Y^\dagger_J |\dot{x}|\,, \tag{9}$$

$$\hat{\mathcal{A}}^B_{1/6} = \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} R^J{}_I Y^\dagger_J Y^I |\dot{x}|\,, \tag{10}$$

with $R^I{}_J = \mathrm{diag}(i, i, -i, -i)$. [Drukker, Plefka, Young, 08][Chen, **JW**, 08][Rey, Suyama, Yamaguchi, 08]

Half-BPS WLs

- These $1/6$-BPS WLs are dual to F-strings with worldsheet AdS_2 in $AdS_4 \times \mathbf{CP}^3$, smearing over a $\mathbf{CP}^1$ inside $\mathbf{CP}^3$. [Drukker, Plefka, Young, 08][Rey, Suyama, Yamaguchi, 08]

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- Similar string solutions with Dirichlet boundary conditions along all directions of $\mathbf{CP}^3$ should correspond to half-BPS Wilson loops invariant under $SU(3) \times U(1)$ inside $SU(4)_R$.
- But no such half-BPS WLs were found among the above $1/6$-BPS WLs. The susy enhancement (from $\mathcal{N} = 3$ to $\mathcal{N} = 6$ at generic k) in the ABJM theory does not apply to the constructions of WLs!

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- They found the half-BPS WLs by including the fermions in the construction.

$$W_{1/2} = \mathrm{Tr}\mathcal{P}\,\exp\left(-i\oint d\tau L_{1/2}(\tau)\right)\,, \quad L_{1/2} = \begin{pmatrix} \mathcal{A} & \bar{f}_1 \\ f_2 & \hat{\mathcal{A}} \end{pmatrix}\,,$$

$$\mathcal{A} = A_\mu \dot{x}^\mu + \frac{2\pi}{k} U_I{}^J Y^I Y_J^\dagger |\dot{x}|\,, \quad \bar{f}_1 = \sqrt{\frac{2\pi}{k}}\bar{\alpha}\bar{\zeta}\psi_1|\dot{x}|\,, \tag{11}$$

$$\hat{\mathcal{A}} = \hat{A}_\mu \dot{x}^\mu + \frac{2\pi}{k} U_I{}^J Y_J^\dagger Y^I |\dot{x}|\,, \quad f_2 = \sqrt{\frac{2\pi}{k}}\psi^{\dagger 1}\eta\beta|\dot{x}|\,, \tag{12}$$

with $\bar{\alpha}\beta = i$, and $U_I{}^J = \mathrm{diag}(i,-i,-i,-i)$.

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- When $\bar{\alpha}^1 \beta_1 = i$, these fermionic $1/6$-BPS WLs become half-BPS WLs.

Local operators

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- When C is symmetric and traceless, $\mathcal{O}_C$ is a chiral primary operator.
- Here we take $\mathcal{O}_C$ to be a generic local operator which is eigen-operator of the planar two-loop anomalous dimension matrix.

Wick contraction

- At tree-level, the correlator $\langle W(\mathcal{C})^B_{1/6}\mathcal{O}_C(0)\rangle$ only gets contributions from

$$\oint\cdots\oint d\tau_{1>2>\cdots>L}\left(\frac{2\pi}{k}\right)^L\langle\mathrm{tr}(R^{\tilde{J}_1}{}_{\tilde{I}_1}Y^{\tilde{I}_1}(x_1)Y^\dagger_{\tilde{J}_1}(x_1)\cdots R^{\tilde{J}_L}{}_{\tilde{I}_L}Y^{\tilde{I}_L}(x_L)Y^\dagger_{\tilde{J}_L}(x_L))C^{J_1\cdots J_L}_{I_1\cdots I_L}\mathrm{tr}(Y^{I_1}(0)Y^\dagger_{J_1}(0)\cdots Y^{I_L}(0)Y^\dagger_{J_L}(0))\rangle\,, \quad (15)$$

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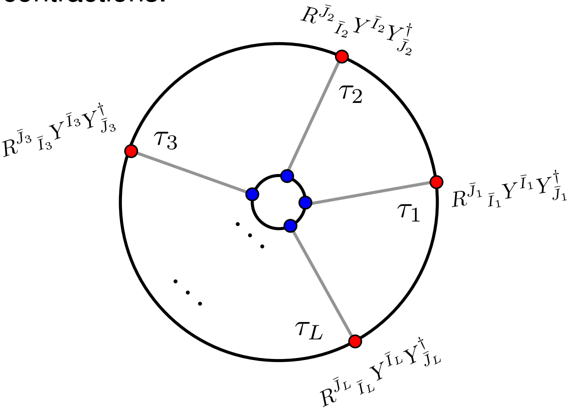
- where $x_i = (R\cos\tau_i, R\sin\tau_i, 0)$, $i = 1, \cdots, L$, and

$$\oint \cdots \oint d\tau_{1>2>\cdots>L} = \int_0^{2\pi} d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{L-1}} d\tau_L\,. \tag{16}$$

- In the large N limit, we only take into account planar Wick contractions.



