Understanding DAHA from physics

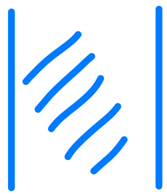
w/ Gukov, Koroteev, Pei, Saberi

Plan

- 1^{st} Talk is to understand rep thy of DAHA $\ddot{H}$

by 2d A-model on $\mathfrak{X} = \mathcal{M}_{flat}(T^2 \setminus pt, SL(2,\mathbb{C}))$

σ-model



$\longrightarrow \mathfrak{X}$

B_{cc} B_L

$\text{A-brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) \cong \mathrm{Rep}(S\ddot{H})$

- 2^{nd} talk is to uplift the story to

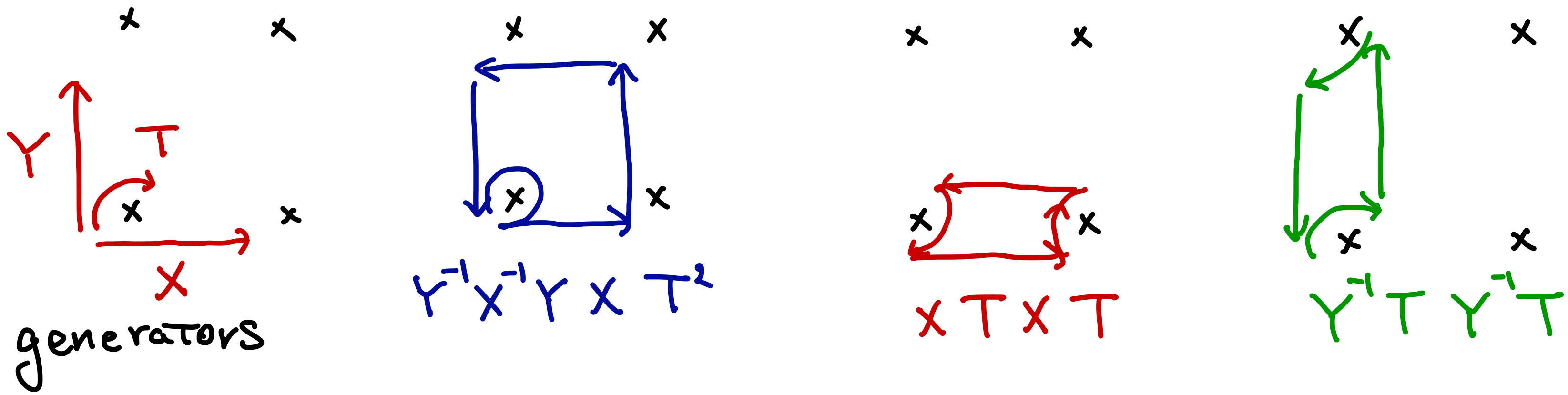
3d modularity $\quad V = K^0(\mathrm{MTC})$

4d $\mathcal{N}=2^*$ thy $\quad$ Coulomb branch, line ops

Double Affine Hecke Algebra (DAHA) Cherednik

- Orbifold fundamental group (Double affine braid group)

$$\pi_1((T^2 \setminus pt)/\mathbb{Z}_2) = \left[T^{\pm}, X^{\pm}, Y^{\pm} \;\middle|\; \begin{array}{c} Y^{-1}X^{-1}YXT^2 = 1 \\ XTXT = 1 \\ Y^{-1}TY^{-1}T = 1 \end{array} \right]$$



$q = t^c$ c: central charge.

- DAHA

$$\ddot{H} = \ddot{H}(sl_2) = \left[T^{\pm}\, X^{\pm}\, Y^{\pm} \;\middle|\; \begin{array}{c} Y^{-1}X^{-1}YXT^2 = q^{-1} \\ TXT = X^{-1} \\ TY^{-1}T = Y \end{array} \quad (T-t)(T+t^{-1}) = 0 \right]$$

Double Affine Hecke Algebra (DAHA), Cherednik

- Symmetry $PSL(2.\mathbb{Z}) \times \Xi$ (Very important!!)

$$PSL(2.\mathbb{Z}) : (X.Y.T) \xrightarrow{\tau_+} (X.\, q^{-\frac{1}{2}} XY.\, T)$$
$$(X.Y.T) \xrightarrow{\tau_-} (q^{\frac{1}{2}} YX.\, Y.\, T)$$
$$(X.Y.T) \xrightarrow{\sigma} (Y^{-1}.\, XT^2.\, T)$$

Ξ : $\xi_1 : X \to -X$, $\xi_2 : Y \to -Y$. sign change.

- Spherical subalgebra. (gauge invariant part).

$$e = \frac{T + t^{-1}}{t + t^{-1}} \qquad \text{idempotent} \qquad e^2 = e.$$

$$S\ddot{H} = e\ddot{H}e = \left[\begin{array}{c|l} x = (X + X^{-1})e & [x.y]_q = (q^{-1} - q)\, z \\ y = (Y + Y^{-1})e & [y.z]_q = (q^{-1} - q)\, x \\ z = (q^{-\frac{1}{2}} Y^{-1} X + q^{\frac{1}{2}} X^{-1} Y)e & [z.x]_q = (q^{-1} - q)\, y \\ & q^{-1}x^2 + qy^2 + q^{-1}z^2 - q^{-\frac{1}{2}} xyz \\ & = (q^{-\frac{1}{2}} t - q^{\frac{1}{2}} t^{-1})^2 + (q^{\frac{1}{2}} + q^{-\frac{1}{2}})^2 \end{array} \right]$$

Spherical DAHA as $\mathcal{O}^q(\mathcal{X})$

$\mathcal{O}$ ✗

- Classical limit $q \to 1$ of central element.

$$x^2+y^2+z^2-xyz = 4+(t-t^{-1})^2 = 2+\mathrm{Tr}\,V \qquad V=\begin{pmatrix} t^2 & 0 \\ 0 & t^{-2}\end{pmatrix}$$

is moduli sp of flat $SL(2,\mathbb{C})$-connections on $T^2\setminus\{pt\}$

Oblomkov.

$$S\ddot{H} \cong \mathcal{O}^q[\mathcal{M}_{flat}(T^2\setminus\{pt\},\ SL(2,\mathbb{C}))]$$

. deformation quantization of coord. ring w.r.t. Ω_J.

- Back to Symmetry.

Weyl.

& $i: t \to t^{-1}$

$$(x,y,z) \xrightarrow{T_+} (x,\ xy-z,\ y)$$
$$(x,y,z) \xrightarrow{T_-} (xy-z,\ y,\ x)$$
$$(x,y,z) \xrightarrow{\sigma} (y,\ x,\ xy-z)$$

$$(x,y,z) \xrightarrow{\xi_1} (-x,\ y,\ -z)$$
$$(x,y,z) \xrightarrow{\xi_2} (x,\ -y,\ -z)$$
$$(x,y,z) \xrightarrow{\xi_3} (-x,\ -y,\ z)$$

Representations of spherical DAHA.

