Exact WKB analysis for the deformed SUSY quantum mechanics

Syo Kamata (NCBJ)

Collaboration with
N. Sueishi (Keio U.), T. Misumi (Kindai U.), and M. Ünsal (NC State U.)

based on arXiv:2111.05922, 2103.06586, and 2008.00379

BIMSA HEP-TH Seminar – June 22th, 2022

Contents

1. Part I : Concept of resurgence theory
   - What is "Resurcence"?
   - Borel resummation
   - Stokes automorphism
   - Simple example : $\frac{df}{dx} + f - 1/x = 0$
2. Part II : Exact-WKB analysis
   - Deformed SUSY potential : $V(x,\hbar) = \frac{1}{2}W'(x) + \hbar\frac{p}{2}W''(x)$
   - Stokes graph, quantization condition, etc...
   - Energy spectra

Concept of resurgence theory

Transseries

Transseries ... Extension of asymptotic expansion to add **transmonomials** such as $e^{-c/\hbar}$, $\log \hbar$, etc.

E.g. Path-integral in 1D (Euclidean) QM

$$
\begin{aligned}
Z(\hbar) &= \int \mathcal{D}x \, e^{-S(x,\hbar)} \\
&= \sum_{j=0}^{\infty} a_n \hbar^j + \sum_{n,k=1}^{\infty} \sum_{j=0}^{\infty} a_{n,k,j} e^{-\frac{nS_B}{\hbar}} \hbar^j (\log \hbar)^k .
\end{aligned}
$$

The path-integral can be expressed by **transseries** generated by **transmonomials**, $(\hbar, e^{-\frac{S_B}{\hbar}}, \log \hbar)$.

- $\hbar$ ... PT fluctuation
- $e^{-\frac{S_B}{\hbar}}$ ... Instanton (Bion) energy
- $\log \hbar$ ... Quasi-zero modes

Resurgence theory

Resurgence Theory

$$Z(\hbar) = \sum_{n\in\mathbb{N}_0} Z^{(n)}(\hbar), \qquad Z^{(n)}(\hbar) = e^{-nS/\hbar} \sum_{k\in\mathbb{N}_0} a_k^{(n)} \hbar^k$$

- Mathematical method to make a relationship among perturbative and nonperturbative sectors by using **Borel resummation theory**.
- If $Z^{(0)}$ (and/or $Z^{(n>0)}$) is a **divergent series**, one can construct the relationship, called **resurgent relation**. In such a case, the information of lower sectors propagate to the higher sectors.

$$Z^{(0)}(\hbar) \quad \Rightarrow \quad Z^{(1)}(\hbar) \quad \Rightarrow \quad Z^{(2)}(\hbar) \quad \Rightarrow \quad \cdots$$

Borel resummation

$$\widehat{f}(\hbar) \sim f(\hbar) = \sum_{n=1}^{\infty} a_n \hbar^n \quad \text{as} \quad \hbar \to 0_+ \qquad (a_n \in \mathbb{R}, \hbar \in \mathbb{R})$$

Borel resummation ($\mathcal{S} := \mathcal{L} \circ \mathcal{B}$) is a method to recover $\widehat{f}(\hbar)$ from $f(\hbar)$:

- **Borel transform** $\mathcal{B} : \hbar^n \mapsto \frac{\xi^{n-1}}{\Gamma(n)}$

$$\mathcal{B}[f](\xi) = \sum_{n=1}^{\infty} a_n \mathcal{B}[\hbar^n] = \sum_{n=1}^{\infty} \frac{a_n}{\Gamma(n)} \xi^{n-1} =: f_B(\xi) \qquad (\xi \in \mathbb{C})$$

- **Laplace integral** $\mathcal{L} : f_B(\xi) \mapsto \widehat{f}(\hbar)$

$$\mathcal{L}[f_B](\hbar) := \int_0^{+\infty} d\xi \, e^{-\xi/\hbar} f_B(\xi) = \widehat{f}(\hbar)$$

Borel resummation

$$\mathcal{B}[\hbar^k] := \frac{\xi^{k-1}}{\Gamma(k)}$$

Asymptotic form

$$\tilde{Z}(\hbar) \sim \sum_{k=0}^{\infty} a_k \hbar^k \quad \in \mathbb{R}[[\hbar]]$$

Borel transform

$$\mathcal{B}[\tilde{Z}](\xi) = \sum_{k=0}^{\infty} b_k \xi^k \quad \in \mathbb{R}[[\xi]]$$

Asymptotic limit

$0 < \hbar \ll 1$

$$\mathcal{L}[F](\hbar) := \int_0^{\infty} F(\xi) e^{-\xi/\hbar} \, d\xi$$

Borel resummation

$$\hat{Z}(\hbar) = \mathcal{L} \circ \mathcal{B}[\tilde{Z}](\hbar)$$

Borel summability

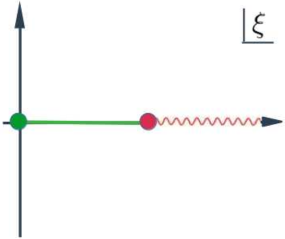
$$\hat{f}(x) \sim f(x) = \sum_{n=1}^{\infty} a_n x^{-n} \quad \text{as} \quad x \to +\infty$$

Radius of convergence : $r_c = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$

- **Convergent series** if $r_c > 0 \;\Rightarrow\; f_B(\xi)$ is analytic for $\forall \xi \in \mathbb{C}$
- **Divergent series** if $r_c = 0 \;\Rightarrow\; f_B(\xi)$ has singularities at $\exists \xi \in \mathbb{C}$

When **divergent series**,
two possibilities exist:

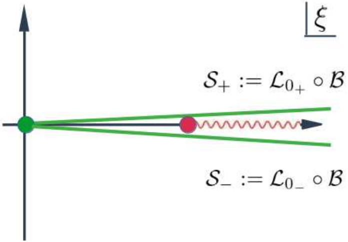
- If $\mathcal{L}$ is integrable,
  $f(x)$ is **Borel summable**
- If $\mathcal{L}$ is not integrable,
  $f(x)$ is **Borel non-summable**



Borel summability

When **Borel non-summable**,
we can avoid the singularities by introducing a complex phase to the integration ray:

$$\mathcal{L} \;\rightarrow\; \mathcal{L}_\theta := \int_0^{+\infty e^{i\theta}} d\xi e^{-x\xi}$$



However, since the singularities give discontinuity between $\mathcal{S}_+$ and $\mathcal{S}_-$, one obtains a result as

$$\mathcal{S}_+[f](x) \neq \mathcal{S}_-[f](x) \neq \widehat{f}(x).$$

Question :
How to obtain $\widehat{f}(x)$ via $\mathcal{S}_\pm$ when $f(x)$ is Borel non-summable?
※ This is a key point to get a resurgent relation.

Stokes automorphism

In order to solve the problem, we make $f_\pm(x)$ from $f(x)$ such that

$$\mathcal{S}_+[f_+](x) = \mathcal{S}_-[f_-](x) = \widehat{f}(x)$$

It is obtained by defining **Stokes automorphism** $\mathfrak{S}$ as

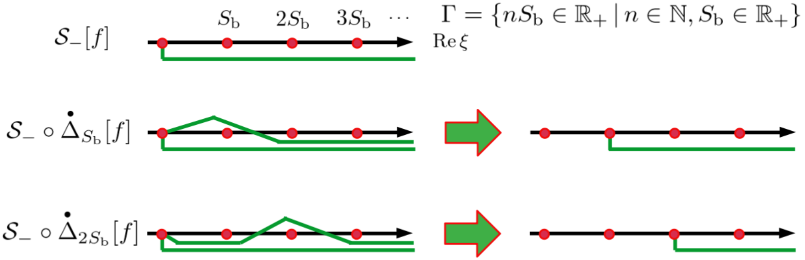
$$\mathcal{S}_+[f](x) = \mathcal{S}_- \circ \mathfrak{S}[f](x) \quad (=: \mathcal{S}_-[f](x) + \mathcal{S}_-[\delta f](x))$$

- Transseries $\to$ Transseries

$$f(x) = \sum_{n,k\in\mathbb{N}_0} a_{n,k} e^{-nx} x^{-k} \ \mapsto \ f'(x) = \sum_{n,k\in\mathbb{N}_0} a'_{n,k} e^{-nx} x^{-k} \quad (a'_{n,k} \in \mathbb{C})$$

- $\mathfrak{S} = \mathrm{id}$ if $f(x)$ is Borel summable (along $\xi \in \mathbb{R}_+$)
- $\delta f(x)$ is calculable by the integration contour going around the singularities

Alien derivative $\dot{\Delta}_w$ **:** Transseries $\to$ Transseries



Collection of all of $\dot{\Delta}_{w\in\Gamma}$ $\Rightarrow$ Stokes automorphism $\mathfrak{S} = \exp\left[\sum_{w\in\Gamma} \dot{\Delta}_w\right]$