Planar Wick contractions between the local operator and the Wilson loop.

Wick contraction

- One can easily obtain

$$\langle W(\mathcal{C})^{B}_{1/6}\mathcal{O}_C(0)\rangle = \frac{\lambda^{2L}k^L}{(L-1)!(2R)^{2L}} C^{J_1\cdots J_L}_{I_1\cdots I_L} R^{I_L}{}_{J_L}\cdots R^{I_1}{}_{J_1}\,, \qquad (17)$$

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- where $\lambda \equiv \frac{N}{k}$ is the 't Hooft coupling of ABJM theory and the tree-level propagators of the scalar fields

$$\langle Y^{I\alpha}{}_{\bar\beta}(x)Y^{\dagger\;\bar\gamma}_{J}{}_{\rho}(y)\rangle = \frac{\delta^I_J\delta^\alpha_\rho\delta^{\bar\gamma}_{\bar\beta}}{4\pi|x-y|}\,, \quad (18)$$

have been used.

Boundary state

- In the spin chain language, we can introduce the following boundary state

$$|\mathcal{B}^{B}_{1/6}\rangle = |\mathcal{B}_R\rangle \,, \tag{19}$$

where, for a four-dimensional matrix R, we define the boundary state $|\mathcal{B}_R\rangle$ through

$$\langle\mathcal{B}_R| \equiv R^{I_1}{}_{J_1} R^{I_2}{}_{J_2} \cdots R^{I_L}{}_{J_L} \langle I_1, J_1, \cdots, I_L, J_L| = \left(R^I{}_J \langle I, J|\right)^{\otimes L} \,, \tag{20}$$

which is a two-site state.

Overlap

- Then the above correlation function can be expressed as

$$\langle W(\mathcal{C})^B_{1/6} \mathcal{O}_C(0) \rangle = \frac{\lambda^{2L} k^L}{(L-1)!(2R)^{2L}} \langle \mathcal{B}^B_{1/6} | \mathcal{O}_C \rangle \,, \tag{21}$$

  where $|\mathcal{O}_C\rangle$ is the spin chain state corresponding to the operator $\mathcal{O}_C$.

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  where $|\mathcal{O}_C\rangle$ is the spin chain state corresponding to the operator $\mathcal{O}_C$.
- Our convention for the Hermitian conjugation and the overlap of the spin chain states is

$$\left(\langle I_1\bar{J}_1\cdots I_L\bar{J}_L|\right)^\dagger = |I_1\bar{J}_1\cdots I_L\bar{J}_L\rangle\,, \tag{22}$$

$$\langle I_1\bar{J}_1\cdots I_L\bar{J}_L|M_1\bar{N}_1\cdots M_L\bar{N}_L\rangle = \delta_{I_1M_1}\delta^{J_1N_1}\cdots \delta_{I_LM_L}\delta_{J_LN_L} \tag{23}$$

Norm

- Let us define the normalization factor $\mathcal{N}_{\mathcal{O}}$ using the two-point function of $\mathcal{O}$ and $\mathcal{O}^\dagger$ as

$$\langle \mathcal{O}(x)\mathcal{O}^\dagger(y)\rangle = \frac{\mathcal{N}_{\mathcal{O}}}{|x-y|^{2\Delta_{\mathcal{O}}}}\,, \tag{24}$$

where $\Delta_{\mathcal{O}}$ is the conformal dimension of $\mathcal{O}$.