- polynomial representations.

$$\mathrm{pol} : S\ddot{H} \longrightarrow \mathrm{End}\left(\mathbb{C}_{q,t}[X, X^{-1}]^{\mathbb{Z}_2} \right)$$

$$x \longmapsto X + X^{-1}$$

$$y \longmapsto \frac{tX - t^{-1}X^{-1}}{X - X^{-1}} q^{\partial_x} + \frac{t^{-1}X - tX^{-1}}{X - X^{-1}} q^{-\partial_x}$$ Macdonald difference op.

$$z \longmapsto X^{-1} \frac{tX - t^{-1}X^{-1}}{X - X^{-1}} q^{\partial_x} + X \frac{t^{-1}X - tX^{-1}}{X - X^{-1}} q^{-\partial_x}$$

- basis of $\mathbb{C}_{q,t}[X, X^{-1}]^{\mathbb{Z}_2}$ is spanned by Macdonald poly $\{P_\ell(x;q,t)\}$

$$y \cdot P_j(x;q,t) = (q^j t + q^{-j} t^{-1}) P_j(x;q,t)$$

- raising and lowering operators

$$R_j = x - q^{j-\frac{1}{2}} t z \qquad R_j \cdot P_j = (1 - q^{-2j} t^{-2}) P_{j+1}$$

$$L_j = x - q^{-j-\frac{1}{2}} t^{-1} z \qquad L_j \cdot P_j = -\frac{(1 - q^{2(j-1)} t^4)(1 - q^{2j})}{q^{2j} t^2 (1 - q^{2(j-1)} t^2)} P_{j-1}$$

Finite-dimensional Rep

$$
\begin{tikzcd}
1 \arrow[r, bend left, "R"] & P_1 \arrow[l, bend left, "L"] \arrow[r, bend left, "R"] & P_2 \arrow[l, bend left, "L"] \arrow[r, bend left, "R"] & \cdots \arrow[l, bend left, "R"] & P_{n-1} \arrow[r, bend left, "R"] & P_n \arrow[l, bend left, "L \ \text{null vector}"] \arrow[r, bend left] & \cdots \arrow[l, bend left]
\end{tikzcd}
$$

$$\rightsquigarrow \quad \mathrm{S\ddot{H}} \curvearrowright V = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (P_n) \quad \text{finite-dim rep.}$$

shortening condition

$$\frac{(1-q^{2j})(1-q^{j-1}t^2)(1+q^{j-1}t^2)}{q^{2j}t^2(1-q^{2(j-1)}t^2)} = 0$$

① $q^{2n} = 1$

② $t^2 = -q^{-k}$ $\quad \rightsquigarrow$ different finite-dim reps

③ $t^2 = q^{-(2\ell-1)}$ $\quad$ we will see geometrically.

Geometry of Hitchin moduli space

- $$\mathcal{M}_{flat}(C, SL(2,\mathbb{C})) \cong \mathcal{M}_H(C, SU(2))$$

$$\left\{ \begin{array}{c} \text{flatness of } SL(2,\mathbb{C})\text{-conn} \\ A+\varphi+\varphi^* \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (E,\Phi) \\ \varphi \in \mathrm{End}(E)\otimes K_C \end{array} \middle| \begin{array}{c} F_A+[\phi,\phi^*]=0 \\ \bar{D}_A\phi=0 \end{array} \right\}$$

non-Abelian Hodge.

Hyperkähler manifold

from C

$\downarrow$

(I, J, K)

$\uparrow$

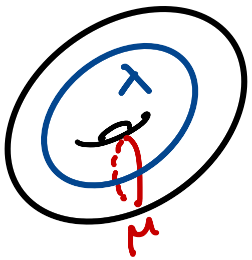
from character variety.

$$\Omega_J = \frac{dx\wedge dy}{2z-xy}.$$

(x,y,z). holom. in J.

- $$x^2+y^2+z^2-xyz = 2+\mathrm{Tr}\,V.$$

no holonomy $V=\begin{pmatrix}1&0\\0&1\end{pmatrix}$



$$\mu\sim\begin{pmatrix}X&*\\0&X^{-1}\end{pmatrix}$$

$$\lambda\sim\begin{pmatrix}Y&*\\0&Y^{-1}\end{pmatrix}$$

$$\begin{array}{ccc} \mathcal{M}_{flat}(T^2, SL(2,\mathbb{C})) & = & \dfrac{\mathbb{C}^\times\times\mathbb{C}^\times}{\mathbb{Z}_2} \\ \cup & & \cup \\ \mathcal{M}_{flat}(T^2, SU(2)) & = & \dfrac{S^1\times S^1}{\mathbb{Z}_2} \end{array}$$

Ramification

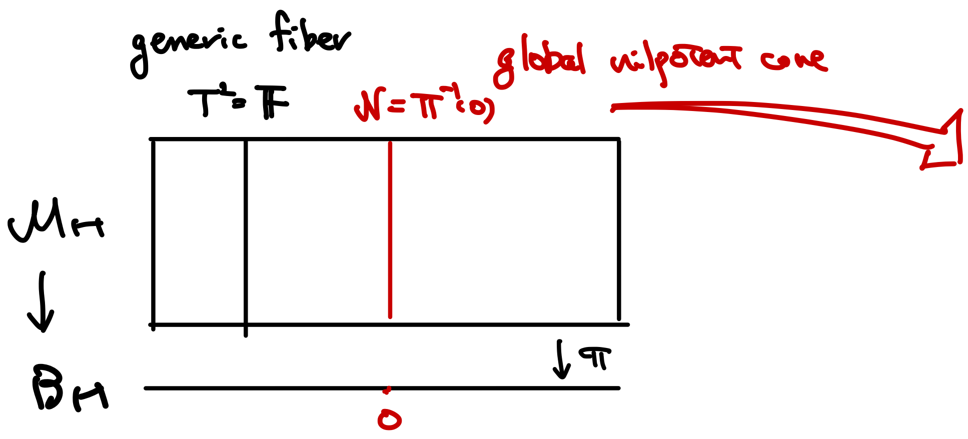
$$A = i\alpha\, d\theta$$

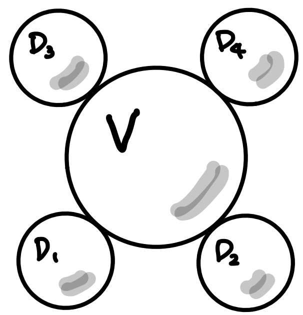
$$\varphi = \frac{1}{2}(\beta + i\gamma)\frac{dz}{z}$$

$t = \exp(-\pi(\gamma + i\alpha))$ complex str on J.

When $\alpha \neq 0$. $\beta = 0 = \gamma$

- Hitchin fibration $\quad \pi : \mathcal{M}_H \longrightarrow \mathcal{B}_H = H^0(C, K_C^{\otimes 2}) = \{\mathrm{Tr}\,\phi^2\}$





affine $\hat{D}_4$ singularity

Kodaira I_0^* type.

$$\mathcal{N} = \mathrm{Bun}_G \cup \bigcup_{i=1}^{4} D_i$$

- 2^{nd} homology of $\mathcal{M}_H$ has relation

$$[\mathbb{F}] = 2[V] + \sum_{i=1}^{4}[D_i]$$

- Exceptional fibers $D_1 \cdots D_4$ show up when we turn on α.