Alien deriavtive $\Rightarrow$ Stoke automorphism

Alien derivative $\dot{\Delta}_w$ is an abstract derivative:

$$\dot{\Delta}_w[f_1 \cdot f_2] = \dot{\Delta}_w[f_1] \cdot f_2 + f_1 \cdot \dot{\Delta}_w[f_2],$$
$$\dot{\Delta}_w[f_{1,B} * f_{2,B}] = \dot{\Delta}_w[f_{1,B}] * f_{2,B} + f_{1,B} * \dot{\Delta}_w[f_{2,B}],$$

where

$$(f_{1,B} * f_{2,B})(\xi) = \int_0^{\xi} d\xi' f_{1,B}(\xi') f_{2,B}(\xi - \xi').$$

- One parameter extension : $\mathfrak{S} \to \mathfrak{S}^{\nu}$ with $\nu \in \mathbb{R}$

The **Stokes automorphism** becomes one-parameterized group defined as

$$\mathfrak{S}^{\nu} = \exp\left[\nu \sum_{w \in \Gamma} \dot{\Delta}_w\right] = 1 + \sum_{k=1}^{\infty} \sum_{\{n_1, \cdots, n_k\} \in \mathbb{N}^k} \frac{\nu^k}{k!} \dot{\Delta}_{w_{n_1}} \cdots \dot{\Delta}_{w_{n_k}},$$

$$w_1 < w_2 < \cdots \in \Gamma = \{\text{singularities along } \xi \in \mathbb{R}_+\}$$

Medien resummation

Since

$$\mathcal{S}_+[f] = \mathcal{S}_- \circ \mathfrak{S}^{+1}[f] \quad \Rightarrow \quad \mathcal{S}_+ \circ \mathfrak{S}^{-1/2}[f] = \mathcal{S}_- \circ \mathfrak{S}^{+1/2}[f],$$
$$(\mathcal{C} \circ \mathcal{S}_- \circ \mathfrak{S}^{\nu} = \mathcal{S}_+ \circ \mathfrak{S}^{-\nu} \circ \mathcal{C}, \qquad \mathcal{C} := \text{complex conj.})$$

one finds that

$$\mathcal{S}_+[f_+] = \mathcal{S}_-[f_-], \quad \text{where} \quad f_\pm(x) := \mathfrak{S}^{\mp 1/2}[f](x).$$

Therefore, $\widehat{f}(x)$ is obtained by **medien resummation** defined as

$$\mathcal{S}_{\rm med} := \mathcal{S}_+ \circ \mathfrak{S}^{-1/2} = \mathcal{S}_- \circ \mathfrak{S}^{+1/2},$$
$$\mathcal{S}_{\rm med}[f](x) = \widehat{f}(x).$$

$\mathcal{S}_{\rm med}$ returns $\widehat{f}(x)$ regardless of Borel summability of $f(x)$.

Resurgent relation is to relate nonperturbative sectors from a perturbative sector (or vice versa). It is essentially the same problem as getting the Stokes automorphism (or alien derivative).

$$\textbf{Input} \quad : \quad f(x) = \sum_{n \in \mathbb{N}} a_n x^{-n},$$

$$\textbf{Output} \quad : \quad \dot{\Delta}_{S_{\rm b}}[f](x) = e^{-S_{\rm b} x} \sum_{n \in \mathbb{N}} a_n^{(1)} x^{-n},$$

$$\left[\frac{1}{2} (\dot{\Delta}_{S_{\rm b}})^2 + \dot{\Delta}_{2S_{\rm b}} \right] [f](x) = e^{-2S_{\rm b} x} \sum_{n \in \mathbb{N}} a_n^{(2)} x^{-n}, \quad \cdots$$

The information of $f(x)$ is propagating to higher sectors.

$$f^{(0)}(x) \quad \Rightarrow \quad f^{(1)}(x) \quad \Rightarrow \quad f^{(2)}(x) \quad \Rightarrow \quad \cdots$$

Simple example

We apply the above discussion to a transseries, $f(x)$, obtained from the ODE by expanding around $x = +\infty$.

Example : linear non-autonomous ODE

$$\frac{df}{dx} + f - \frac{1}{x} = 0, \qquad (*)$$

Exact solution : $\widehat{f}(x) = e^{-x}\mathrm{Ei}(x) + \sigma e^{-x}$ with $\sigma \in \mathbb{R}$

Transmonomial :
Transmonomial in the NP sector can be found by the linearized eq.

$$\frac{d\delta f}{dx} + \delta f(x) \sim 0 \quad \Rightarrow \quad \delta f(x) = \sigma e^{-x}.$$

Ansatz : $f(x) = f_{\mathrm{pt}}(x) + \sigma f_{\mathrm{np}}(x)$

$$f_{\mathrm{pt}}(x) := \sum_{k\in\mathbb{N}}^{\infty} a_k^{(0)} x^{-k}, \qquad f_{\mathrm{np}}(x) := \sum_{k\in\mathbb{N}_0}^{\infty} a_k^{(1)} e^{-x} x^{-k}.$$

Simple example

Example : linear non-autonomous ODE

$$\frac{df}{dx} + f - \frac{1}{x} = 0. \qquad (*)$$

Substituting the ansatz into $(*)$, the coefficients are obtained as

$$a^{(0)}_{k\geq 1} = \Gamma(k), \quad a^{(1)}_{k} = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

The PT sector is a **divergent series**:

$$r_c = \lim_{k\to\infty} \left| \frac{a^{(0)}_{k}}{a^{(0)}_{k+1}} \right| = \lim_{k\to\infty} \frac{1}{k} = 0.$$

The NP sector is a monomial.

Simple example

We act **Borel transform** $\mathcal{B}$ to $f_{\rm pt}(x)$:

$$f_{{\rm pt},B}(\xi) := \mathcal{B}[f_{\rm pt}](\xi) = \sum_{k\in\mathbb{N}} \frac{a_k^{(0)}}{\Gamma(k)}\xi^{k-1} = \sum_{k\in\mathbb{N}} \xi^{k-1} = \frac{1}{1-\xi}.$$

A set of singularities along $\xi \in \mathbb{R}_+$ is $\Gamma = \{1\}$, and thus, $f_{\rm pt}(x)$ is **Borel non-summable**.

The **Stokes automorphism** is obtained by

$$\begin{aligned} \mathcal{S}_+[f_{\rm pt}] &= \mathcal{S}_- \circ \mathfrak{S}[f_{\rm pt}] \\ &= \mathcal{S}_-[f_{\rm pt}] - \oint_{\xi=1} d\xi e^{-x\xi} \frac{1}{1-\xi} = \mathcal{S}_-[f_{\rm pt}] - 2\pi i e^{-x} \\ \Rightarrow \quad & \mathfrak{S}[f_{\rm pt}] = f_{\rm pt} - 2\pi i e^{-x} = f_{\rm pt} - 2\pi i \cdot f_{\rm np}. \qquad (\mathcal{S}_\pm[f_{\rm np}] = f_{\rm np}) \end{aligned}$$

The action of the **alien derivative** to $f_{\rm pt}$ is

$$(\dot{\Delta}_{w=1})^n[f_{\rm pt}] = \begin{cases} -2\pi i \cdot f_{\rm np} & \text{for} \quad n = 1 \\ 0 & \text{for} \quad n > 1 \end{cases}$$

Simple example

From the calculation of alien derivative, the one-parametrized Stokes automorphism is obtained by

$$\begin{aligned}\mathfrak{S}^{\nu}[f_{\rm pt}] &= \exp\left[\nu\dot{\Delta}_{w=1}\right][f_{\rm pt}] \\ &= \left(1+\nu\dot{\Delta}_{w=1}\right)[f_{\rm pt}] = f_{\rm pt} - 2\pi\nu i\cdot f_{\rm np}\end{aligned}$$

$$\Rightarrow \quad f_{\pm} = \mathfrak{S}^{\mp 1/2}[f] = f_{\rm pt} + (\pm\pi i + \sigma)\, f_{\rm np}. \qquad (f = f_{\rm pt} + \sigma f_{\rm np})$$

We find that

$$\mathcal{S}_{\pm}[f_{\rm pt}] = e^{-x}{\rm Ei}(x) \mp \pi i, \qquad \mathcal{S}_{\pm}[f_{\rm np}] = e^{-x}.$$

Therefore,

$$\mathcal{S}_{\rm med}[f](x) = \mathcal{S}_{\pm}\circ\mathfrak{S}^{\mp 1/2}[f](x) = \mathcal{S}_{\pm}[f_{\pm}](x) = \widehat{f}(x).$$

Resurgent relation : $\dot{\Delta}_{w=1}[f_{\rm pf}] = -2\pi i\cdot f_{\rm np} \propto e^{-x}$.

