SL$(3, \mathbb{Z})$ modularity of $\mathcal{N} = 4$ SYM

Yang Lei

KITS-UCAS

Introduction

- Black hole entropy and temperature should be understood from quantum gravity point of view.

$$S = \frac{Ac^3}{4G\hbar}$$

- Strominger and Vafa's work in 1996
- $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence: modularity relation
- How can one generalize these understandings to higher dimensional AdS black holes?

AdS_3/CFT_2

The first example of counting the microscopic states of black hole can be understood as AdS_3/CFT_2. The Bekenstein-Hawking entropy of BTZ black hole can be written as

$$S = \frac{A}{4G} = \pi\sqrt{\frac{\ell(\ell\mathcal{M}+\mathcal{J})}{2G}} + \pi\sqrt{\frac{\ell(\ell\mathcal{M}-\mathcal{J})}{2G}}$$

The AdS/CFT correspondence relates the conformal dimension to energy of black hole, and spin to the angular momentum

$$\mathcal{M} \leftrightarrow \Delta, \qquad \mathcal{J} \leftrightarrow s$$

We can also split the 2d CFT into the left moving modes and the right moving modes as

$$L_0 - \frac{c}{24} = \ell\mathcal{M} + \mathcal{J}, \qquad \bar{L}_0 - \frac{c}{24} = \ell\mathcal{M} - \mathcal{J}$$

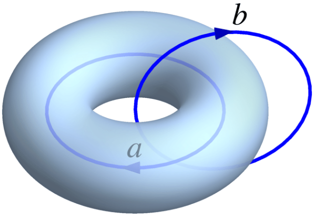
Then understanding the entropy reduces to understanding the Cardy formula from CFT_2

$$S = 2\pi\sqrt{\frac{c}{6}\left(L_0 - \frac{c}{24}\right)} + 2\pi\sqrt{\frac{c}{6}\left(\bar{L}_0 - \frac{c}{24}\right)}$$

Modularity

The 2d partition function $Z[\tau,\bar{\tau}] = \mathrm{Tr}\, e^{2\pi i\tau L_0} e^{-2\pi i\bar{\tau}\bar{L}_0}$ satisfies the $\mathrm{SL}(2,\mathbb{Z})$ modularity

$$Z_0\left[\tau\right] = Z_0\left[-\frac{1}{\tau}\right], \quad Z_0\left[\tau\right] = Z[\tau]e^{-2\pi i\tau\frac{c}{24}}$$



This indicates the asymptotic expansion of partition function near $\tau \to 0$ is

$$Z_0[\tau] \to e^{\frac{2\pi i c}{24\tau}+\frac{2\pi i c\tau}{24}} \times (1+...)$$

Cardy formula

The degeneracy of states with given energy is determined by inverse Legendre transformation

$$\rho(\Delta) = \int e^{-2\pi i\tau\Delta} e^{\frac{2\pi i c}{24\tau} + \frac{2\pi i c\tau}{24}} Z\left[-\frac{1}{\tau}\right] d\tau$$

Since $Z\left[-\frac{1}{\tau}\right] \to 1$, we can evaluate this integral by saddle point approximation

$$\tau = i\sqrt{\frac{c}{24\Delta}}$$

Then we get the Cardy formula

- This means in the high temperature, the universal behavior of $Z_0[\tau]$

$$\text{Casimir energy} + \mathbb{1}$$

- The modularity originates from conformal symmetry

How to generalize this to higher dimensions?

Large AdS black hole

We consider black holes in $\text{AdS}_5 \times S^5$, which includes two angular momenta and three R-charges Q_1, Q_2, Q_3. Our first task is of course to find the large AdS_5 black hole solution. This is a rather difficult problem since Einstein gravity is a non-linear equation.

- (Chong, Cvetic, Lv, Pope): non-supersymmetric solutions but some equal charge/angular momentum

2.1 The Non-Extremal Black Holes

Since there are no solution-generating techniques available for constructing non-extremal rotating black holes in gauged supergravities, our procedure for obtaining them depends to a large extent on a combination of guesswork and conjecture, followed by an explicit verification that the equations of motion are indeed satisfied. Here, we simply present the outcome of this process.