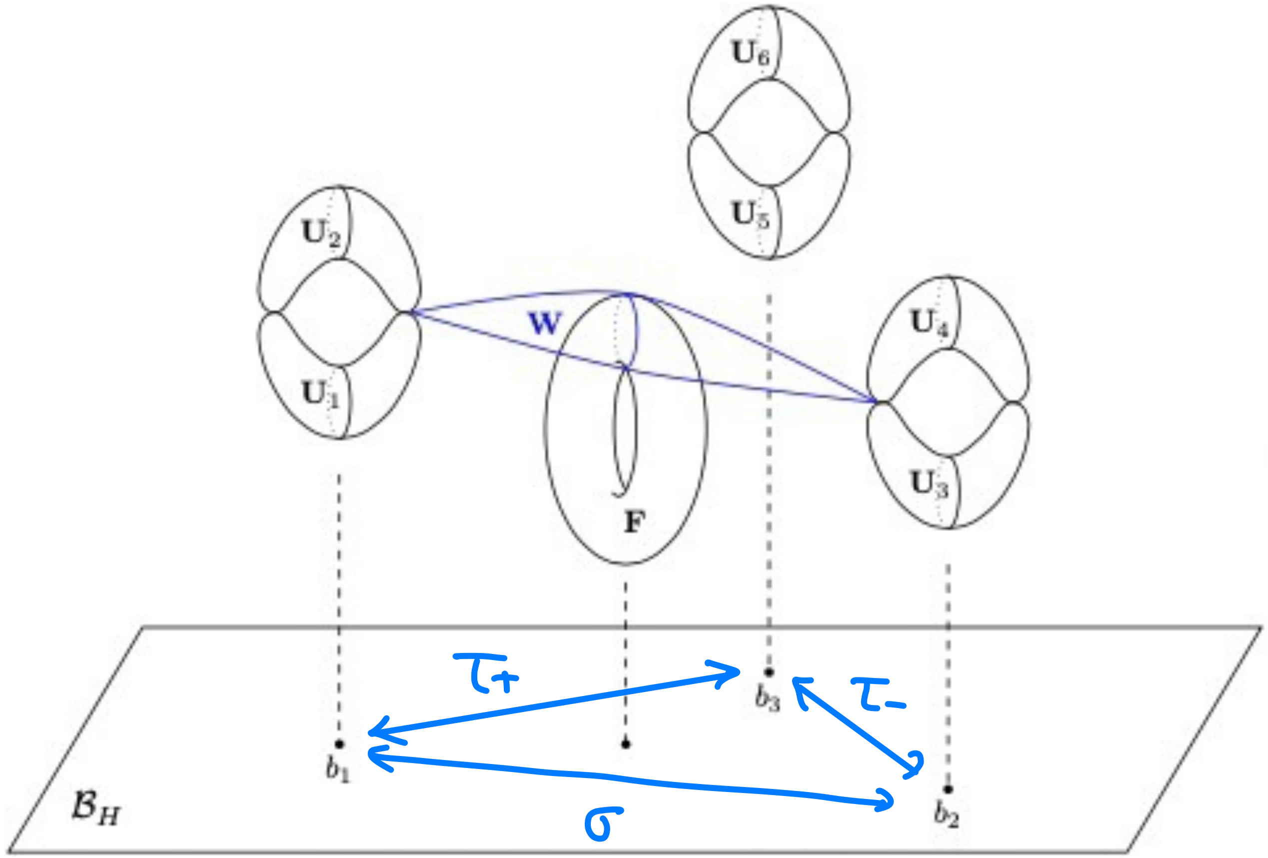
Geometry of Hitchin moduli space

At generic ramification $\beta, \gamma \neq 0$.

$$\alpha = \int_{D_i} \frac{\omega_I}{\pi} \qquad \beta = \int_{D_i} \frac{\omega_J}{\pi} \qquad \gamma = \int_{D_i} \frac{\omega_K}{\pi}$$

$$\int_F \frac{\omega_I}{2\pi} = 1$$

$$\int_F \frac{\omega_J}{2\pi} = 0 = \int_F \frac{\omega_K}{2\pi}.$$



$$U_1 = V + D_1 + D_2 \qquad U_3 = V + D_1 + D_3 \qquad U_5 = V + D_1 + D_4$$

$$U_2 = V + D_3 + D_4 \qquad U_4 = V + D_2 + D_4 \qquad U_6 = V + D_2 + D_3$$

as. H_2 homology class

A-model on $\mathcal{X} = \mathcal{M}_{flat}(T^2 \setminus pt, SL(2,\mathbb{C}))$

- 2d A-model on $\Sigma \to (\mathcal{X}, \omega_{\mathcal{X}})$

- Σ can have boundary. Boundary condition is described by A-branes.

- Usual A-branes are "flat unitary bundle E over Lagrangian L" $\mathcal{B}_L = \begin{pmatrix} E \\ \downarrow \\ L \end{pmatrix}$

- Kapustin-Orlov some A-branes are supported on coisotropic submfd. $\mathcal{Z}$
  (i.e. submfd locally defined by Poisson commuting fns)

- One distinguished object "canonical coisotropic brane $\mathcal{B}_{cc}$"

$$\mathcal{B}_{cc} = \begin{pmatrix} \mathcal{L} \\ \downarrow \\ \mathcal{M}_H \end{pmatrix} \qquad c_1(\mathcal{L}) = F$$

$$F + B + i\omega_{\mathcal{X}} = \frac{\Omega_J}{i\hbar} \rightsquigarrow q = e^{2\pi i \hbar}$$

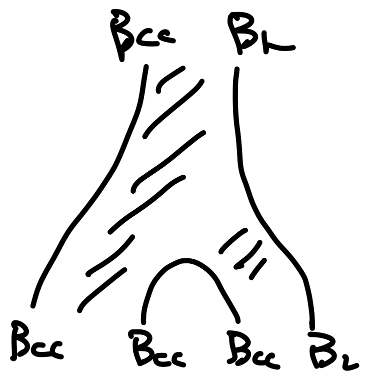
$$\tau = \exp(-\pi(\gamma + i\alpha))$$

Brane quantization   Gukov – Witten.

- (B_{cc}, B_{cc})-string : deformation quantization of holom. fns w.r.t. J over M_H.

- joining of (B_{cc}, B_{cc})-string and (B_{cc}, B_L)-string gives rise to another (B_{cc}, B_L)-string.

$$
\begin{array}{ccc}
\mathrm{Hom}(B_{cc}, B_{cc}) & \longleftrightarrow & S\ddot{H} \\
\circlearrowright & & \circlearrowright \\
\mathrm{Hom}(B_{cc}, B_L) & \longleftrightarrow & V_L \\
\cap & & \cap \\
A\text{-brane}(\mathfrak{X}, \omega_{\mathfrak{X}}) & \longleftrightarrow & \mathrm{Rep}(S\ddot{H})
\end{array}
$$



- Given A-brane, there is the corresponding module V_L

compact A-brane $B_L \longleftrightarrow$ finite-dimensional module V_L

$$\dim V = \int_L \mathrm{ch}(B_{cc} \otimes B_L^{-1})\, \mathrm{Td}(L) \quad \text{Hirzebruch–Riemann–Roch}$$

Brane with compact support & finite-dim reps

- generic fiber F

HRR $\rightsquigarrow$ $\frac{1}{\hbar} = 2n$ $\rightsquigarrow$ shortening cond ① $q^{2n} = 1$

encode holonomy position

$$\mathrm{Hom}(B_{cc}, B_F) = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (X^{2n} + X^{-2n} - a - a^{-1})$$