Exact-WKB analysis

Exact-WKB analysis for the deformed SUSY potential

In this talk, we consider the deformed SUSY potentials.

$$\left[-\hbar^2 \frac{d^2}{dx^2} + Q(x,\hbar)\right]\psi(x) = 0, \quad Q(x) = 2(V(x,\hbar) - E),$$

where

$$\begin{aligned} V(x,\hbar) &= \frac{1}{2}W'(x)^2 + \hbar\frac{p}{2}W''(x), \qquad p \in \mathbb{R} \\ \text{Double-well} &: \quad W'(x) = x^2 - \frac{1}{4}, \\ \text{Triple-well} &: \quad W'(x) = \frac{1}{2}x(x^2-1). \end{aligned}$$

When $p = 0 \Rightarrow$ No quantum deformation
$p = 1 \Rightarrow$ SUSY limit

We consider the energy spectra and contribution from NP objects.

Exact-WKB analysis : Outline

Goal : (1) See Borel summability of the energy spectra, $E(\hbar)$
(2) Obtain transseies of $E(\hbar)$
(leading order of $O(\hbar)$ of PT and NP sectors)

1. Put ansatz of transseries to the wavefunction, $\psi(\hbar e^{\pm i\theta})$.
2. Get the monodromy matrix by analytic continuation, $\psi^{\rm I} = \mathcal{M}^{\pm}\psi^{\rm II}$.
3. Get the quantization condition $\mathfrak{D}^{\pm}(E,\hbar) = 0$ from $\mathcal{M}^{\pm}$.
   (cf. Bohr–Sommerfeld)
4. Consider the Stokes automorphism $\mathfrak{S}$ for $\mathfrak{D}^{\pm}$ (DDP formula), and then get $\mathfrak{D}$ using $\mathfrak{S}$.
5. Observe $E^{\pm}(\hbar)$ and $E(\hbar)$ by solving $\mathfrak{D}^{\pm} = 0$ and $\mathfrak{D} = 0$, respectively.

Exact-WKB analysis

Consider the Schrödinger equation given by $(\hbar, x \in \mathbb{C}, E \in \mathbb{R}_+)$
(See e.g.**[T.Kawai et al. AMS, c2005]** in technical details.)

$$\left[-\hbar^2 \frac{d^2}{dx^2} + Q(x,\hbar)\right]\psi(x) = 0, \quad Q(x) = 2(V(x,\hbar) - E),$$
$$Q(x,\hbar) = \sum_{n\in\mathbb{N}_0} Q_n(x)\hbar^n.$$

Put ansatz of a formal solution

$$\psi(x,\hbar) = e^{\int^x S(x,\hbar)dx},$$
$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \hbar^2 S_2(x) + \cdots$$

where $S(x,\hbar)$ satisfies the nonlinear Riccati equation.

$$S(x,\hbar)^2 + \frac{\partial S(x,\hbar)}{\partial x} = \hbar^{-2}Q(x,\hbar), \qquad \boxed{S_{-1}(x) = \pm\sqrt{Q_0(x)}}.$$

Exact-WKB analysis

Since the Sch eq is the 2nd order diff eq, two independent solutions exist:

$$S^{\pm}(x,\hbar) = \pm S_{\rm odd}(x,\hbar) + S_{\rm even}(x,\hbar)\,,$$
$$S_{\rm odd/even}(x,\hbar) := \frac{S^{+}(x,\hbar) \mp S^{-}(x,\hbar)}{2}\,,$$
$$S_{\rm odd}(x,\hbar) = \sum_{n=-1}^{\infty} S_{{\rm odd},n}(x)\hbar^{n}, \quad S_{\rm even}(x,\hbar) = \sum_{n=0}^{\infty} S_{{\rm even},n}(x)\hbar^{n}\,.$$

By the Riccati eq, one finds

$$S_{\rm even}(x,\hbar) = -\frac{1}{2}\frac{\partial \log S_{\rm odd}(x,\hbar)}{\partial x}\,,$$

hence, the formal solution can be expressed only by $S_{\rm odd}(x,\hbar)$.

$$\psi_a^{\pm}(x,\hbar) = \frac{e^{\pm\int_a^x S_{\rm odd}(x,\hbar)dx}}{\sqrt{S_{\rm odd}(x,\hbar)}} = e^{\pm\frac{\xi_0(x)}{\hbar}}\sum_{n=0}^{\infty}\psi_{a,n}^{\pm}(x)\hbar^{n+1/2}.$$

Exact-WKB analysis

Let us look at the Borel resummation of the wavefunction.

$$\mathcal{S}_\theta[\psi_a^\pm(x)](\hbar) = \int_{\mp\xi_0(x)}^{\infty e^{i\theta}} e^{-\frac{\xi}{\hbar}} \mathfrak{B}[\psi_a^\pm(x)](\xi) d\xi, \quad \theta = \arg(\hbar),$$

$$\mathfrak{B}[\psi_a^\pm(x)](\xi) = \sum_{n=0}^{\infty} \frac{\psi_{a,n}^\pm(x)}{\Gamma\left(n+\frac{1}{2}\right)} (\xi \pm \xi_0(x))^{n-\frac{1}{2}}, \quad \xi_0(x) = \int_a^x dx\, S_{\mathrm{odd},-1}(x).$$

The Borel summablity is determined from

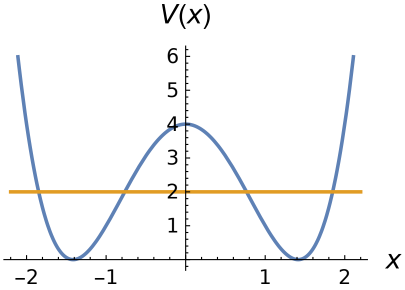
$$\frac{\xi_0(x)}{\hbar} = \frac{1}{\hbar}\int_a^x dx\, S_{\mathrm{odd},-1}(x) = \frac{1}{\hbar}\int_a^x dx\, \sqrt{Q_0(x)}.$$

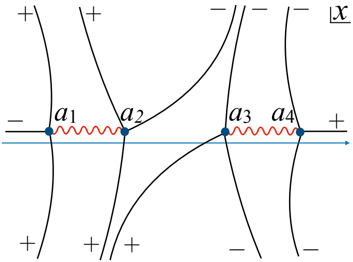
Stokes phenomenon (i.e. **Borel non-summable**) happens when

$$\mathrm{Im}\frac{\xi_0(x)}{\hbar} = -\mathrm{Im}\frac{\xi_0(x)}{\hbar} = 0.$$

Since the Borel summability is relevant to $\frac{1}{\hbar}\int S_{\mathrm{odd},-1}$, it is natural to see the Riemann surface defined by $\frac{1}{\hbar}\int S_{\mathrm{odd},-1}$, called **Stokes graph**.

Example: Double-well potential



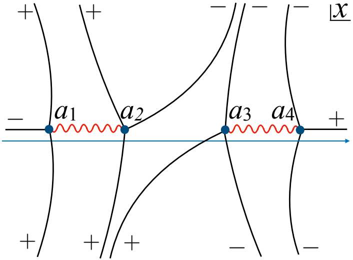


The Stokes graph

Since the Borel summability is relevant to $\frac{1}{\hbar}\int S_{\mathrm{odd},-1}$, it is natural to see the Riemann surface defined by $\frac{1}{\hbar}\int S_{\mathrm{odd},-1}$, called **Stokes graph**.

Constitutive ingredients:

- Turning point $(a_1, a_2, \cdots)$
  Def: $Q_0(x) = 0$
  $(V_0(x) - E = 0)$
- Stokes line (black line)
  Def: $\mathrm{Im}\frac{1}{\hbar}\int S_{\mathrm{odd},-1} = 0$
  $\pm$ labels $\int S_{\mathrm{odd},-1} \to \pm\infty$
- Branch cut (red wave)
  $+S_{\mathrm{odd}}(x,\hbar) \leftrightarrow -S_{\mathrm{odd}}(x,\hbar)$

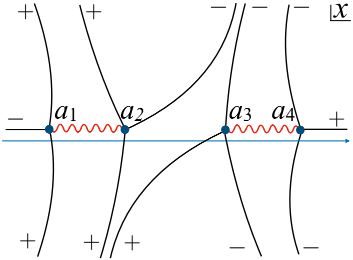


Example of the Stokes graph.
Double-well with $\arg(\hbar) > 0$.