- (Kunduri, Lucietti, Reall): only BPS black hole solutions.

$$E = J_1 + J_2 + Q_1 + Q_2 + Q_3$$

  The most general BPS black hole was found in 0601156.
- (Wu): the most general non-supersymmetric solution

Solution

$$ds^2 = -(H_1H_2H_3)^{-2/3}(dt + \omega_\phi d\phi + \omega_\psi d\psi)^2 + (H_1H_2H_3)^{1/3} h_{mn}dx^m dx^n, \tag{74}$$

where

$$H_I = 1 + \frac{\sqrt{\Xi_a\Xi_b}(1+g^2\mu_I) - \Xi_a\cos^2\theta - \Xi_b\sin^2\theta}{g^2r^2}, \tag{75}$$

$$\begin{aligned} h_{mn}dx^m dx^n \;=\; & r^2\left\{\frac{dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta} + \frac{\cos^2\theta}{\Xi_b^2}\left[\Xi_b + \cos^2\theta\left(\rho^2g^2 + 2(1+bg)(a+b)g\right)\right]d\psi^2\right. \\ & + \frac{\sin^2\theta}{\Xi_a^2}\left[\Xi_a + \sin^2\theta\left(\rho^2g^2 + 2(1+ag)(a+b)g\right)\right]d\phi^2 \\ & \left.+\frac{2\sin^2\theta\cos^2\theta}{\Xi_a\Xi_b}\left[\rho^2g^2 + 2(a+b)g + (a+b)^2g^2\right]d\psi d\phi\right\}, \end{aligned} \tag{76}$$

$$\begin{aligned} \Delta_r &= r^2[g^2r^2 + (1+ag+bg)^2], \qquad \Delta_\theta = 1 - a^2g^2\cos^2\theta - b^2g^2\sin^2\theta, \\ \Xi_a &= 1 - a^2g^2, \qquad \Xi_b = 1 - b^2g^2, \qquad \rho^2 = r^2 + a^2\cos^2\theta + b^2\sin^2\theta, \end{aligned} \tag{77}$$

$$\begin{aligned} \omega_\psi &= -\frac{g\cos^2\theta}{r^2\Xi_b}\left[\rho^4 + (2r_m^2 + b^2)\rho^2 + \frac{1}{2}\left(\beta_2 - a^2b^2 + g^{-2}(a^2 - b^2)\right)\right], \\ \omega_\phi &= -\frac{g\sin^2\theta}{r^2\Xi_a}\left[\rho^4 + (2r_m^2 + a^2)\rho^2 + \frac{1}{2}\left(\beta_2 - a^2b^2 - g^{-2}(a^2 - b^2)\right)\right], \end{aligned} \tag{78}$$

and

$$r_m^2 = g^{-1}(a+b) + ab \tag{79}$$

$$\beta_2 = \Xi_a\Xi_b(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3) - \frac{2\sqrt{\Xi_a\Xi_b}\,(1 - \sqrt{\Xi_a\Xi_b})}{g^2}(\mu_1 + \mu_2 + \mu_3) + \frac{3\,(1-\sqrt{\Xi_a\Xi_b})^2}{g^4} \tag{80}$$

The scalars are

$$X^I = \frac{(H_1H_2H_3)^{1/3}}{H_I}. \tag{81}$$

The vectors are:

$$A^I = H_I^{-1}(dt + \omega_\psi d\psi + \omega_\phi d\phi) + U_\psi^I d\psi + U_\phi^I d\phi \tag{82}$$

Puzzle

In the most general BPS black hole solution, the entropy is

$$S = \frac{A}{4G} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2}{2}(J_1 + J_2)}$$

with the extremality condition

$$\left(Q_1Q_2 + Q_2Q_3 + Q_1Q_3 - \frac{N^2}{2}(J_1 + J_2)\right)\left(Q_1 + Q_2 + Q_3 + \frac{N^2}{2}\right)$$
$$= \frac{N^2}{2}J_1J_2 + Q_1Q_2Q_3$$

How to understand this entropy from the dual $\mathcal{N} = 4$ SYM calculation?

Fail trials

In 2005, Maldacena, Raju, Minwalla and Kinney studied the superconformal index, but they found the superconformal index in the large N limit scales as $\mathcal{O}(N^0)$ while black hole entropy scales as $\mathcal{O}(N^2)$. This mismatch was believed due to cancellation between fermionic and bosonic degrees of freedom.

Should AdS/CFT be modified in the non-perturbative regium?