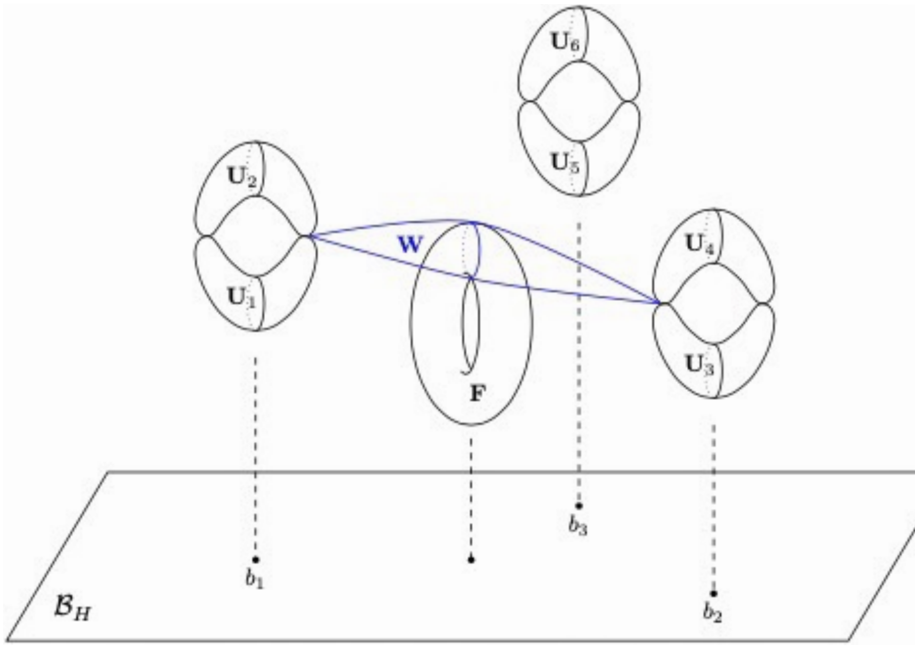
- irreducible comp U_1

$$\mathrm{Hom}(B_{cc}, B_{U_1}) = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (P_n)$$

$$0 \to U_2 \to \overset{a=-1}{F} \to U_1 \to 0$$

$$0 \to U_2 \to \zeta_2 \cdot i(\overset{a=-1}{F}) \to U_1 \to 0$$

generates $\mathrm{Ext}^1(U_1, U_2)$



Brane with compact support & finite-dim reps

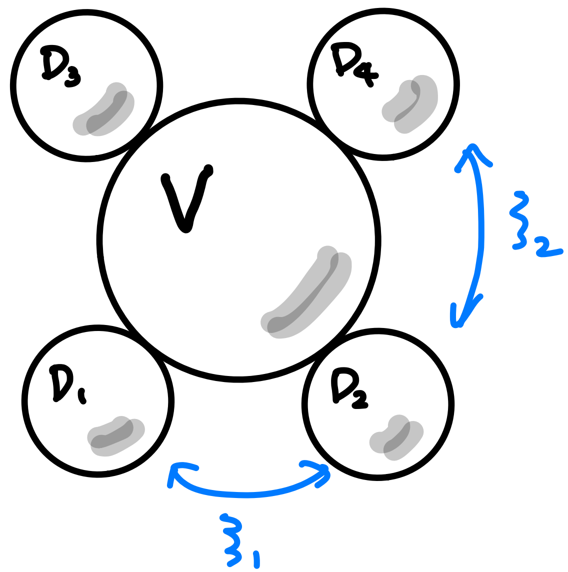
- moduli space of G-bundle V

HRR $\rightsquigarrow$ shortening cond ② $\tau^2 = -q^{-k}$

$$\mathrm{Hom}(B_{cc}, B_V) = \mathbb{C}_{q,\tau}[X^{\pm}]^{\mathbb{Z}_2} / (P_k)$$

- Exceptional divisor D_i

HRR $\rightsquigarrow$ shortening cond ③ $\tau^2 = q^{-2l+1}$

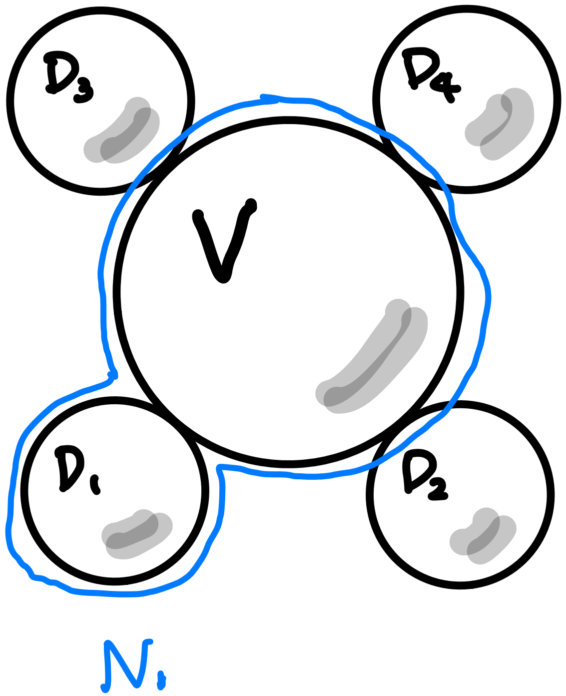


$$\begin{array}{c} \mathrm{Hom}(B_{cc}, B_{D_1}) \\ \oplus \\ \mathrm{Hom}(B_{cc}, B_{D_2}) \end{array} = \mathbb{C}_{q,t}[X^{\pm}]^{\mathbb{Z}_2} / (P_{2l})$$

Bound states of branes & Morphism structure

$$0 \to V \to N_1 \to D_1 \to 0$$

generates $\mathrm{Ext}^1(D_1, V)$



This gives evidence

$$\text{A-brane}(\mathcal{X}, \omega_{\mathcal{X}}) \cong \mathrm{Rep}(s\ddot{l}_2)$$

Advantages.

- new reps
- geometric understanding.

Fukaya.

- New phenomena in A-brane
- lift to higher-dim physics

2nd half of the talk.

PSL(2.$\mathbb{Z}$) action

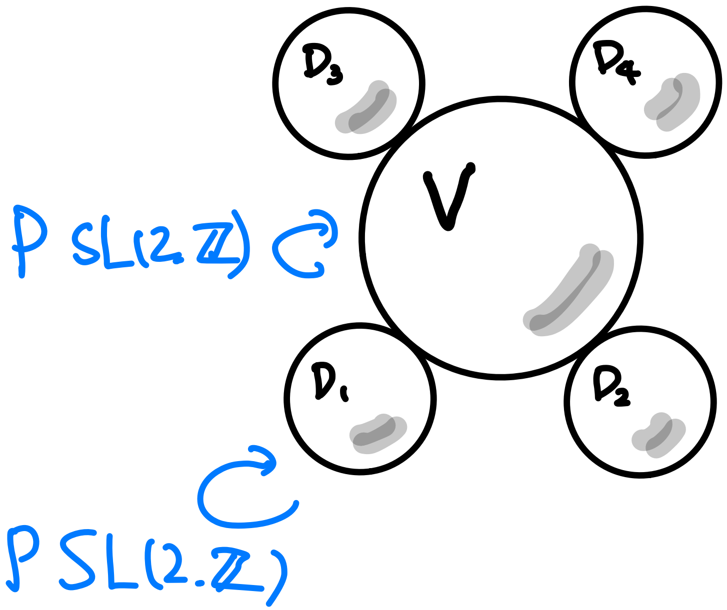
Sït & target space $\mathfrak{X}$ enjoy PSL(2.$\mathbb{Z}$) action

Compact Lagrangian V & D_1 are invariant under PSL(2.$\mathbb{Z}$)

$PSL(2.\mathbb{Z}) \circlearrowright Hom(B_{cc}.B_V)$

$\rightsquigarrow$

$PSL(2.\mathbb{Z}) \circlearrowright Hom(B_{cc}.B_{D_1})$



question

Which modular rep?

answer from 3d physics

3d / 3d Correspondence

$$
\begin{array}{ccccccccc}
 & S^1 & \times & \mathbb{R}^2_q & \times & \mathbb{R}^2_t & \times & T^*M_3 \\
N\ M5 & \underbrace{S^1 \times \mathbb{R}^2_q}_{3d\ \mathcal{N}=2\ T[M_3]} & & & \times & pt & \times & \underbrace{M_3}_{\mathbb{C}\ CS.}
\end{array}
$$

M_3: Seifert mfd.