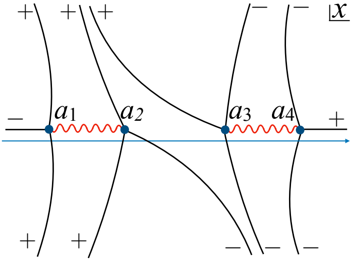
The Stokes graph

For getting the Q.C, we consider the analytic continuation of $\psi(x)$ for a given Stokes graph and a B.C. To do it, we have to know the effect of crossing Stokes line for $\psi(x)$.

$$\psi^{\mathrm{I}}(x) = \mathcal{M}^{\pm}\psi^{\mathrm{II}}(x), \qquad \psi := (\psi_+, \psi_-)^{\top}.$$



$\arg(\hbar) > 0$



$\arg(\hbar) < 0$

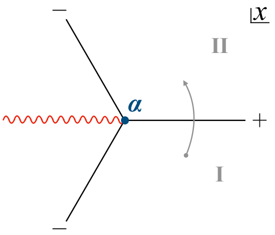
Connection formula for the Airy-type

Consider crossing the Stokes line from I to II.

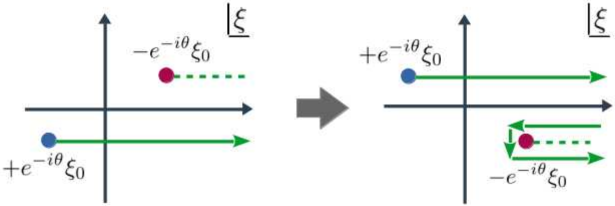
$$\psi = (\psi_+, \psi_-)^\top,$$

$$\psi_a^{\mathrm{I}} = M_+ \psi_a^{\mathrm{II}}, \qquad \psi_a^{\mathrm{I}} = M_- \psi_a^{\mathrm{II}},$$

$$M_+ = \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix}, \quad M_- = \begin{pmatrix} 1 & 0 \\ +i & 1 \end{pmatrix}.$$



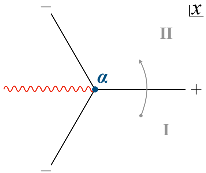
$\left(\frac{\xi_0(x)}{\hbar} = \frac{1}{\hbar}\int_a^x dx\, S_{\mathrm{odd},-1}(x)\right)$



Connection formula for the Airy-type

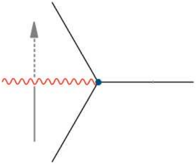
Connection matrix

$$M_+ = \begin{pmatrix} 1 & +i \\ 0 & 1 \end{pmatrix}, \quad M_- = \begin{pmatrix} 1 & 0 \\ +i & 1 \end{pmatrix},$$



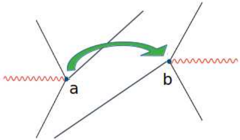
Branchcut matrix

$$T = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$



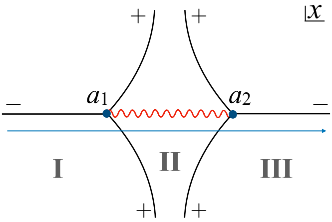
Normalization matrix (Voros multiplier)

$$N_{ba} = \begin{pmatrix} e^{+\int_a^b dx\, S_{\rm odd}(x,\hbar)} & 0 \\ 0 & e^{-\int_a^b dx\, S_{\rm odd}(x,\hbar)} \end{pmatrix}.$$



Example 1: Harmonic oscillator $V(x) = \frac{1}{2}x^2$

$$
\begin{aligned}
\mathrm{I} \to \mathrm{II} &: \quad \psi^{\mathrm{I}}_{a_1} = M_+ \psi^{\mathrm{II}}_{a_1} \\
(1 \to 2) &: \quad \psi^{\mathrm{I}}_{a_1} = N_{a_1,a_2} \psi^{\mathrm{II}}_{a_2} \\
\mathrm{II} \to \mathrm{III} &: \quad \psi^{\mathrm{II}}_{a_2} = M_+ \psi^{\mathrm{III}}_{a_2} \\
(2 \to 1) &: \quad \psi^{\mathrm{III}}_{a_2} = N_{a_2,a_1} \psi^{\mathrm{III}}_{a_1} \\
&\Rightarrow \quad \psi^{\mathrm{I}}_{a_1} = \begin{pmatrix} 1 & i(1+A) \\ 0 & 1 \end{pmatrix} \psi^{\mathrm{III}}_{a_1}.
\end{aligned}
$$



Normalizability for $\psi_a = (\psi_{+,a}, \psi_{-,a})^\top$:

$$
\begin{aligned}
&\psi^{\mathrm{I}}_{a_1}(x,\hbar) \to 0 \quad \text{as} \quad x \to \pm\infty \\
\Rightarrow \quad &\psi^{\mathrm{I}}_{-,a_1}(x,\hbar) = 0 \quad \text{and} \quad \boxed{D(\hbar) := i(1 + A(\hbar)) = 0}
\end{aligned}
$$

where $A(\hbar) := e^{2\int_{a_1}^{a_2} dx\, S_{\mathrm{odd}}(x,\hbar)} = e^{\oint_A dx\, S_{\mathrm{odd}}(x,\hbar)}$.

Example 1: Harmonic oscillator $V(x) = \frac{1}{2}x^2$

Since $Q(x) = x^2 - 2E$ with $E > 0$, the turning points are given by $a_1 = -\sqrt{2E}, a_2 = +\sqrt{2E}$. Hence,

$$2\int_{a_1}^{a_2} dx\, S_{\mathrm{odd}}(x,\hbar) = -\frac{2\pi i E}{\hbar}.$$

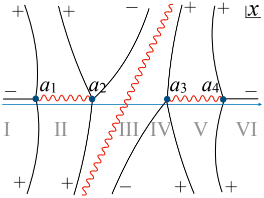
From the quantization condition, i.e. $D = i(1+A) = 0$,

$$1 + e^{-\frac{2\pi i E}{\hbar}} = 0 \quad \Rightarrow \quad E = \left(\frac{1}{2} + n\right)\hbar, \quad n \in \mathbb{Z}$$

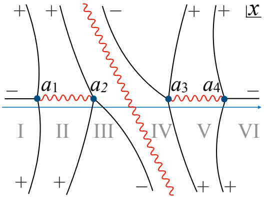
From the positive energy condition,

$$\boxed{E = \left(\frac{1}{2} + n\right)\hbar, \quad n \in \mathbb{N}_0}$$

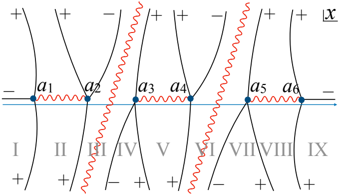
Stokes graph



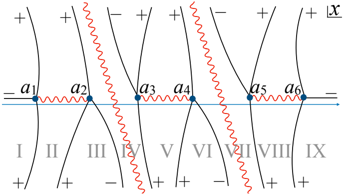
Double-well $(\operatorname{Im}\hbar > 0)$



Double-well $(\operatorname{Im}\hbar < 0)$

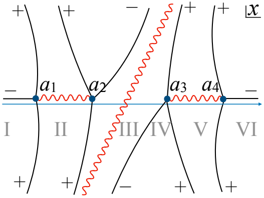


Triple-well $(\operatorname{Im}\hbar > 0)$

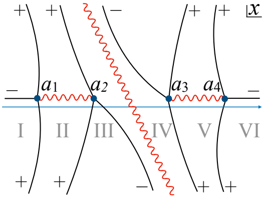


Triple-well $(\operatorname{Im}\hbar < 0)$

Quantization condition



Double-well $(\mathrm{Im}\,\hbar > 0)$



Double-well $(\mathrm{Im}\,\hbar < 0)$

$$\psi_{\mathrm{I},a_1}(x,\hbar) = \mathcal{M}\psi_{\mathrm{VI},a_1}(x,\hbar),$$

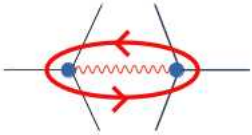
$$\mathcal{M} = \begin{cases} M_+ N_{a_1,a_2} M_+ N_{a_2,a_3} T M_- M_+ N_{a_3,a_4} M_+ N_{a_4,a_1} =: \mathcal{M}^+ & \text{for } \theta > 0 \\ M_+ N_{a_1,a_2} M_+ M_- N_{a_2,a_3} T M_+ N_{a_3,a_4} M_+ N_{a_4,a_1} =: \mathcal{M}^- & \text{for } \theta < 0 \end{cases}.$$

Cycle expression (Voros multiplier)

For the WKB analysis, it is convenient to introduce **cycle** expression, which is known as the Voros multipliers.