Resolution

It was first realized by (Hosseini, Hristov, Zaffroni) that this entropy can be acquired by inverse Legendre transformation of the partition function

$$\ln Z = -i\pi \frac{N^2}{2} \frac{\phi_1\phi_2\phi_3}{\tau\sigma}$$

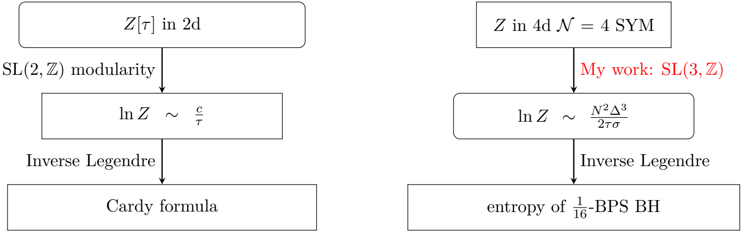
under the chemical potential constraint $\phi_1 + \phi_2 + \phi_3 - \tau - \sigma = 1$ which is equivalent to extremization of the entropy functional

$$S = -i\pi \frac{N^2}{2} \frac{\phi_1\phi_2\phi_3}{\tau\sigma} - 2\pi i(\tau J_1 + \sigma J_2 + Q_1\phi_1 + Q_2\phi_2 + Q_3\phi_3) \\ - \pi i\Lambda(\phi_1 + \phi_2 + \phi_3 - \tau - \sigma - 1)$$

Breakthrough

- Chemical potentials are complex
- The leading order of partition function is very similar to the known supersymmetric Casimir energy

The analogy



Our work is to find the analogue of $SL(2,\mathbb{Z})$ modularity in 4d, which turns out to be $SL(3,\mathbb{Z})$ modularity. Other related researches in studying this includes Bethe Ansatz and Matrix model methods.

Contents of $\mathcal{N}=4$ SYM

The set of letters of $\mathcal{N}=4$ SYM

- 6 indepedent gauge components $F_{\pm,0}, \bar{F}_{\pm,0}$
- 6 complex scalars $Z, W, X, \bar{Z}, \bar{W}, \bar{X}$
- 16 complex fermions $\chi_i, \bar{\chi}_i$, $i=1,..,8$
- 4 components of covariant derivatives $d_{1,2}$ and $\bar{d}_{1,2}$

The letters are specified by dimension E, $SO(4)$ spin (J_1, J_2) and R-charges (Q_1, Q_2, Q_3). The BPS letters are those satisfying

$$E = J_1 + J_2 + Q_1 + Q_2 + Q_3$$

Supersymmetric partition function

$$I = \text{tr}(e^{-\beta E - Q_1\Delta_1 - Q_2\Delta_2 - Q_3\Delta_3 - J_1\tau - J_2\sigma})$$

Elliptic Gamma function

The integral form of superconformal index is

$$I_N = \frac{\kappa_N}{N!} \prod_{k=1}^{N-1} \oint_{|x_k|=1} \frac{dx_k}{2\pi i x_k} \prod_{1\leq i\neq j\leq N} \frac{\prod_{a=1}^{3} \Gamma(x_{ij} f_a)}{\Gamma(x_{ij})} .$$

where the elliptic Gamma function is defined as

$$\Gamma(x) \equiv \Gamma(z;\tau,\sigma) = \prod_{m,n=0}^{\infty} \frac{1 - x^{-1} p^{m+1} q^{n+1}}{1 - x p^m q^n}$$

where $q = e^{2\pi i\tau}$, $p = e^{2\pi i\sigma}$, $x = e^{2\pi i z}$ The analogue function in 2d CFT is q-θ function

$$\theta(z;\tau) = \prod_{n=0}^{\infty} (1 - x q^n)(1 - x^{-1} q^{n+1})$$

Modularity

The θ-function has the following $SL(2,\mathbb{Z})$ modularity transformation:

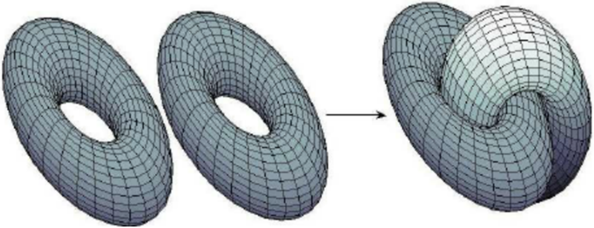
$$\theta\left(\frac{z}{\tau};-\frac{1}{\tau}\right)=e^{i\pi B(z,\tau)}\theta(z;\tau)\,,$$
$$B(z,\tau)=\frac{z^2}{\tau}+z\left(\frac{1}{\tau}-1\right)+\frac{1}{6}\left(\tau+\frac{1}{\tau}\right)-\frac{1}{2}\,.$$