Aganagic–Shakirov, Yoshida–Sugiyama, Gukov–Putrov–Vafa

$$Z = \mathrm{Tr}\,(-1)^F q^{J_3+S} t^{R-S}$$

e.g. $M_3 = L(p,1)$

$$\hat{Z}_b = \int_{|x|=1} \frac{dx}{2\pi i x} \prod_{\alpha\in\Delta_+} \frac{(x^\alpha;q)_\infty}{(t^2 x^\alpha;q)_\infty} \Theta_{b,p}(x;q)$$

generalized Θ-fn

$\mathcal{M}_{\mathrm{flat}}(L(p,1), G)$

e.g. $M_3 = S^3$

Macdonald

$$S_{\mu\nu} = \int_{|x|=1} \frac{dx}{2\pi i x} \prod_{\alpha\in\Delta_+} \frac{(x^\alpha;q)_\infty}{(t^2 x^\alpha;q)_\infty} \Theta(x;q)\, P_\lambda(x)\, \overline{P_\mu(x)}$$

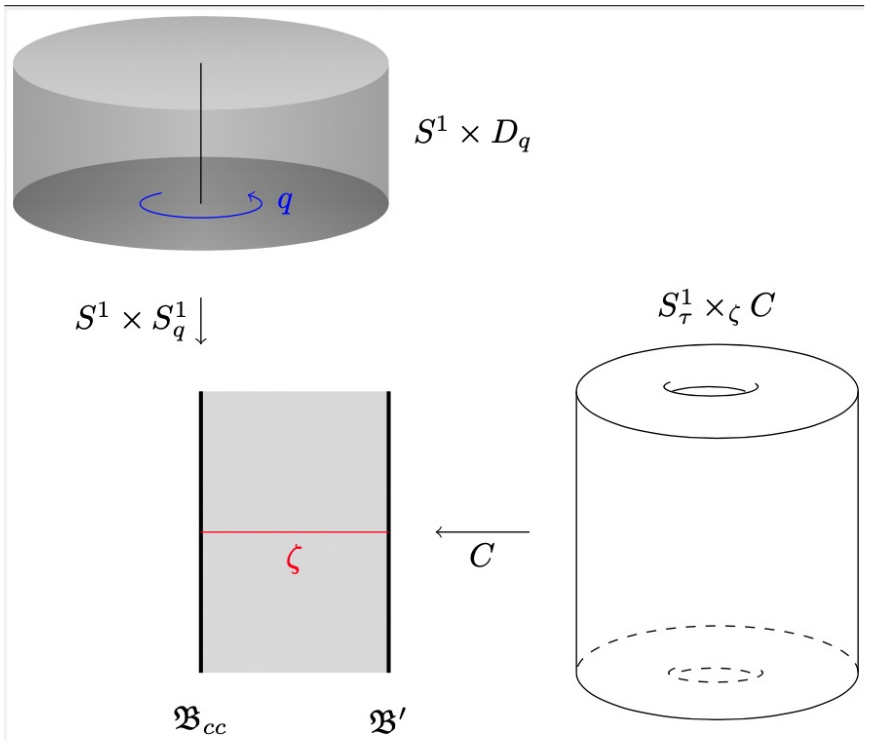
Connecting 2d σ-model to 3d Physics

$M_3 = S^1_\tau \times_\varsigma C \qquad \varsigma \in SL(2,\mathbb{Z}) \qquad$ mapping tori

6d (2,0) thy on $S^1 \times \mathbb{R}^2_q \times S^1_\tau \times_\varsigma C$

$S^1 \times S^1_q \times C \swarrow \qquad \searrow S^1 \times_\varsigma C$

2d σ-model $S^1_\tau \times I \to \mathcal{M}_H(C_p, G) \simeq$ 3d $\mathcal{N}=2$ thy $T[S^1_\tau \times_\varsigma C]$ on $S^1 \times \mathbb{R}^2_q$.



$PSL(2.\mathbb{Z})$ on $\mathrm{Hom}(B_{cc}, B_v)$

shortening condition ② $t^2 = -q^{-k}$ $\rightsquigarrow$ $t = \exp\left(\frac{c\pi i}{k+2c}\right)$

$q = \exp\left(\frac{\pi i}{k+2c}\right)$

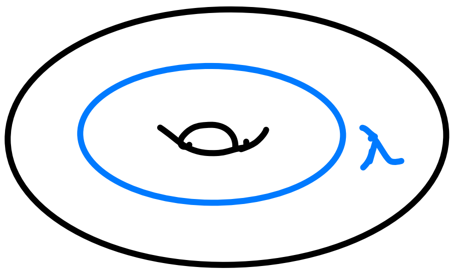
rank $(k+1)$

$$S_{ij} = P_i(tq^j; q, t)\, P_j(t^{-1}; q, t)\Big|_{②} \quad \circlearrowright \; \mathrm{Hom}(B_{cc}, B_v)$$

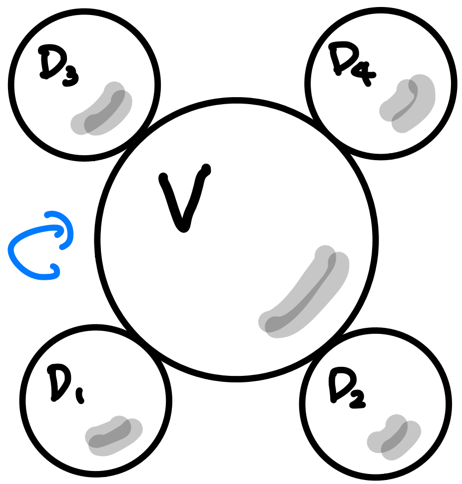
$$T_{ij} = \delta_{ij}\, q^{-\frac{j^2}{2}}\, t^{-j}\Big|_{②}$$

Refined CS.

$\xrightarrow[t=q]{}$ Verlinde module.



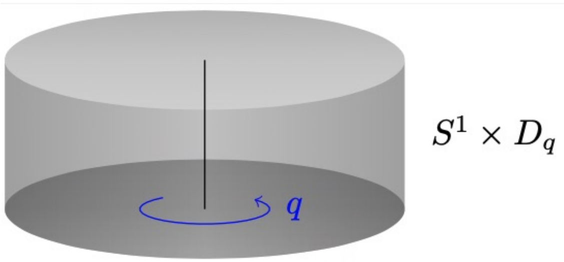
$PSL(2.\mathbb{Z}) \circlearrowright$



$PSL(2,\mathbb{Z})$ on $\mathrm{Hom}(\mathcal{B}_{cc}, \mathcal{B}_{\rho_1})$

D. Xie.

$(A_1, A_{2(\ell-1)})$ Argyres-Douglas thy $\quad T[C_{\text{wild}}, SU(2)]$



$\cong$

$\varphi(z)\,dz \sim z^{\frac{2\ell-1}{2}} \sigma_3\, dz$

$t\, q^{\frac{2\ell-1}{2}} \varphi(x_0+\beta, z) = \varphi(x_0, z)$

Kozcaz - Shakirov - Yan

$\longrightarrow$ shortening cond ③ $\quad T^2 q^{\frac{2\ell-1}{2}} = 1$.

(rank ℓ)

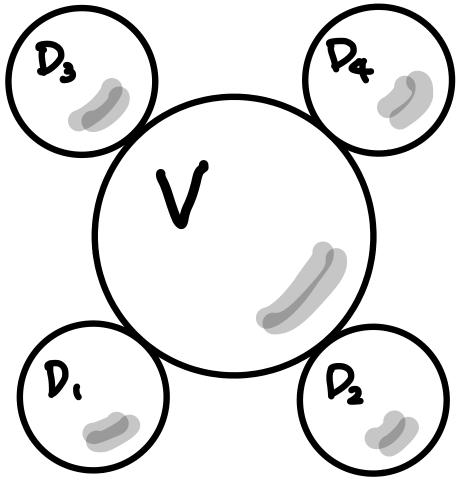
$S_{ij}|_{③}$ modular matrices

$T_{ij}|_{③}$ of Virasoro Minimal model

$\cong$

chiral alg of $(A_1, A_{2\ell-2})$ thy.