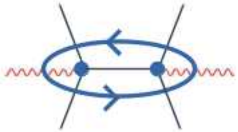
- ***A*-cycle (PT)**

$$A(\hbar) := e^{\oint_A dx\, S_{\rm odd}(x,\hbar)},$$
$$\frac{1}{\hbar}\oint_A dx\, \sqrt{2(V(x)-E)} \in i\mathbb{R}.$$



- ***B*-cycle (NP)**

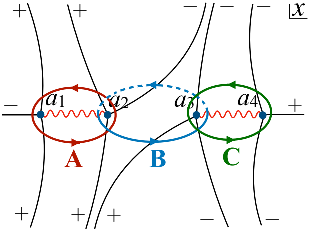
$$B(\hbar) := e^{\oint_B dx\, S_{\rm odd}(x,\hbar)},$$
$$\frac{1}{\hbar}\oint_B dx\, \sqrt{2(V(x)-E)} \in \mathbb{R}.$$



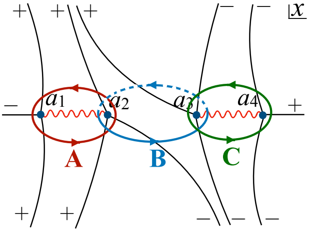
Note: We take the orientation such that $B(\hbar) \propto e^{-\frac{S_{\rm b}}{\hbar}}$.

Delabaere-Dillinger-Pham (DDP) formula

Cycles have the resurgent relation called **DDP formula**.



$\arg(\hbar) > 0$



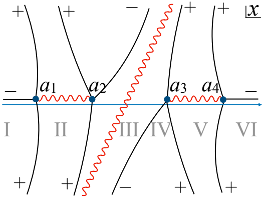
$\arg(\hbar) < 0$

In the case of the double-well potential, it is given by

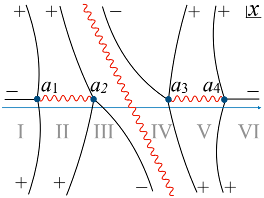
$$\mathcal{S}_+[A] = S_-[A](1+\mathcal{S}[B])^{-1}, \qquad \mathcal{S}_+[C] = S_-[C](1+\mathcal{S}[B])^{+1},$$
$$\mathcal{S}_+[B] = \mathcal{S}_-[B] =: \mathcal{S}[B].$$

(See DDP paper for details and generic cases.
E.g.**[E. Delabaere et al., JMP 38(12), '97]**)

Quantization condition



Double-well ($\mathrm{Im}\,\hbar > 0$)



Double-well ($\mathrm{Im}\,\hbar < 0$)

$$\mathcal{M}^+ = \begin{pmatrix} 1 + B^{-1} + A_1^{-1}B^{-1} & i(1+A_1)(1+A_2^{-1}) + iA_1A_2^{-1}B \\ -iA_1^{-1}B^{-1} & 1 + A_2^{-1} \end{pmatrix},$$

$$\mathcal{M}^- = \begin{pmatrix} B^{-1}(1 + A_1^{-1}) & i(1+A_1)(1+A_2^{-1}) + iB \\ -iA_1^{-1}B^{-1} & 1 + A_2^{-1} + B \end{pmatrix},$$

Quantization condition

We set the boundary condition (normalizability):

$$\lim_{|x|\to\infty} \psi_{\mathrm{I}_{a_1}}(x, |\hbar| e^{i0^\pm}) = 0 \quad \Rightarrow \quad \mathfrak{D}^\pm(E,\hbar) := \mathcal{M}^\pm_{12} = 0.$$

Double-well :

$$\mathfrak{D}^\pm(E,\hbar) = (1+A_1)\left(1+A_2^{-1}\right) + \begin{cases} A_1 B & \text{for } + \\ {A_2}^{-1} B & \text{for } - \end{cases},$$

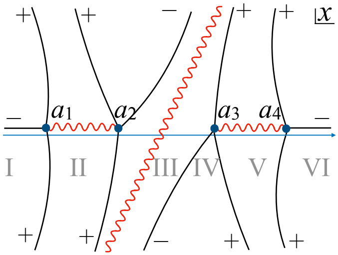
Triple-well :

$$\begin{aligned} \mathfrak{D}^\pm(E,\hbar) &= (1+A_1)(1+A_2^{-1})(1+A_3) \\ &+ \begin{cases} A_1(1+A_3)B_1 + (1+A_1)A_3 B_2 + A_1 A_3 B_1 B_2 & \text{for } + \\ A_2^{-1}(1+A_3)B_1 + (1+A_1)A_2^{-1} B_2 + A_2^{-1} B_1 B_2 & \text{for } - \end{cases}. \end{aligned}$$

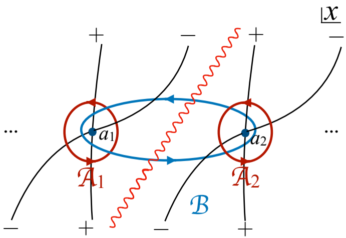
Quantization condition

In the Airy case, we implicitly assumed $E = O(\hbar^0)$ in the Sch eq. However, the solution of the energy spectrum is generally $E = O(\hbar^1)$. By replacing the energy in the Sch eq as $E \to E\hbar$, the Stokes graph changes.

$$\text{Airy : } (A, B) \longrightarrow \text{Degenerate Weber : } (\mathfrak{A}, \mathfrak{B})$$



Airy-type



Degenerate Weber-type

Quantization condition

We can obtain $\mathcal{M}^{\pm}$ of the DW-type by replacing cycles $(A, B) \to (\mathfrak{A}, \mathfrak{B})$.
The DDP formula using the cycle expression does *not* change.

$$
\begin{aligned}
\mathfrak{A}_\ell(E,\hbar) &= e^{2\pi i F_\ell(E,\hbar)}, \\
\mathfrak{B}_\ell(E,\hbar) &= 2\pi\mathfrak{B}_0 \prod_{\ell'=0}^{1} \frac{C_{\ell+\ell' -}(E,\hbar)}{C_{\ell+\ell' +}(E,\hbar)} \frac{e^{(-1)^{\ell'}\pi i F_{\ell+\ell'}(E,\hbar)}\hbar^{F_{\ell+\ell'}(E,\hbar)}}{\Gamma(1/2 - F_{\ell+\ell'}(E,\hbar))},
\end{aligned}
$$

where

$$
\begin{aligned}
&F_\ell(E,\hbar) = -\mathrm{Res}_{a=a_\ell} S_{\mathrm{odd}}(E,\hbar), \\
&\mathfrak{B}_0 = e^{-S_{\mathrm{B}}/\hbar}.
\end{aligned}
$$

Due to the form of SUSY-type potential, all $\mathfrak{A}_\ell$ can be expressed by $\bar{\mathfrak{A}}$:

$$\text{E.g. Double-well}: \; \mathfrak{A}_\ell(E,\hbar) = e^{(-1)^\ell \pi i p}\bar{\mathfrak{A}}(E,\hbar).$$

The ℓ-dependence enters only into the overall phase.

Quantization condition

Double-well :

$$\mathfrak{A}_\ell(E,\hbar) = e^{(-1)^\ell \pi i p}\bar{\mathfrak{A}}(E,\hbar) = e^{-2\pi i[E+(-1)^{\ell+1}p/2]} + O(\hbar),$$

$$\mathfrak{B}(E,\hbar) = \bar{\mathfrak{B}}(E,\hbar) = \frac{2\pi\mathfrak{B}_0}{\Gamma(E+\frac{1+p}{2})\Gamma(E+\frac{1-p}{2})}\left(\frac{\hbar}{2}\right)^{-2E}(1+O(\hbar)),$$

$$\mathfrak{B}_0 = e^{-\frac{1}{3\hbar}},$$

Triple-well :

$$\mathfrak{A}_{1,3}(E,\hbar) = e^{+\pi i p}\bar{\mathfrak{A}}(E,\hbar) = e^{-2\pi i(E-p/2)} + O(\hbar),$$

$$\mathfrak{A}_2(E,\hbar) = e^{-\pi i p}\bar{\mathfrak{A}}^2(E,\hbar) = e^{-2\pi i(2E+p/2)} + O(\hbar),$$

$$\mathfrak{B}_\ell(E,\hbar) = \bar{\mathfrak{B}}(E,\hbar) = \frac{2\pi\mathfrak{B}_0}{\Gamma(E+\frac{1-p}{2})\Gamma(2E+\frac{1+p}{2})}\cdot\frac{\hbar^{-3E}}{2^{E-\frac{p}{2}}}(1+O(\hbar)),$$

$$\mathfrak{B}_0 = e^{-\frac{1}{4\hbar}}.$$

DDP formula and Stokes automorphism

From the DDP formula,

$$\mathcal{S}_+[\bar{\mathfrak{A}}] = \mathcal{S}_-[\bar{\mathfrak{A}}](1+\mathcal{S}[\bar{\mathfrak{B}}])^{-1} \quad \Leftrightarrow \quad \mathcal{S}_+[\bar{\mathfrak{A}}] = \mathcal{S}_- \circ \mathfrak{S}[\bar{\mathfrak{A}}]$$

$$\Rightarrow \quad \mathfrak{S}[\bar{\mathfrak{A}}] = \bar{\mathfrak{A}}(1+\bar{\mathfrak{B}})^{-1}.$$