As $\tau\to 0$, $\theta(z/\tau,-1/\tau)\to 1$ approaches to 1, which is equivalent to the identity operator dominates. Then we can approximate the q-θ function by purely the phase factor, which is equivalent to the Casimir operator in 2d.
The modularity satisfied by elliptic Gamma function is [Felder, 99]

$$\Gamma\left(z;\tau,\sigma\right)\Gamma\left(z;\frac{\tau}{\sigma},\frac{1}{\sigma}\right)\Gamma\left(z;\frac{\sigma}{\tau},\frac{1}{\tau}\right)=e^{-i\pi Q(z-1;\tau,\sigma)}\,,$$
$$Q(z;\tau,\sigma)=\frac{z^3}{3\tau\sigma}-\frac{\tau+\sigma-1}{2\tau\sigma}z^2+\frac{\tau^2+\sigma^2+3\tau\sigma-3\tau-3\sigma+1}{6\tau\sigma}z$$
$$+\frac{1}{12}(\tau+\sigma-1)(\tau^{-1}+\sigma^{-1}-1)$$

Geometric background

The three dimensional manifold has the following topological operation:



Given two solid torus $D_2 \times S^1$. We can glue $(1,0)$ circle of one torus with $(0,1)$ circle of the other one. The gluing results in manifold S^3.

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = S \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The procedure is known as Heegaard splitting.

Four dimensional manifold

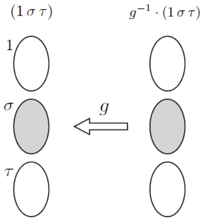
In four dimensions, we are using two three solid tori $D_2 \times T^2$ to do Heegaard splitting gluing. There are three S^1, which results in $\mathrm{SL}(3,\mathbb{Z})$ modularity. We then use S_{23} to glue two solid torus to acquire $S^3 \times S^1$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = S_{23} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad S_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

which is written as

$$\mathcal{M}_g = (D_2 \times T^2) \times_g (D_2 \times T^2)$$

When $g = \mathbb{1}$, $M_g = S^2 \times T^2$.



Physical interpretation

Consider the $\mathcal{N}=1$ chiral multiplet defined on $D_2 \times T^2$. We can use localization method to show the partition function is

$$B_L = \Gamma\left(\frac{z}{\sigma}, \frac{\tau}{\sigma}, -\frac{1}{\sigma}\right)$$

These are called the holomorphic block. Similarly the partition function on $S^3 \times S^1$ can also be shown as $\Gamma(z,\tau,\sigma)$. Therefore the modularity of SL$(3,\mathbb{Z})$ is

$$Z_{S_{23}}(z,\tau,\sigma) = e^{i\phi(S_{23})} B_L(z,\tau,\sigma) B_R(S_{23}\cdot(z,\tau,\sigma))$$

This is called the holomorphic factorization.

Partition function relations

The partition function in 4d depends on the group element we choose. It has been shown they satisfy the following

$$Z_{g_1 g_2}(\vec{\tau}) = e^{i\phi_{g_1,g_2}} Z_{g_1}(\vec{\tau}) \cdot Z_{g_2}(g_1^{-1}\vec{\tau})$$

Recall the 2d partition function satisfies

$$Z(z,\tau) = e^{i\phi_g} Z(g^{-1} \cdot (z,\tau))$$

They are distinguished as automorphic form of degree 0 and 1.

Matrix to modularity

SL$(3,\mathbb{Z})$ matrix relation leads to SL$(3,\mathbb{Z})$ modularity!