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  where $\Delta_\mathcal{O}$ is the conformal dimension of $\mathcal{O}$.
- At tree level and the planar limit, we have

$$\mathcal{N}_\mathcal{O} = \left(\frac{N}{4\pi}\right)^{2L} L\langle\mathcal{O}|\mathcal{O}\rangle\,. \tag{25}$$

WL one-point function

- We define the Wilson-loop one-point function as

$$\langle\langle \mathcal{O} \rangle\rangle_{W(\mathcal{C})} \equiv \frac{\langle W(\mathcal{C})\mathcal{O}\rangle}{\sqrt{\mathcal{N}_{\mathcal{O}}}} \,. \tag{26}$$

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- Then for $W^B_{1/6}$ we have

$$\langle\langle \mathcal{O} \rangle\rangle_{W(\mathcal{C})^B_{1/6}} = \frac{\pi^L \lambda^L}{R^{2L}(L-1)!\sqrt{L}} \frac{\langle \mathcal{B}^B_{1/6} | \mathcal{O} \rangle}{\sqrt{\langle \mathcal{O} | \mathcal{O} \rangle}} . \tag{27}$$

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- The computation of the Wilson loop one-point function thus amounts to the calculation of

$$\frac{\langle \mathcal{B}^B_{1/6} | \mathcal{O} \rangle}{\sqrt{\langle \mathcal{O} | \mathcal{O} \rangle}} \,, \tag{28}$$

which will be performed by integrability in some cases.

Other boundary states from WLs

- For $\hat{W}(\mathcal{C})^B_{1/6}$, the boundary state is

$$\langle\hat{\mathcal{B}}^B_{1/6}| = R^{I_1}{}_{J_L} R^{I_2}{}_{J_1} \cdots R^{I_L}{}_{J_{L-1}} \langle I_1, J_1, \cdots, I_L, J_L| \,. \qquad (29)$$

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- We can rewrite $|\hat{\mathcal{B}}^B_{1/6}\rangle$ as

$$|\hat{\mathcal{B}}^B_{1/6}\rangle = U_{\text{even}} |\mathcal{B}^B_{1/6}\rangle \tag{30}$$

where U_{even} is the shift operator which shifts all even site to the left by two units and leave the odd sites untouched.
In another word, the action of U_{even} on the state $|I_1, J_1, I_2, J_2, \cdots, I_{L-1}, J_{L-1}, I_L, J_L\rangle$ gives $|I_1, J_2, I_2, J_3, \cdots, I_{L-1}, J_L, I_L, J_1\rangle$.

Other boundary states from WLs

- The boundary state from $W^F_{1/6}$ is

$$|\mathcal{B}^F_{1/6}\rangle = (1 + U_{\text{even}})|\mathcal{B}_U\rangle\,, \tag{31}$$

with $U = \text{diag}(i, i - 2\bar{\alpha}^1\beta_1, -i, -i)$.

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- The boundary state from $W_{1/2}$ is

$$|\mathcal{B}_{1/2}\rangle = |\mathcal{B}^F_{1/6}\rangle|_{\bar{\alpha}^1\beta_1 = i} \tag{32}$$

ABJM spin chain

- The operator $\mathcal{O}_C = C^{J_1\cdots J_L}_{I_1\cdots I_L}\mathrm{Tr}(Y^{I_1}Y^{\dagger}_{J_1}\cdots Y^{I_L}Y^{\dagger}_{J_L})$ can be mapped to a state $|C\rangle := C^{J_1\cdots J_L}_{I_1\cdots I_L}|I_1\bar{J}_1\cdots I_L\bar{J}_L\rangle$ on an alternating closed $SU(4)$ spin chain with length $2L$.

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- The Hilbert space of this chain is $\mathbf{C}^{8L} = \otimes_{i=1}^{2L} \mathbf{C}^4$.
- The odd site of the chain is in the $\mathbf{4}$ representation of $SU(4)$, while the even site is in the $\bar{\mathbf{4}}$ representation.

Hamiltonian

- The planar two-loop anomalous dimensional matrix can be map to the following Hamiltonian on the above chain ([Minahan, Zarembo, 08][Bak, Rey, 08]),

$$\mathbb{H} = \frac{\lambda^2}{2} \sum_{l=1}^{2L} (2 - 2\mathrm{P}_{l,l+2} + \mathrm{P}_{l,l+2}\mathrm{K}_{l,l+1} + \mathrm{K}_{l,l+1}\mathrm{P}_{l,l+2}) \ , \tag{33}$$

where P_{ab} and K_{ab} are permutation and trace operators acting on the a-th and b-th sites. We denote the set of orthonormal basis of the Hilbert space at each site by $|i\rangle$, $i = 1, \cdots, 4$. The two operators act as

$$\mathrm{P}|i\rangle \otimes |j\rangle = |j\rangle \otimes |i\rangle, \qquad \mathrm{K}|i\rangle \otimes |j\rangle = \delta_{ij} \sum_{k=1}^{4} |k\rangle \otimes |k\rangle \, . \tag{34}$$