One can make sure that

$$\mathfrak{S}[\mathfrak{D}^+(\bar{\mathfrak{A}},\bar{\mathfrak{B}})] \;=\; \mathfrak{D}^+(\mathfrak{S}[\bar{\mathfrak{A}}],\mathfrak{S}[\bar{\mathfrak{B}}]) \;=\; \mathfrak{D}^-(\bar{\mathfrak{A}},\bar{\mathfrak{B}}).$$

The formal exact form of Q.C. is expressed by

$$\widehat{\mathfrak{D}}_{\rm ex} = \mathcal{S}_+[\mathfrak{D}^+] = \mathcal{S}_-[\mathfrak{D}^-].$$

DDP formula and Stokes automorphism

If one wants to obtain the transseries $\mathfrak{D}$ of the formal exact solution $\widehat{\mathfrak{D}}_{\rm ex}$ as

$$\widehat{\mathfrak{D}}_{\rm ex}(\hbar) \sim \mathfrak{D}(\hbar) \quad \text{as} \quad \hbar \to 0_+,$$

one can obtain $\mathfrak{D}$ using $\mathfrak{S}^{\nu}$ and $\mathcal{S}_{\rm med}$ without the explicit form of $\widehat{\mathfrak{D}}_{\rm ex}(\hbar)$:

$$\mathfrak{S}^{\nu}[\bar{\mathfrak{A}}] = \bar{\mathfrak{A}}(1+\bar{\mathfrak{B}})^{-\nu}$$
$$\mathcal{S}_{\rm med} := \mathcal{S}_{+} \circ \mathfrak{S}^{-1/2} = \mathcal{S}_{-} \circ \mathfrak{S}^{+1/2}$$

If there exists $\mathfrak{D}$ such that $\mathfrak{S}^{\mp 1/2}[\mathfrak{D}] = \mathfrak{D}^{\pm}$, then

$$\mathfrak{D} = \mathfrak{S}^{\pm 1/2}[\mathfrak{D}^{\pm}],$$

which is the transseries of $\widehat{\mathfrak{D}}_{\rm ex}$ expanded around $\hbar = 0$.

Quantization condition

Q.C. with discontinuity, $\mathfrak{D}^{\pm}$

Double-well :

$$\mathfrak{D}^{\pm} = \left(1 + e^{-\pi i p}\bar{\mathfrak{A}}\right)\left(1 + e^{-\pi i p}\bar{\mathfrak{A}}^{-1}\right) + e^{-\pi i p}\bar{\mathfrak{A}}^{\pm 1}\bar{\mathfrak{B}},$$

Triple-well :

$$\mathfrak{D}^{\pm} = \prod_{\varepsilon\in\{-1,+1\}}\left[(1 + e^{\mp\pi i p}\bar{\mathfrak{A}}^{\mp 1})(1 + i\varepsilon e^{\pm\pi i \frac{p}{2}}\bar{\mathfrak{A}}^{\mp 1}) + \bar{\mathfrak{B}}\right],$$

Q.C. without discontinuity, $\mathfrak{D} = \mathfrak{S}^{\pm 1/2}[\mathfrak{D}^{\pm}]$

Double-well :

$$\mathfrak{D} = 1 + e^{-2i\pi p} + \left(e^{-\pi i p}\bar{\mathfrak{A}} + e^{-\pi i p}\bar{\mathfrak{A}}^{-1}\right)\left(1 + \bar{\mathfrak{B}}\right)^{1/2},$$

Triple-well :

$$\mathfrak{D} = \prod_{\varepsilon\in\{-1,+1\}}\left[e^{3\pi i p/2} - i\varepsilon + \left(e^{\pi i p/2}\bar{\mathfrak{A}}^{-1} - i\varepsilon e^{\pi i p}\bar{\mathfrak{A}}\right)\left(1 + \bar{\mathfrak{B}}\right)^{1/2}\right].$$

Energy spectra

Let us obtain the energy spectra by solving the quantization condition. Here, we consider the leading order of $\hbar$ and introduce $\delta(\hbar) = O(e^{-c/\hbar})$ as a nonperturbative part:

$$E = E_{\rm pt} + \delta(\hbar),$$

where the perturbative part $E_{\rm pt}$ is given by

$$\text{Double-well} \quad : \qquad E_{\rm pt} = k + \frac{1 \mp p}{2}, \qquad (k \in \mathbb{N}_0)$$

$$\text{Triple-well} \quad : \qquad E_{\rm pt} = \begin{cases} \frac{1}{2}\left(k + \frac{1-p}{2}\right) & \text{for the inner-vacuum} \\ k + \frac{1+p}{2} & \text{for the outer-vacua} \end{cases}.$$

We substitute E into $\mathfrak{D}^{\pm}$ and $\mathfrak{D}$ to obtain $\delta(\hbar)$.
Since $\mathfrak{D}^{\pm}$ still has the discontinuity, it gives Borel summability of the energy spectra, while $\mathfrak{D}$ gives a transseries of $\widehat{\mathfrak{D}}_{\rm ex}$.

Double-well :

$$\delta_{p\in\mathbb{Z}}(\hbar) = \mathcal{P}\sqrt{\frac{\mathfrak{B}_0}{\pi\hbar\Gamma(1+k)\Gamma(1+k-p)}}\left(\frac{\hbar}{2}\right)^{-k+\frac{p}{2}} - \frac{\mathfrak{B}_0}{\pi\hbar\Gamma(1+k)\Gamma(1+k-p)}\left(\frac{\hbar}{2}\right)^{-2k+p}\Phi_{\pm1}(k,p) + O(\mathfrak{B}_0^{3/2}),$$

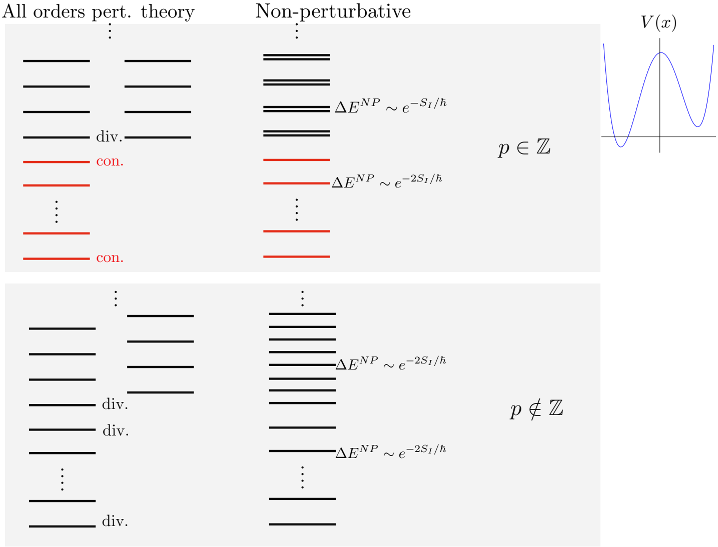
$$\delta_{p\notin\mathbb{Z}}(\hbar) = (-1)^{1+k}\frac{\mathfrak{B}_0 e^{\pm\pi ip}\Gamma(-k+p)}{\pi\hbar\Gamma(1+k)}\left(\frac{\hbar}{2}\right)^{-2k+p} - \frac{2\mathfrak{B}_0^2 e^{\pm2\pi ip}\Gamma(-k+p)^2}{\pi^2\hbar^2\Gamma(1+k)^2}\left(\frac{\hbar}{2}\right)^{-4k+2p}\Phi_{\pm1}(k,p) + O(\mathfrak{B}_0^3),$$

where $\mathcal{P} \in \{-1,+1\}$ is the parity, and

$$\Phi_n(k,p) := \frac{\psi^{(0)}(1+k) + \psi^{(0)}(1+k-p)}{2} + \log\frac{\hbar}{2} + \pi n i,$$

with the polygamma function $\psi^{(n)}(x)$.