A class of modularity

In my work considering $Y^3=1$ relations, where $Y=AS_{23}$ such that $A\in H$. I found the following modularity by studying the SL$(3,\mathbb{Z})$ matrix relations

$$\Gamma(z;\tau,\sigma)=e^{-i\pi Q'_{\mathbf{m}}(mz;\tau,\sigma)}\Gamma\left(\tfrac{z}{m\tau+n};\tfrac{\sigma-\tau}{m\tau+n},\tfrac{\tau-n_1}{m\tau+n}\right)\Gamma\left(\tfrac{z}{m\sigma+n};\tfrac{\tau-\sigma}{m\sigma+n},\tfrac{\sigma-n_1}{m\sigma+n}\right)$$

which indicates the following high temperature limit

$$\tau=\sigma\to\frac{1}{m}-n_0$$

If we consider $Y_1Y_2Y_3=1$, we get

$$\Gamma(z+\sigma,\tau,\sigma)=e^{-i\pi Q'_{\mathbf{m}}}\frac{\Gamma\left(\frac{z}{m\sigma+n_1},\frac{\tau-n_2(k_1\sigma+l_1)}{m\sigma+n_1},\frac{k_1\sigma+l_1}{m\sigma+n_1}\right)}{\Gamma\left(\frac{z}{m\tau+n_2},\frac{-\sigma+n_1(k_2\tau+l_2)}{m\tau+n_2},\frac{k_2\tau+l_2}{m\tau+n_2}\right)}$$

which gives Cardy limit $(\tau,\sigma)\to(-\frac{n_2}{m},-\frac{n_1}{m})$

Phase polynomial

The phase is determined by a single Q-polynomial up to constant

$$Q'_m = \frac{1}{m} Q(mz, m\tau + n_2, m\sigma + n_1) + f(m, n_1, n_2)$$

In the case $n_1 = n_2 = n$, we can show

$$f(m, n) = 2s(n, m) + \frac{(m-1)(m-5)}{12m}$$

We introduce the famous Dedekind sum defined for coprime pair (n, m)

$$s(n, m) = \frac{1}{4m} \sum_{\mu=1}^{m-1} \cot \frac{\pi\mu}{m} \cot \frac{\pi n \mu}{m}$$

But we do not know how to work out the constant with $n_1 \neq n_2$

Dedekind sum

- The general $SL(2,\mathbb{Z})$ action on the η function is

$$\eta\left(\frac{a\tau+b}{c\tau+d}\right)=\epsilon(a,b,c,d)\ \sqrt{c\tau+d}\ \eta(\tau)\,,$$

where

$$\epsilon(a,b,c,d)=\begin{cases}\exp\left(i\pi\left[\frac{a+d}{12c}-s(d,c)-\frac{1}{4}\right]\right) & \text{for} \quad c\neq 0\,,\\ \exp(i\pi b/12) & \text{for} \quad c=0\,,\end{cases}$$

- Classification of lens spaces

Farey tail

In AdS$_3$/CFT$_2$,

$$\tau \to 0, \quad \text{BTZ black hole}$$

$$\tau \to \infty \quad \text{thermal AdS.}$$

We need to sum over all the saddle point in the partition function.
The 2d partition function was studied by [Dijkgraaf, et al, 2000]. They found the moduli τ can be related to gravitational saddle point by approaching any rational number. These are called the SL$(2,\mathbb{Z})$ family of black holes. They are very crucial in understanding information paradox.

$$\mathcal{Z}_\chi(\beta,\omega) = -2\pi i \sum_{(c,d)=1, c\geq 0} \sum_{\mu=1}^{k} \sum_{4km-\mu^2<0} \tilde{c}_\mu\left(4km-\mu^2; \mathrm{Sym}^k(K3)\right)$$
$$(c\tau+d)^{-3} \exp\left[2\pi i \left(m - \frac{\mu^2}{4k}\right)\frac{a\tau+b}{c\tau+d}\right] \exp[-2\pi i k \frac{c\omega^2}{c\tau+d}] \Theta^+_{\mu,k}\left(\frac{\omega}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right)$$

Question

Can one do this in 4d SCFT?

$SL(3,\mathbb{Z})$ family: sum over non-perturbative saddles

We should have independent Lorentz spin related to chemical potentials τ,σ. We need to construct modularity whose high temperature limits are defined by

$$\lim_{\tau,\sigma\to(\mathbb{Q},\mathbb{Q})}\Gamma(z,\tau,\sigma)$$

Consider following relations

$$A_1 S_{23} T_{23}^{-e_1} S_{23} ... T_{23}^{-e_t} S_{23} A_2 S_{23} A_3 S_{23} = 1$$

the modularity becomes

$$\Gamma(z,\tau,\sigma) = e^{-i\pi Q_t(z,\tau,\sigma)} \mathfrak{L}_t\left(\frac{z}{k^0-m\sigma}, \frac{\tau+e_0(n_1\sigma-l^0)}{k^0-m\sigma}, \frac{n_1\sigma-l^0}{k^0-m\sigma}\right) \times \Gamma\left(\frac{z}{mp_t\tau+w_t}, \frac{(q_t+k^{t+1}p_t n w_{t-1})\tau+l^{t+1}w_{t-1}}{mp_t\tau+w_t}, \frac{k^{t+1}p_t\tau+\sigma+\frac{l^{t+1}-l^0}{n_1}}{mp_t\tau+w_t}\right)$$

where $\mathfrak{L}_t$ is called the Lens space partition function defined in the following way.