Integrability

- In the algebraic Bethe ansatz (ABA) approach, we introduce the following R-matrices

$$\begin{aligned} R_{12}^{\bullet\bullet}(u) &= R_{12}^{\circ\circ}(u) = u + P_{12} \equiv R_{12}(u)\,, \\ R_{12}^{\bullet\circ}(u) &= R_{12}^{\circ\bullet}(u) = -u - 2 + K_{12} \equiv \bar{R}_{12}(u)\,, \end{aligned} \tag{35}$$

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- where $\bullet$ denotes the states in the $\mathbf{4}$ representation of $SU(4)_R$, while $\circ$ denotes the states in the $\bar{\mathbf{4}}$ representation.
- These R-matrices satisfy a set of Yang-Baxter equations and the following crossing symmetry relation,

$$R_{12}(u)^{t_1} = \bar{R}_{12}(-u-2), \qquad \bar{R}_{12}(u)^{t_1} = R_{12}(-u-2)\,. \tag{36}$$

Integrability

- Using these R-matrices one can constructed two transfer matrices $\tau(u)$ and $\bar{\tau}(u)$, satisfying

$$[\tau(u), \tau(v)] = [\tau(u), \bar{\tau}(v)] = [\bar{\tau}(u), \bar{\tau}(v)] = 0\,. \tag{37}$$

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- They are generating functions of commuting conserved charges, among whom there is the Hamiltonian.
- This proves the integrability of two-loop ABJM spin chain. [Minahan, Zarembo, 08][Bak, Rey, 08]

Bethe roots

- Eigenstates of $\mathbb{H}$ can be constructed using R-matrices and the states are parameterized by three set of Bethe roots,

$$u_1, \cdots, u_{K_{\mathbf{u}}}, \tag{38}$$

$$v_1, \cdots, v_{K_{\mathbf{v}}}, \tag{39}$$

$$w_1, \cdots, w_{K_{\mathbf{w}}}. \tag{40}$$

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$$w_1, \cdots, w_{K_{\mathbf{w}}}. \tag{40}$$

- One selection rule for $\langle \mathcal{B}_R | \mathbf{u}, \mathbf{v}, \mathbf{w} \rangle$ being nonzero is that $K_{\mathbf{u}} = K_{\mathbf{v}} = K_{\mathbf{w}} = L$.

Bethe ansatz equations

- These Bethe roots should satisfy the following Bethe ansatz equations,

$$1=\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L\prod_{\substack{k=1\\k\neq j}}^{K_{\mathrm{u}}}S\left(u_j,u_k\right)\prod_{k=1}^{K_{\mathrm{w}}}\tilde{S}\left(u_j,w_k\right),\tag{41}$$

$$1=\prod_{\substack{k=1\\k\neq j}}^{K_{\mathrm{w}}}S\left(w_j,w_k\right)\prod_{k=1}^{K_{\mathrm{u}}}\tilde{S}\left(w_j,u_k\right)\prod_{k=1}^{K_{\mathrm{v}}}\tilde{S}\left(w_j,v_k\right),\tag{42}$$

$$1=\left(\frac{v_j+\frac{i}{2}}{v_j-\frac{i}{2}}\right)^L\prod_{\substack{k=1\\k\neq j}}^{K_{\mathrm{v}}}S\left(v_j,v_k\right)\prod_{k=1}^{K_{\mathrm{w}}}\tilde{S}\left(v_j,w_k\right),\tag{43}$$

Bethe ansatz equations

- In the previous page, the S-matrices $S(u,v)$ and $\tilde{S}(u,v)$ are given by

$$S(u,v) \equiv \frac{u-v-i}{u-v+i}, \quad \tilde{S}(u,v) \equiv \frac{u-v+\frac{i}{2}}{u-v-\frac{i}{2}}. \tag{44}$$

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- The cyclicity property of the single trace operator is equivalent to the zero momentum condition

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- The eigenvalues of $\tau(u), \bar{\tau}(u), \mathbb{H}$ on the Bethe state $|\mathbf{u},\mathbf{v},\mathbf{w}\rangle$ can be expressed in terms of the Bethe roots, $\mathbf{u},\mathbf{v},\mathbf{w}$.