$PSL(2,\mathbb{Z}) \circlearrowright$

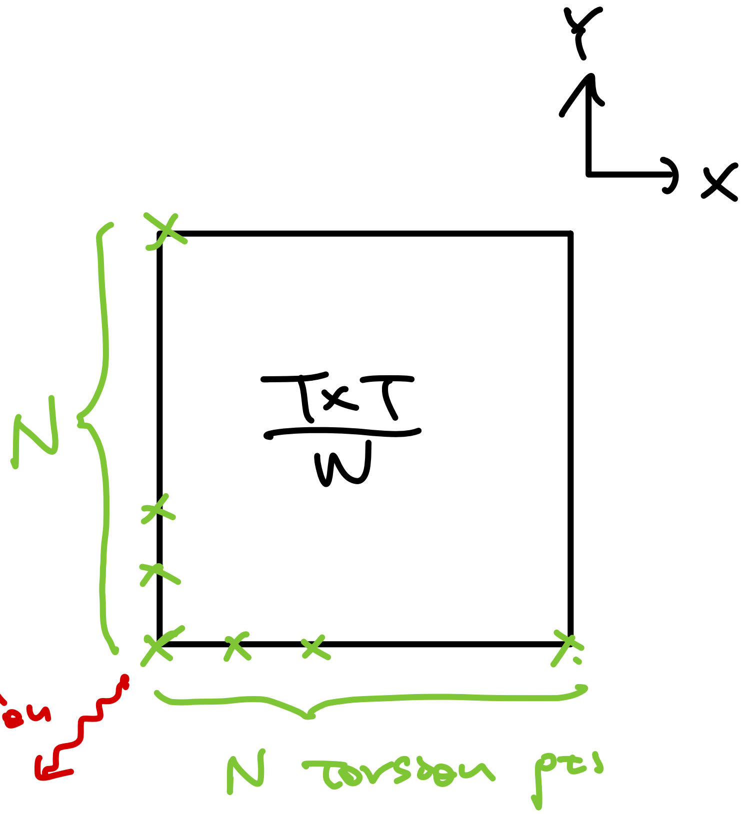


Higher rank generalization

$$\mathcal{M}_{flat}(T^2, SL(N,\mathbb{C})) = \frac{T_{\mathbb{C}} \times T_{\mathbb{C}}}{W}$$

$$\cup \qquad\qquad\qquad\qquad \cup$$

$$\mathcal{M}_{flat}(T^2, SU(N)) = \frac{T \times T}{W}$$



after ramification

shortening cond $t^N q^M = 1$.

$PSL(2,\mathbb{Z}) \circlearrowleft$  analogous to $\mathbb{P}_1$

$$K_0(MTC(A_{N-1}, A_{M-1})) \cong \mathrm{Hom}(B_{cc}, B_{\mathbb{P}_1^{(N)}}) \cong \mathcal{H}_{bottom}(T_{N,M})$$

Gorsky-Oblomkov

-Rasmussen-Shende.

4d $\mathcal{N}=2$ of class S & Coulomb branch

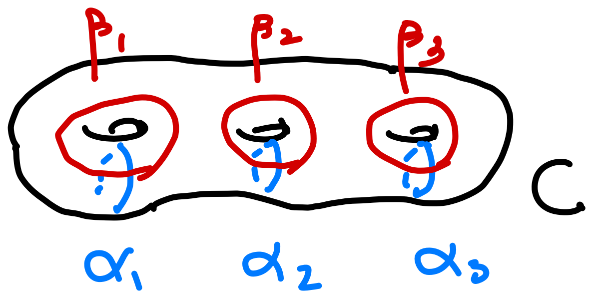
Thy. $T[C, G, L]$

Gaiotto

Gaiotto-Moore-Neitzke

Tachikawa.

Hitchin System



$$\begin{array}{c} \Sigma \hookrightarrow T^*C \\ \downarrow \\ C \end{array}$$

symplectic basis

$$(\alpha_1 \cdots \alpha_g \ \beta_1 \cdots \beta_g) \in H_1(C)$$

$$\alpha_i \cdot \beta_j = \delta_{ij} \qquad \alpha_i \cdot \alpha_j = 0 = \beta_i \cdot \beta_j$$

$\downarrow$ maximal isotropic sublattice.

Coulomb branch

$$\mathcal{M}_C(C, G, L) \cong \mathcal{M}_H(C, G)/L^\vee$$

$$L \subset (H^1(C, Z(G)), \omega)$$

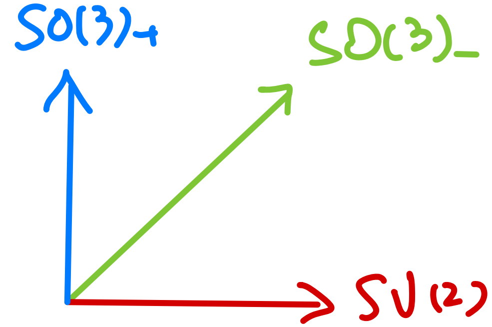
$$\pi : \mathcal{M}_C(C, G, L) \to B_u.$$

Completely integrable system.

4d $\mathcal{N}=2^*$ thy of Type A_1

$C_p = T^2 \setminus pt.$ $\qquad H^1(C, \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \supset$ $(1,0)$ $(0,1)$ M.I.L $(1,1)$