Lens space

- $S^3 \times S^1$: represented as Hopf surface

$$(z_1, z_2) \sim (\mathrm{p} z_1, \mathrm{q} z_2)\,, \quad (z_1, z_2) \in \mathbb{C}^2 \setminus \{(0,0)\}\,, \quad 0 < |\mathrm{p}| \leq |\mathrm{q}| < 1\,,$$

with $\mathrm{p} = e^{2\pi i(\sigma_1 + i\sigma_2)}$ and $\mathrm{q} = e^{2\pi i(\tau_1 + i\tau_2)}$. Heegaard splitting of S^3: gluing $(1,0)$ circle of $D_2 \times T^2$ with $(0,1)$ circle of another $D_2 \times T^2$.

- $L(p,q) \times S^1$: additional Lens quotient:

$$(z_1, z_2) \sim (e^{\frac{q 2\pi i}{p}} z_1, e^{\frac{-2\pi i}{p}} z_2)\,.$$

Heegaard splitting of $L(p,q)$: gluing $(1,0)$ circle of $D_2 \times T^2$ with (q,p) circle of another $D_2 \times T^2$.

Lens data

Define

$$\Delta_i = S_{23} \prod_{j=1}^{i} \left(T_{23}^{-e_j} S_{23} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & u_i & -v_i \\ 0 & -p_i & q_i \end{pmatrix} .$$

The entries will satisfy the recursive relation

$$\begin{aligned} p_i &= e_i p_{i-1} - p_{i-2} \,, \qquad & q_i &= -p_{i-1} \,, \\ u_i &= e_i u_{i-1} - u_{i-2} \,, \qquad & v_i &= -u_{i-1} \,. \end{aligned}$$

with the initial conditions

$$p_0 = 1 \,, \quad p_1 = e_1 \,, \quad u_0 = 0 \,, \quad u_1 = -1 \,.$$

Define

$$w_i \equiv e_0 p_i + u_i = -(e_0 q_{i+1} + v_{i+1})$$

Continued fraction

$$-\frac{p_t}{q_t} = [e_t; \ldots, e_1]^- = e_t - \frac{1}{e_{t-1} - \cdots}$$

$$\frac{u_t}{p_t} = [0; e_1, e_2, ...e_t] \qquad \frac{w_t}{p_t} = [e_0; e_1, e_2, ...e_t]$$

The Lens partition function is

$$\mathfrak{L}_t(z, \tau, \sigma) = Z_{\Delta_t} = \prod_{i=0}^{t} \Gamma(z + \sigma_i, \tau_i, \sigma_i)$$

where $\sigma_{i+1} = \tau_i = p_i \tau + u_i \sigma$. Then the Cardy limit data is

$$(\tau, \sigma) = \left(-\frac{[e_0; e_1, ...]}{m}, \frac{k^0}{m}\right)$$

Then we scan all the rational pairs

2d vs 4d

	2d SCFT	4d SCFT
Basic block	$\theta(z,\tau)$	$\Gamma(z,\tau,\sigma)$
Modular group	$\mathrm{SL}(2,\mathbb{Z})$	$\mathrm{SL}(3,\mathbb{Z})$
Modularity	$Z[\vec{\tau}] = e^{i\phi} Z[g^{-1}\vec{\tau}]$	$Z_{g_1 g_2}(\vec{\tau}) = e^{i\phi} Z_{g_1}(\vec{\tau}) \cdot Z_{g_2}(g_1^{-1}\vec{\tau})$
Phase	Quadratic poly	cubic poly
Farey tail	Sum over (c,d)	Sum over rational pair (gluing)
Casimir energy	$\ln Z = \frac{c\Delta_1\Delta_2}{2\omega} \sim \frac{c}{12}$	$\ln Z = \frac{N^2}{2}\frac{\Delta_1\Delta_2\Delta_3}{\tau\sigma}$

Essential question

How to compute the phase Q_t? This is crucial for computing the entropy of dual black hole. Let's crack this problem by considering more fundamental mathematical background of the modularity.