Energy spectra $\mathfrak{D}^{\pm} = 0$ (double-well)



Triple-well (inner-vacuum) : $\quad (p = (2q + 1)/3)$

$$\delta^{\varepsilon=(-1)^q}_{k+q\in 2\mathbb{Z}+1}(\hbar) = \mp i \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(2+k-q)} \hbar^{-\frac{1}{2}(1+3k-q)}}{\Gamma(1+k)\Gamma(\frac{1}{2}+\frac{k}{2}-\frac{q}{2})} + \frac{\mathfrak{B}_0^2 2^{-(2+k-q)} \hbar^{-(1+3k-q)}}{\Gamma(1+k)^2\Gamma(\frac{1}{2}+\frac{k}{2}-\frac{q}{2})^2} \Psi^{(1)}_{\pm 2}\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^3),$$

$$\delta^{\varepsilon=(-1)^{q+1}}_{k+q\in 2\mathbb{Z}+1}(\hbar) = \mathcal{P}\sqrt{\frac{\mathfrak{B}_0}{\pi\Gamma(1+k)\Gamma(\frac{1}{2}+\frac{k}{2}-\frac{q}{2})}} 2^{-\frac{1}{4}(2+k-q)} \hbar^{-\frac{1}{4}(1+3k-q)} - \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(2+k-q)} \hbar^{-\frac{1}{2}(1+3k-q)}}{2\pi\Gamma(1+k)\Gamma(\frac{1}{2}+\frac{k}{2}-\frac{q}{2})} \Psi^{(1)}_{\pm 2}\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^{3/2}),$$

where $\mathcal{P} \in \{-1, +1\}$ is the parity, and

$$\Psi^{(1)}_n(k,p) := 2\psi^{(0)}(1+k) + \psi^{(0)}\left(\frac{3}{4} + \frac{k}{2} - \frac{3p}{4}\right) + \log 2 + 3\log\hbar + \pi n i,$$

with $\psi^{(n)}(x)$ is the polygamma function.

Energy spectra $\mathfrak{D}^{\pm} = 0$ (triple-well, inner-vacuum, $k + q \notin 2\mathbb{Z} + 1$)

Triple-well (inner-vacuum) : $(p = (2q+1)/3)$

$$\delta_{k+q\notin 2\mathbb{Z}+1}(\hbar) = \mp i \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(2+k-q)} e^{\mp \frac{\pi i}{2}(k-q)} \hbar^{-\frac{1}{2}(1+3k-q)} \Gamma(\frac{1}{2} - \frac{k}{2} + \frac{q}{2})}{\pi \Gamma(1+k)} + \frac{\mathfrak{B}_0^2 2^{-(2+k-q)} e^{\mp \pi i(k-q)} \hbar^{-(1+3k-q)} \Gamma(\frac{1}{2} - \frac{k}{2} + \frac{q}{2})^2}{\pi^2 \Gamma(1+k)^2} \cdot \Psi^{(1)}_{\pm 2}\left(k, \frac{2q+1}{3}\right),$$

where

$$\Psi^{(1)}_n(k,p) := 2\psi^{(0)}(1+k) + \psi^{(0)}\left(\frac{3}{4} + \frac{k}{2} - \frac{3p}{4}\right) + \log 2 + 3\log\hbar + \pi n i,$$

with $\psi^{(n)}(x)$ is the polygamma function.

Energy spectra $\mathfrak{D}^{\pm} = 0$ (triple-well, outer-vacua, $q \in \mathbb{Z}$)

Triple-well (outer-vacua) : $(p = (2q+1)/3)$

$$\delta_{q\in\mathbb{Z}}^{\varepsilon=(-1)^q}(\hbar) = \mp i \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(3+2k)} \hbar^{-(2+3k+q)}}{\Gamma(1+k)\Gamma(2+2k+q)} + \frac{\mathfrak{B}_0^2 2^{-(3+2k)} \hbar^{-2(2+3k+q)}}{\Gamma(1+k)^2 \Gamma(2+2k+q)^2} \Psi_{\pm 2}^{(2)}\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^3),$$

$$\delta_{q\in\mathbb{Z}}^{\varepsilon=(-1)^{q+1}}(\hbar) = \mathcal{P}\sqrt{\frac{\mathfrak{B}_0}{\pi\Gamma(1+k)\Gamma(2+2k+q)}} 2^{-\frac{1}{4}(3+2k)} \hbar^{-\frac{1}{2}(2+3k+q)} - \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(3+2k)} \hbar^{-(2+3k+q)}}{\pi\Gamma(1+k)\Gamma(2+2k+q)} \Psi_{\pm 2}^{(2)}\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^{3/2}),$$

where $\mathcal{P} \in \{-1, +1\}$, and

$$\Psi_n^{(2)}(k,p) := \psi^{(0)}(1+k) + 2\psi^{(0)}\left(\frac{3}{2} + 2k + \frac{3p}{2}\right) + \log 2 + 3\log\hbar + \pi n i,$$

with $\psi^{(n)}(x)$ is the polygamma function.

Energy spectra $\mathfrak{D}^{\pm} = 0$ (triple-well, outer-vacua, $q \notin \mathbb{Z}$)

Triple-well (outer-vacua) : $(p = (2q+1)/3)$

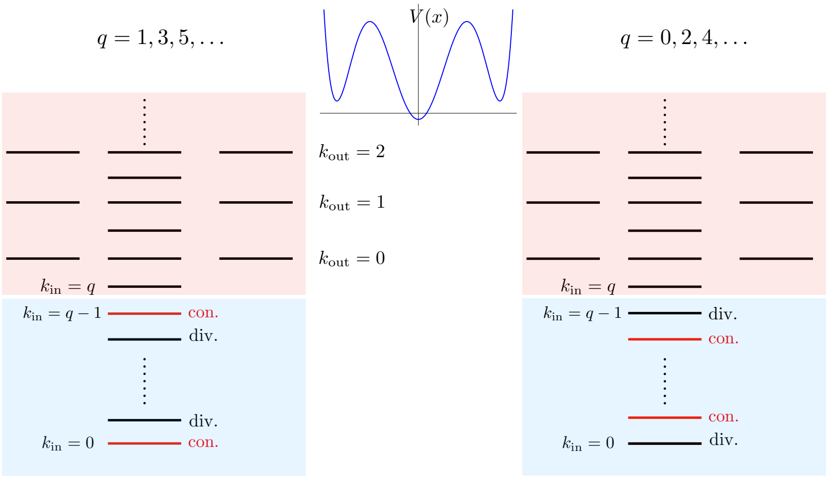
$$
\begin{aligned}
\delta_{q\notin\mathbb{Z}}(\hbar) = & -\frac{\mathfrak{B}_0 2^{-\frac{1}{2}(3+2k)}\hbar^{-(2+3k+q)}\Gamma(-1-2k-q)}{\pi\Gamma(1+k)}\left(\varepsilon - e^{\mp\pi i q}\right) \\
& -\frac{\mathfrak{B}_0^2 2^{-(3+2k)}\hbar^{-2(2+3k+q)}\Gamma(-1-2k-q)^2}{\pi^2\Gamma(1+k)^2}\left(\varepsilon - e^{\mp\pi i q}\right)^2 \\
& \cdot\Psi^{(2)}_{\pm 2}\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^3),
\end{aligned}
$$

where $\varepsilon \in \{+1, -1\}$ and

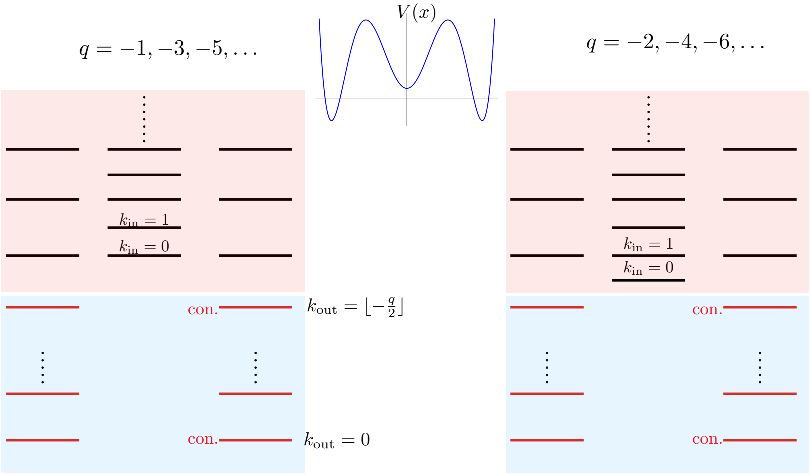
$$
\Psi^{(2)}_n(k,p) := \psi^{(0)}(1+k) + 2\psi^{(0)}\left(\frac{3}{2} + 2k + \frac{3p}{2}\right) + \log 2 + 3\log\hbar + \pi n i,
$$

with $\psi^{(n)}(x)$ is the polygamma function.