Numerical solution

- The BAEs and zero momentum condition can be solved using rational Q-system. [Marboe, Volin, 16][Gu, Jiang, Sperling, 22].

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- The BAEs and zero momentum condition can be solved using rational Q-system. [Marboe, Volin, 16][Gu, Jiang, Sperling, 22].
- The Bethe states can be constructed using the algorithm in [Yang, Jiang, **JW**, Komatsu, 21] based on coordinate Bethe ansatz.

IBS from WLs

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Theorem

If a four-dimensional matrix $K(u)$ satisfies the following boundary Yang-Baxter equation,

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v) = K_2(v)R_{12}(u+v) \\ K_1(u)R_{12}(u-v)\,, \qquad (46)$$

the boundary state

$$|\mathcal{B}_M\rangle \equiv M^{I_1}{}_{J_1} M^{I_2}{}_{J_2} \cdots M^{I_L}{}_{J_L} |I_1, J_1, \cdots, I_L, J_L\rangle = \left(M^I{}_J |I, J\rangle\right)^{\otimes L} \,, \qquad (47)$$

with $M = K(-1)^*$ is integrable in the sense explained in the next page.

-

A key selection rule

- When the condition of the theorem is satisfied, we have that $|\mathcal{B}_M\rangle$ satisfying the following untwisted integrable condition,

$$\tau(-u-2)|\mathcal{B}_M\rangle = \tau(u)|\mathcal{B}_M\rangle. \quad (48)$$

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- This leads to the pairing condition which states that $\langle\mathcal{B}_M|\mathbf{u},\mathbf{v},\mathbf{w}\rangle$ is non-zero only when the selection rule

$$\mathbf{u} = -\mathbf{v}\,, \qquad \mathbf{w} = -\mathbf{w} \tag{49}$$

is satisfied.

IBS from WLs

- Using this theorem, we can prove that the boundary state from bosonic $1/6$-BPS Wilson loop, $|\mathcal{B}_R\rangle$ is integrable.
- We just take $K(u) = R$. (Notice this R is the one appearing in the definition of $|\mathcal{B}_R\rangle$, it is not the R-matrices in the ABA approach.)
- Similarly we proved that the half-BPS WLs give integrable boundary state.

Non-integrable boundary states

- For the boundary state from a generic(*) fermionic $1/6$-BPS WL, we perform the following $SO(4) \subset SU(4)_R$ transformation [Gombor, Bajnok, 20]

$$M_{g(\theta)} = g(\theta) M g(\theta)^{-1} \,, \tag{50}$$

with

$$g(\theta) = \begin{pmatrix} \cos^2\theta & \sin\theta & 0 & \sin\theta\cos\theta \\ -\sin\theta\cos^2\theta & \cos^2\theta & \sin\theta & -\sin^2\theta\cos\theta \\ \sin^2\theta\cos\theta & -\sin\theta\cos\theta & \cos\theta & \sin^3\theta \\ -\sin\theta & 0 & 0 & \cos\theta \end{pmatrix} , \tag{51}$$

where θ satisfies $0 < \theta < \frac{\pi}{2}$.

Non-integrable boundary states

- Since all R-matrices are $SU(4)_R$ invariant, $(1 + U_{even})|\mathcal{B}_M\rangle$ is integrable if and only if $(1 + U_{even})|\mathcal{B}_{M_{g(\theta)}}$ is.

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- We found the following set of Bethe roots with $L=3, K_{\mathbf{u}}=K_{\mathbf{w}}=1, K_{\mathbf{v}}=2,$

$$\begin{aligned} u_1 &= 0.866025, \quad w_1 = 0.866025, \\ v_1 &= -0.198072, \quad v_2 = 0.631084\,. \end{aligned} \tag{52}$$

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- Notice that this set of Bethe roots does not satisfy the selection rule: $\mathbf{u}=-\mathbf{v}, \mathbf{w}=-\mathbf{w}$.

Non-integrable boundary states

- We found that for these Bethe roots, the Bethe states $|\mathbf{u}, \mathbf{w}, \mathbf{v}\rangle$ has nonzero overlap with $(1 + U_{even})|\mathcal{B}_{M_{g(\theta)}}\rangle$ when $\bar{\alpha}^1\beta_1 \neq 0, i$.