- What is the generalization of modularity?
- Geometric origin of modularity

Jacobi group and $\mathrm{SL}(2,\mathbb{Z})$ modularity

Recall that the Jacobi group is given by $\mathcal{J} = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$.

$$\chi\left(z + m\tau + n; \tau\right) = e^{-2\pi i k(m^2\tau + 2mz)}\chi(z;\tau)\,,$$
$$\chi\left(\frac{z}{c\tau+d}; \frac{a\tau+b}{c\tau+d}\right) = e^{2\pi i k\frac{cz^2}{c\tau+d}}\chi(z;\tau)\,.$$

This identifies degree 0 Jacobi forms as elements of the zeroth group cohomology $H^0(\mathcal{J}, N/M)$ of the Jacobi group $\mathcal{J}$. N is the set of meromorphic function while M is the set of nowhere vanishing meromorphic functions. i.e. phases. They form short exact sequence

$$1 \to M \to N \to N/M \to 1\,.$$

Co-chain

Focus on the k-cochain group $C^k(G,A)$, where $A = N, M, N/M$. The group $C^k(G,A)$ consists of k-cocycles $\xi : G^k \to A$ such that $\xi_{g_1,\ldots,g_k} = 1$ if $g_j = 1$ for some j. Furthermore, one defines $C^0(G,A) = A$. To construct the relevant cohomology groups, we now define a coboundary operator $\delta = \delta_k : C^k(G,A) \to C^{k+1}(G,A)$ via:

$$(\delta\xi)_{g_1,\ldots,g_{k+1}}(\boldsymbol{\rho}) = \xi_{g_1,\ldots,g_k}(\boldsymbol{\rho}) \left(\xi_{g_2,\ldots,g_{k+1}}(g_1^{-1}\boldsymbol{\rho}) \prod_{j=1}^{k} \xi_{g_1,\ldots,g_j g_{j+1},\ldots,g_{k+1}}(\boldsymbol{\rho})^{(-1)^j} \right)^{(-1)^{k+1}}$$

So $\delta^2 = 1$ as one can verify.

Furthermore, for $k = 0$ one defines δ on $\chi \in C^0(G, A)$ as:

$$(\delta\chi)_g(\boldsymbol{\rho}) = \frac{\chi(\boldsymbol{\rho})}{\chi(g^{-1}\boldsymbol{\rho})}$$

The coboundary operator allows us to define cohomology in the usual way:

$$H^k(G, A) = \frac{\ker \delta_k}{\text{im } \delta_{k-1}}\,,\ k \geq 1\,, \quad H^0(G, A) = \ker \delta_0$$

If $\chi \in C^0(G, N) = N$ A degree 0 automorphic form of type $\xi_g \in C^1(G, M)$ corresponds to such a function χ which obeys:

$$(\delta\chi)_g(\boldsymbol{\rho}) = \frac{\chi(\boldsymbol{\rho})}{\chi(g^{-1}\boldsymbol{\rho})} = \xi_g(\boldsymbol{\rho})\,,$$

Since ξ_g is taking value in M, we notice that this equation abstracts the property associated to degree 0 Jacobi forms when $G = \mathcal{J}$. It also follows that such χ can be thought of as elements in $H^0(\mathcal{J}, N/M)$, since they are annihilated by δ modulo M.

Degree 1

Having set up the general framework, let us now increase the rank of the cohomology by one, and consider the action of δ_*. We take a one-cocycle $X_g \in C^1(G,N)$. Given the above, it follows that if $[X_g] \in H^1(G,N/M)$, it should satisfy:

$$\delta(X_{g_1}(\boldsymbol{\rho}))_{g_2} = \frac{X_{g_1}(\boldsymbol{\rho})X_{g_2}(g_1^{-1}\boldsymbol{\rho})}{X_{g_1g_2}(\boldsymbol{\rho})} = \xi_{g_1,g_2}(\boldsymbol{\rho})\,,$$

where $[\xi_{g_1,g_2}] \in H^2(G,M)$. It will be important in the following that for $H^1(G,N/M)$, there is a notion of trivializable or "exact" elements. Indeed, such classes can be written as:

$$[X_g] = [(\delta B)_g] = \left[\frac{B(\boldsymbol{\rho})}{B(g^{-1}\boldsymbol{\rho})}\right]\,, \tag{1}$$

with $B \in C^0(G,N)$.