Energy spectra $\mathfrak{D}^{\pm} = 0$ (triple-well, $q \in \mathbb{N}_0$)



Energy spectra $\mathfrak{D}^{\pm} = 0$ (triple-well, $q \in \mathbb{Z}_-$)



Energy spectra $\mathfrak{D} = 0$ (double-well)

Double-well :

$$\delta_{p\in\mathbb{Z}}(\hbar) = \mathcal{P}\sqrt{\frac{\mathfrak{B}_0}{\pi\hbar\Gamma(1+k)\Gamma(1+k-p)}}\left(\frac{\hbar}{2}\right)^{-k+\frac{p}{2}}$$

$$-\frac{\mathfrak{B}_0}{\pi\hbar\Gamma(1+k)\Gamma(1+k-p)}\left(\frac{\hbar}{2}\right)^{-2k+p}\Phi_0(k,p) + O(\mathfrak{B}_0^{3/2}),$$

$$\delta_{p\notin\mathbb{Z}}(\hbar) = (-1)^{1+k}\frac{\mathfrak{B}_0\Gamma(-k+p)}{\pi\hbar\Gamma(1+k)}\left(\frac{\hbar}{2}\right)^{-2k+p}\cos(\pi p)$$

$$-\frac{2\mathfrak{B}_0^2\Gamma(-k+p)^2}{\pi^2\hbar^2\Gamma(1+k)^2}\left(\frac{\hbar}{2}\right)^{-4k+2p}\cos^2(\pi p)$$

$$\cdot\left[\Phi_0(k,p) - \frac{3\pi}{2}\tan(\pi p)\right] + O(\mathfrak{B}_0^3),$$

Triple-well (inner-vacuum) :

$$\delta^{\varepsilon=(-1)^q}_{k+q\in 2\mathbb{Z}+1}(\hbar) = 0,$$

$$\delta^{\varepsilon=(-1)^{q+1}}_{k+q\in 2\mathbb{Z}+1}(\hbar) = \mathcal{P}\sqrt{\frac{\mathfrak{B}_0}{\pi\Gamma(1+k)\Gamma(\frac{1}{2}+\frac{k}{2}-\frac{q}{2})}}2^{-\frac{1}{4}(2+k-q)}\hbar^{-\frac{1}{4}(1+3k-q)} - \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(2+k-q)}\hbar^{-\frac{1}{2}(1+3k-q)}}{2\pi\Gamma(1+k)\Gamma(\frac{1}{2}+\frac{k}{2}-\frac{q}{2})}\Psi^{(1)}_0\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^{3/2}),$$

Triple-well (outer-vacua) :

$$\delta^{\varepsilon=(-1)^q}_{q\in\mathbb{Z}}(\hbar) = 0,$$

$$\delta^{\varepsilon=(-1)^{q+1}}_{q\in\mathbb{Z}}(\hbar) = \mathcal{P}\sqrt{\frac{\mathfrak{B}_0}{\pi\Gamma(1+k)\Gamma(2+2k+q)}}2^{-\frac{1}{4}(3+2k)}\hbar^{-\frac{1}{2}(2+3k+q)} - \frac{\mathfrak{B}_0 2^{-\frac{1}{2}(3+2k)}\hbar^{-(2+3k+q)}}{2\pi\Gamma(1+k)\Gamma(2+2k+q)}\Psi^{(2)}_0\left(k, \frac{2q+1}{3}\right) + O(\mathfrak{B}_0^{3/2}),$$

Triple-well (inner-vacuum) :

$$\delta_{k+q\notin 2\mathbb{Z}+1}(\hbar) = -\frac{\mathfrak{B}_0 2^{-\frac{1}{2}(2+k-q)}\hbar^{-\frac{1}{2}(1+3k-q)}\Gamma(\frac{1}{2}-\frac{k}{2}+\frac{q}{2})}{\pi\Gamma(1+k)}\sin\frac{\pi(k-q)}{2}$$
$$-\frac{\mathfrak{B}_0^2 2^{-(2+k-q)}\hbar^{-(1+3k-q)}\Gamma(\frac{1}{2}-\frac{k}{2}+\frac{q}{2})^2}{\pi^2\Gamma(1+k)^2}\sin^2\frac{\pi(k-q)}{2}$$
$$\cdot\left[\Psi_0^{(1)}\left(k,\frac{2q+1}{3}\right)-3\pi\cot\frac{\pi(k-q)}{2}\right]+O(\mathfrak{B}_0^3),$$

Triple-well (outer-vacua) :

$$\delta_{q\notin\mathbb{Z}}(\hbar) = -\varepsilon\frac{\mathfrak{B}_0 2^{-\frac{1}{2}(3+2k)}\hbar^{-(2+3k+q)}\Gamma(-1-2k-q)}{\pi\Gamma(1+k)}\left(\tan\frac{\pi q}{2}\right)^{\varepsilon}\sin(\pi q)$$
$$-\frac{\mathfrak{B}_0^2 2^{-(3+2k)}\hbar^{-2(3k+q+2)}\Gamma(-1-2k-q)}{\pi^2\Gamma(1+k)^2}\left(\tan\frac{\pi q}{2}\right)^{2\varepsilon}\sin^2(\pi q)$$
$$\cdot\left[\Psi_0^{(2)}\left(k,\frac{2q+1}{3}\right)-3\pi\varepsilon\left(\cot\frac{\pi q}{2}\right)^{\varepsilon}\right]+O(\mathfrak{B}_0^3),$$

Summary

- Resurgent theory
  - Making a relationship among PT and NP sectors. The information of sectors propagates to the other sectors.
  - Stokes automorphism/Alien derivative $\Rightarrow$ Resurgent relation

- Exact-WKB analysis for $V(x,\hbar) = \frac{1}{2}W'(x) + \frac{1}{2}p\hbar W''(x)$
  - Double-well and triple-well potentials
  - Energy spectrum ... Borel summability
    ... Transseries w/o discontinuity
  - Comparison with the path-integral (See arXiv:2111.05922)
  - P-NP relation (Dunne-Ünsal relation) (See arXiv:2111.05922)

References (See also our paper and references within.)

[1] J. Ecalle, "Les Fonctions Resurgentes," Vol. I - III (Publ. Math. Orsay, 1981).

[2] A. Voros, "The return of the quartic oscillator. The complex WKB method," Ann. de l'I. H. Poincare, A 39, 211 (1983).

[3] D. Sauzin, "Introduction to 1-summability and resurgence, " arXiv:1405.0356 [math.DS].

[4] T. Kawai and Y. Takei, "Algebraic Analysis of Singular Perturbation Theory", Providence, R.I. : American Mathematical Society, c2005.

[5] E. Delabaere, H. Dillinger and F. Pham, "Exact semiclassical expansions for one-dimensional quantum oscillators," Journal of Mathematical Physics (New York), 38(12) (1997), 6126-6184.

[6] E. Delabaere and F. Pham, "Resurgent methods in semiclassical asymptotics", Ann. Inst. H. Poincare, **71**(1999), 1-94.

[7] J. Zinn-Justin and U. D. Jentschura, "Multi-instantons and exact results I: Conjectures, WKB expansions, and instanton interactions," Annals Phys. **313**, 197 (2004).

[8] M. Sato and T. Tanaka, "N fold supersymmetry in quantum mechanics: Analyses of particular models," J. Math. Phys. **43**, 3484-3510 (2002).

Backup slides

Other expressions

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)$$
Schrödinger eq.

↓ exact-WKB

$D(E) = 0$
exact quantization condition

→ $\frac{1}{2\pi i}\oint p dx \simeq (N + \frac{1}{2}) + O(e^{-\frac{S}{\hbar}})$
generalized Bohr-Sommerfeld

↓

$G(E) = -\partial_E \log D(E)$

→ expand → $G(E) \simeq \sum_n \sum_{p.p.o.} (-1)^n e^{in\oint pdx}$
Gutzwiller trace formula

↓ inverse Laplace transform

$Z(\beta) = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} G(E) e^{-\beta E} dE$

→ calculate the residues → $Z(\beta) = \sum_n e^{-\beta E_n}$
spectral summation form

↓ integral by parts and expand $\log D(E)$

$$Z(\beta) = \sum_n a_n \hbar^n + e^{-\frac{S_1}{\hbar}} \sum_n b_n \hbar^n + e^{-\frac{S_2}{\hbar}} \sum_n c_n \hbar^n + \dots$$
path-integral(trans-series) form

Structure of ODE and Borel summability

The strucutre of ODE is quite essential to the transseries and the resurgent relation:

$$\frac{df}{dx} = G(f,x), \quad f(x) \sim 0 \quad \text{as} \quad x \to +\infty \qquad (f(x) \in \mathbb{R}).$$

For example...

- $G(f,x) = G(f) \in \mathbb{R}[[f]] \quad \Rightarrow$ **Convergent series**

- $G(f,x) = \alpha f + \beta/x + O(f^2, f/x, 1/x^2)$ with $\alpha \in \mathbb{R}_+$ and $\beta \neq 0$

  $\Rightarrow$ **Divergent series** and **Borel summable**

- $G(f,x) = \alpha f + \beta/x + O(f^2, f/x, 1/x^2)$ with $\alpha \in \mathbb{R}_-$ and $\beta \neq 0$

  $\Rightarrow$ **Divergent series** and **Borel non-summable**