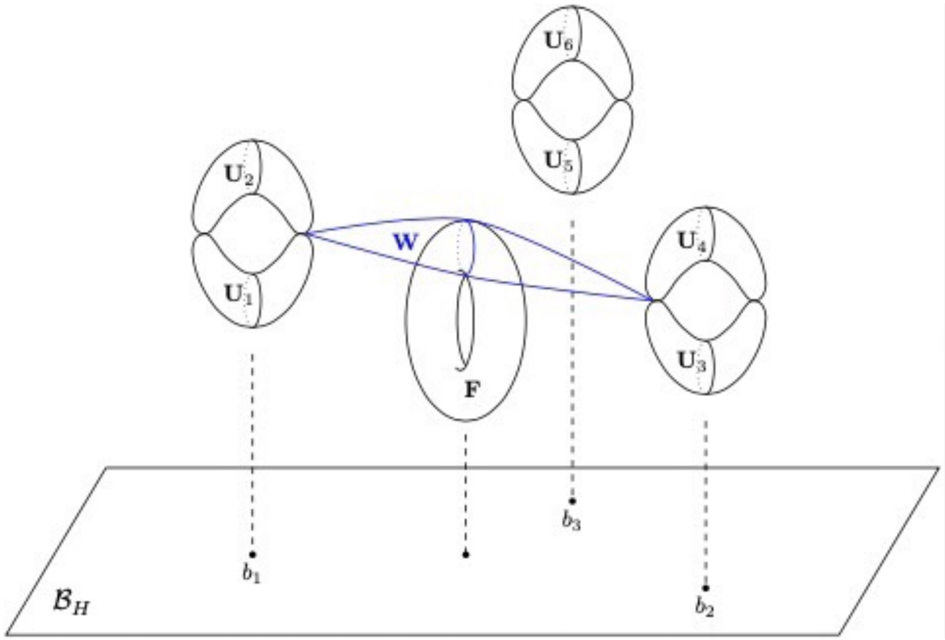
$G = SU(2)$

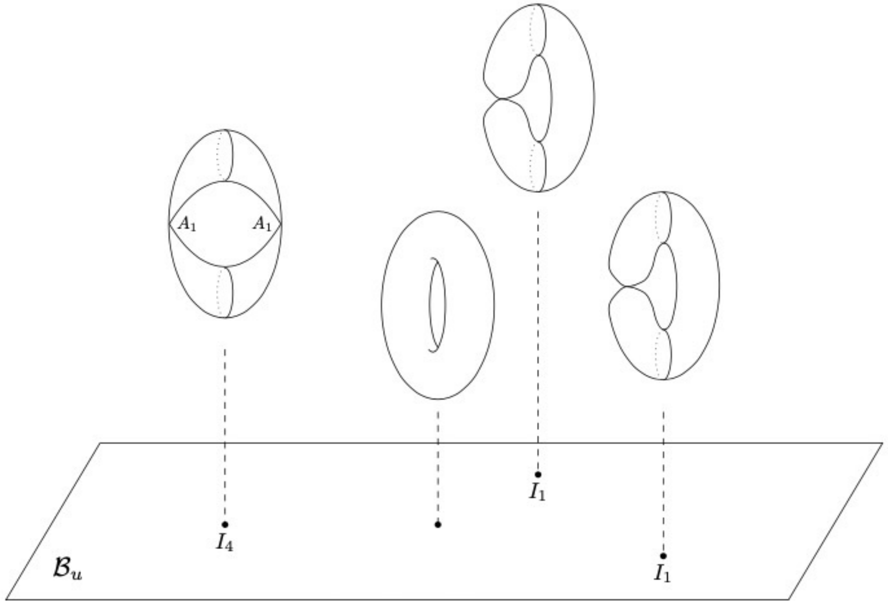


$$
\begin{tikzcd}
 & \mathrm{SU}(2) \arrow[loop above, "T"] \arrow[dl, leftrightarrow, "S"'] \arrow[dr, leftrightarrow, "TST"] & \\
\mathrm{SO}(3)_+ \arrow[loop left, "TST"] \arrow[rr, leftrightarrow, "T"'] & & \mathrm{SO}(3)_- \arrow[loop right, "S"]
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathcal{M}_H(C_p, \mathrm{SU}(2)) \arrow[dl, "\xi_2"'] \arrow[d, "\xi_1"] \arrow[dr, "\xi_3"] & \\
\mathcal{M}_C(C_p, \mathrm{SU}(2)) \arrow[dr, "\xi_1"'] & \mathcal{M}_C(C_p, \mathrm{SO}(3)_+) \arrow[d, "\xi_2"] & \mathcal{M}_C(C_p, \mathrm{SO}(3)_-) \arrow[dl, "\xi_1"] \\
 & \mathcal{M}_H(C_p, \mathrm{SO}(3)) &
\end{tikzcd}
$$

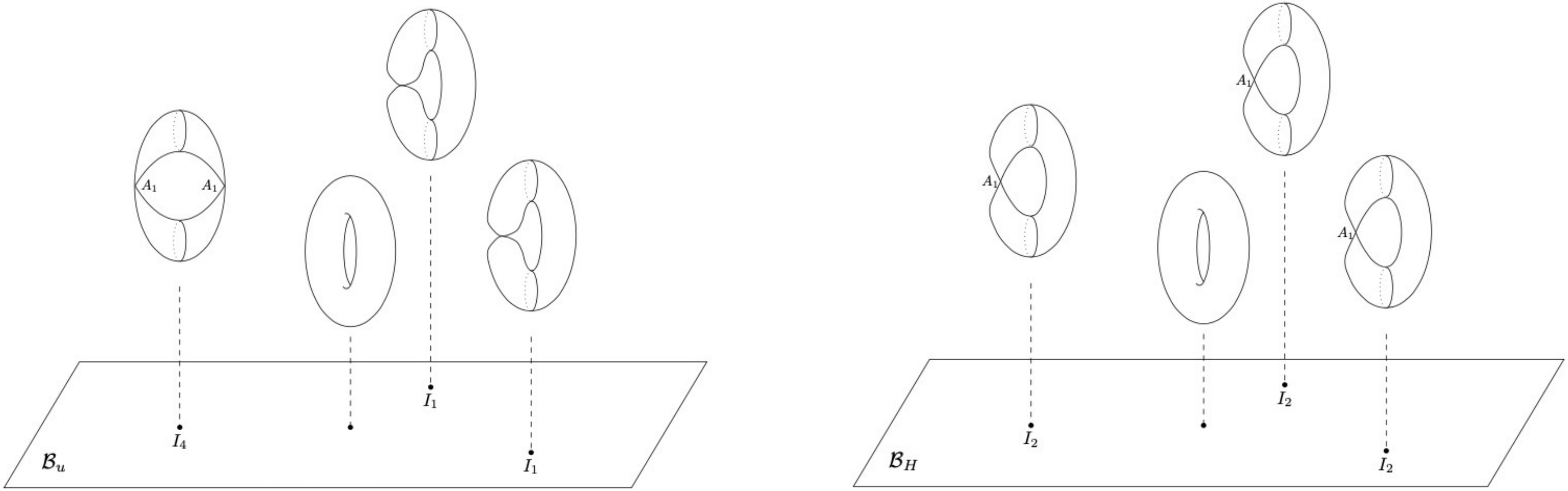
Geometry of Coulomb branch of 4d $\mathcal{N}=2^*$ of type A_1





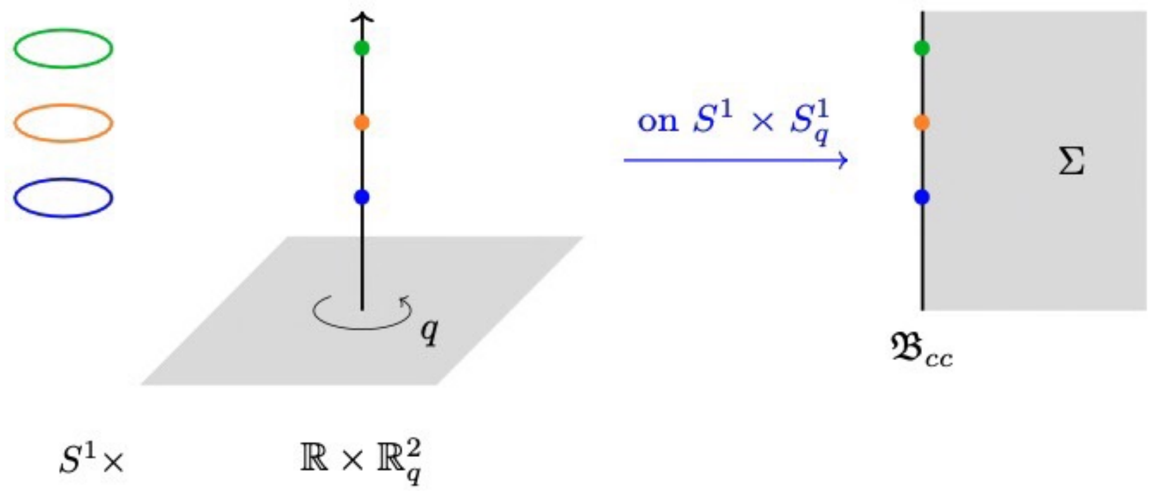
$$\mathcal{M}_H(C_p.SU_2) \xrightarrow{\mathbb{Z}_2} \mathcal{M}_C(C_p.SU_2)$$

$\mathcal{M}_H(C, SO(3))$



$$\mathcal{M}_c(C, SU(2), L) \xrightarrow{3_1} \mathcal{M}_H(C, SO(3))$$

Algebra of line operators (quantized Coulomb branch)