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- So for generic $\bar{\alpha}^1$ and β_1 satisfying $\bar{\alpha}^1 \beta_1 \neq 0, i$, the boundary state from the fermonic $1/6$-BPS WL is not integrable.

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- So for generic $\bar{\alpha}^1$ and β_1 satisfying $\bar{\alpha}^1\beta_1 \neq 0, i$, the boundary state from the fermonic $1/6$-BPS WL is not integrable.
- Notice that when $\bar{\alpha}^1\beta_1 = i$, the WL is the half-BPS one.

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- So for generic $\bar{\alpha}^1$ and β_1 satisfying $\bar{\alpha}^1\beta_1 \neq 0, i$, the boundary state from the fermonic $1/6$-BPS WL is not integrable.
- Notice that when $\bar{\alpha}^1\beta_1 = i$, the WL is the half-BPS one.
- And when $\bar{\alpha}^1\beta_1 = 0$, the WL is essential the bosonic $1/6$-BPS one.

Overlaps

- We obtained the following formula for the normalized overlap between $|\mathcal{B}_R\rangle$ and a Bethe state,

$$\frac{|\langle\mathcal{B}_R|\mathbf{u},\mathbf{v},\mathbf{w}\rangle|^2}{\langle\mathbf{u},\mathbf{v},\mathbf{w}|\mathbf{u},\mathbf{v},\mathbf{w}\rangle} = \prod_{i=1}^{K_{\mathbf{w}}/2} \frac{w_i^2}{w_i^2 + 1/4} \times \frac{\det G^+}{\det G^-}. \tag{53}$$

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- Here the Bethe roots satisfy the pairing condition, $G^{\pm}$ are Gaudin determinants depending on $\mathbf{u},\mathbf{v},\mathbf{w}$.
- This result was obtained using [Gombor, Bajnok, 20][Gombor, Kristjansen, 22] and passed non-trivial checks based on numerical computations.

Overlaps

- For another bosonic $1/6$-BPS WL, we have

$$\frac{\langle\widehat{\mathcal{B}}_R|\mathbf{u},\mathbf{v},\mathbf{w}\rangle}{\sqrt{\langle\mathbf{u},\mathbf{v},\mathbf{w}|\mathbf{u},\mathbf{v},\mathbf{w}\rangle}}=\prod_{j=1}^{K_{\mathbf{u}}}\frac{u_j+i/2}{u_j-i/2}\frac{\langle\mathcal{B}_R|\mathbf{u},\mathbf{v},\mathbf{w}\rangle}{\sqrt{\langle\mathbf{u},\mathbf{v},\mathbf{w}|\mathbf{u},\mathbf{v},\mathbf{w}\rangle}}. \tag{54}$$

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- Hence there is a relative phase between these two boundary state.

Overlaps

For half-BPS WLs, we have

$$\frac{|\langle \mathcal{B}_{1/2}|\mathbf{u}, -\mathbf{u}, \mathbf{w}\rangle|^2}{\langle \mathbf{u}, -\mathbf{u}, \mathbf{w}|\mathbf{u}, -\mathbf{u}, \mathbf{w}\rangle} = \left|1 + \prod_{j=1}^{K_{\mathbf{u}}} \left(\frac{u_j + i/2}{u_j - i/2}\right)^2\right|^2 \frac{|\langle \mathcal{B}_U|\mathbf{u}, -\mathbf{u}, \mathbf{w}\rangle|^2}{\langle \mathbf{u}, -\mathbf{u}, \mathbf{w}|\mathbf{u}, -\mathbf{u}, \mathbf{w}\rangle}\,. \tag{55}$$

$$\frac{|\langle \mathcal{B}_U|\mathbf{u}, -\mathbf{u}, \mathbf{w}\rangle|^2}{\langle \mathbf{u}, -\mathbf{u}, \mathbf{w}|\mathbf{u}, -\mathbf{u}, \mathbf{w}\rangle} = (-1)^L \prod_{i=1}^{K_{\mathbf{u}}} (u_i^2 + \frac{1}{4}) \prod_{j=1}^{[K_{\mathbf{w}}/2]} \frac{1}{w_i^2(w_i^2 + 1/4)} \frac{\det G_+}{\det G_-}\,. \tag{56}$$

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- For generic fermionic $1/6$-BPS WLs, the corresponding boundary states are not integrable.
- We computed the norm of the overlap of the integrable boundary states from WLs and the Bethe states.

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Thanks for Your Attention !