Ito-Okuda-Taki

Hayashi-Okuda-Yoshida

- $SU(2)$: $\mathcal{O}^q(\mathcal{M}_C(\mathcal{C}_p, SU(2))) \cong S\ddot{H}^{\mathfrak{Z}_2}$

$$x = (X + X^{-1})e \qquad y^2 - 1 = (Y^2 + 1 + Y^{-2})e$$

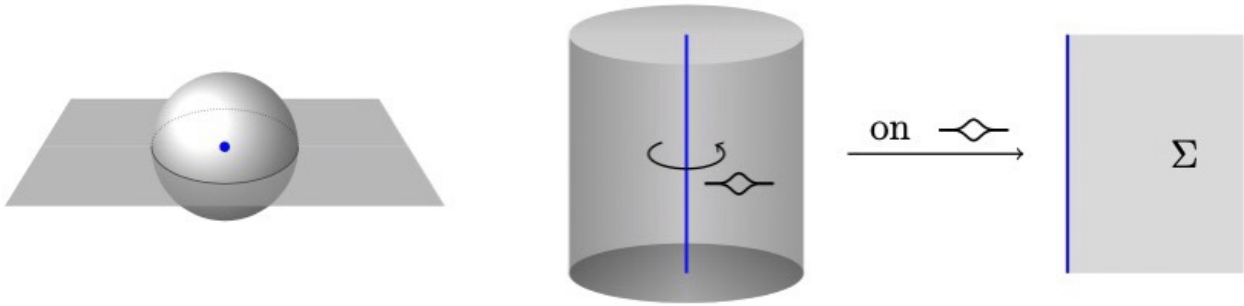
- $SO(3)_+$: $\mathcal{O}^q(\mathcal{M}_C(\mathcal{C}_p, SO(3)_+)) \cong S\ddot{H}^{\mathfrak{Z}_1}$

$$x^2 - 1 = (X^2 + 1 + X^2)e \qquad y = (Y + Y^{-1})e$$

- $SO(3)_-$: $\mathcal{O}^q(\mathcal{M}_C(\mathcal{C}_p, SO(3)_-)) \cong S\ddot{H}^{\mathfrak{Z}_3}$

$$x^2 - 1 = (X^2 + 1 + X^2)e \qquad z = (q^{-\frac{1}{2}} Y^{-1} X + q^{\frac{1}{2}} X^{-1} Y)e$$

Category of line operators



Kapustin-Witten
BFM
BFN

(B.B.B) Wilson
(B.A.A) 't Hooft.

Compactify 4d $\mathcal{N}=4$ ($\mathcal{N}=2^*$) thy around line operators

$\downarrow$ raviolo

(B.A.A) brane in $\mathcal{M}_H$(raviolo, G)

$\mathcal{O} = \mathbb{C}[\![z]\!]$

$K = \mathbb{C}(\!(z)\!)$

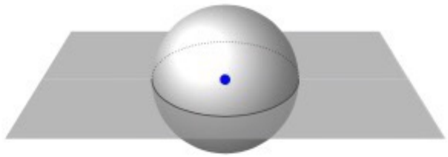
$Gr_{G_\mathbb{C}} = G_\mathbb{C}^K / G_\mathbb{C}^\mathcal{O}$

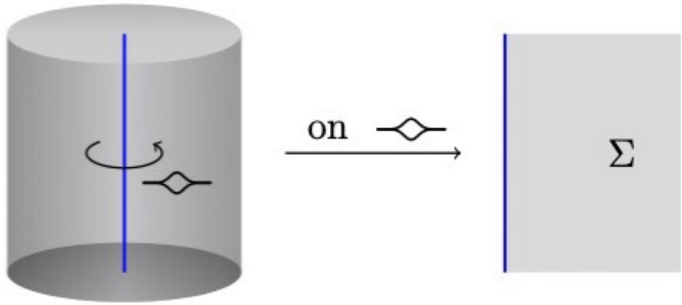
affine Grassmannian Steinberg variety.

$$R = \{ (x, [g]) \in \mathfrak{g}_\mathbb{C}^\mathcal{O} \times Gr_{G_\mathbb{C}} \mid \mathrm{Ad}_g^{-1}(x) \in \mathfrak{g}^\mathcal{O} \}$$

$\rightarrow$ Category of line op $\simeq D^b \mathrm{Coh}^{G_\mathbb{C}^\mathcal{O}}(R)$.

Category of line operators





(B.A.A)

taking Grothendieck group

$$K^{G_{\mathbb{C}}^{\mathcal{O}}}(R) = \mathcal{O}(T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee})^{W}$$

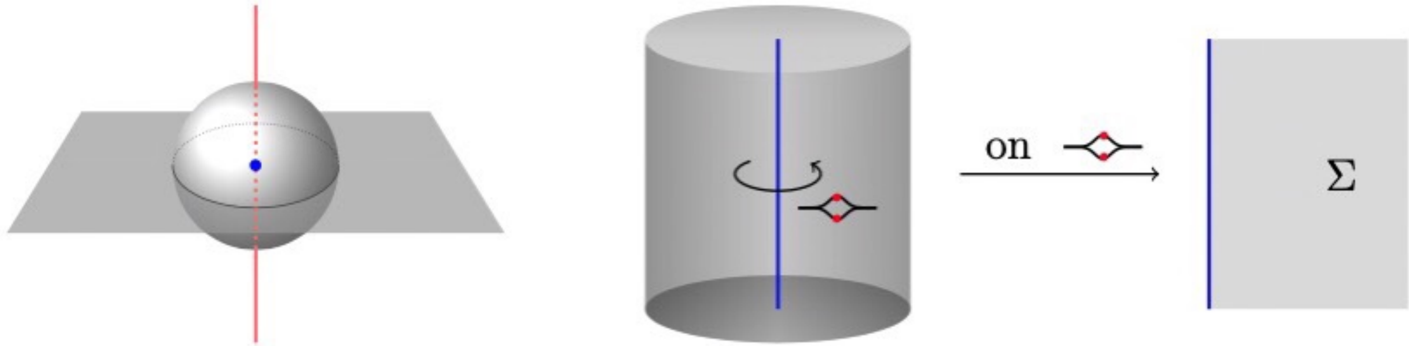
Coulomb branch of 4d $\mathcal{N}=4$ $= \dfrac{T_{\mathbb{C}} \times T_{\mathbb{C}}^{\vee}}{W} \subset \dfrac{T_{\mathbb{C}} \times T_{\mathbb{C}}}{W}$

$\rightsquigarrow$ Turning on Ω-def (q,t)

$$K^{(G_{\mathbb{C}}^{\mathcal{O}} \times \mathbb{C}_t^{\times}) \rtimes \mathbb{C}_q^{\times}}(R) \cong S\ddot{H}(W)^{\iota}$$

Including surface operator

Vasserot



If we include a surface operator, raviolo has 2 ramifications.

$$R \rightsquigarrow Z = \{(x,[g]) \in \mathrm{Lie}(I) \times Fl_{G_\mathbb{C}} \mid \mathrm{Ad}_g^{-1}(x) \in \mathrm{Lie}(I)\}.$$

affine Flag Steinberg variety

$$Fl_{G_\mathbb{C}} = G_\mathbb{C}^{K}/I.$$

$$\rightsquigarrow K^{(I\times \mathbb{C}_t^\times)\rtimes \mathbb{C}_q^\times}(Z) \cong \ddot{H}(W)^L$$

$$\begin{array}{ccc} G_\mathbb{C}^{\mathcal{O}} & \xrightarrow{p} & G_\mathbb{C} \\ \cup & & \cup \\ I = p^{-1}(B) & \longrightarrow & B \end{array}$$

alg of line operators on a surface operators.