Trivialization subgroup

$$\Gamma(z+\sigma;\tau,\sigma) = e^{-i\pi Q(z+\sigma;\tau,\sigma)} \frac{\Gamma\left(\frac{z}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)}{\Gamma\left(\frac{z}{\tau};-\frac{\sigma}{\tau},-\frac{1}{\tau}\right)}\,. \tag{2}$$

Then, the equation can be written as follows:

$$X_{S_{23}}(\boldsymbol{\rho}) \cong \frac{B^S(\boldsymbol{\rho})}{B^S(S_{23}^{-1}\boldsymbol{\rho})}\,, \tag{3}$$

where we have defined the function $B^S(\boldsymbol{\rho})$:

$$B^S(\boldsymbol{\rho}) \equiv B(S_{13}\boldsymbol{\rho}) = \Gamma\left(\frac{z}{\sigma};\frac{\tau}{\sigma},-\frac{1}{\sigma}\right)\,, \qquad B(\boldsymbol{\rho}) = \Gamma(z;\tau,\sigma)\,. \tag{4}$$

g sits in a subgroup of modular group $\mathrm{SL}(3,\mathbb{Z}) \ltimes \mathbb{Z}^3$

$$F_S \equiv SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^2 \qquad \text{with} \quad SL(2,\mathbb{Z}) = \{S_{23}, T_{23}\}\,, \quad \mathbb{Z}^2 = \{T_{12}, T_{13}\}\,.$$

Chain of trivilization

Consider $g = g_1 g_2 \cdots g_n$

$$
\begin{aligned}
X_g(\boldsymbol{\rho}) &\cong X_{g_1}(\boldsymbol{\rho}) \cdots X_{g_n}(g_{n-1}^{-1} \cdots g_1^{-1} \boldsymbol{\rho}) \\
&\cong \frac{B^S(\boldsymbol{\rho})}{B^S(g_1^{-1}\boldsymbol{\rho})} \cdots \frac{B^S(g_{n-1}^{-1} \cdots g_1^{-1}\boldsymbol{\rho})}{B^S(g_n^{-1} \cdots g_1^{-1}\boldsymbol{\rho})} = \frac{B^S(\boldsymbol{\rho})}{B^S(g^{-1}\boldsymbol{\rho})} \,,
\end{aligned}
$$

Lens modularity

$$\begin{aligned}\mathfrak{L}_t(z,\tau,\sigma) &= \prod_{i=0}^{t}\Gamma(z+\sigma_i,\tau_i,\sigma_i) = \prod_{i=0}^{t} e^{-i\pi Q_i'}\frac{\Gamma_{i+}}{\Gamma_{i-}} \\ &= \left(\prod_{i=0}^{t} e^{-i\pi Q_i'}\right)\frac{\Gamma_{0+}}{\Gamma_{0-}}\frac{\Gamma_{1+}}{\Gamma_{1-}}\cdots\frac{\Gamma_{t+}}{\Gamma_{t-}} \\ &= \left(\prod_{i=0}^{t} e^{-i\pi Q_i'}\right)\frac{\Gamma_{0+}}{\Gamma_{t-}}\end{aligned}$$

This results in

$$\begin{aligned}\mathfrak{L}_t(z,\tau,\sigma) &= \prod_{i=0}^{t}\Gamma(z+\sigma_i,\tau_i,\sigma_i) \\ &= e^{-i\pi \mathbf{P}_t(z,\tau,\sigma)}\frac{\Gamma\left(\frac{z}{m\sigma+n_1},\frac{\tau-n_1 e_0(k^0\sigma+l^0)}{m\sigma+n_1},\frac{k^0\sigma+l^0}{m\sigma+n_1}\right)}{\Gamma\left(\frac{z}{m\tau_t+n_1 w_t},\frac{-\sigma_t+n_1 w_{t-1}(k^{t+1}\tau_t+l^{t+1})}{m\tau_t+n_1 w_t},\frac{k^{t+1}\tau_t+l^{t+1}}{m\tau_t+n_1 w_t}\right)}\end{aligned}$$

Moduli data v.s. geometry

There are two $\mathsf{SL}(2,\mathbb{Z})$ conditions from the modularity

$$n_1 k^0 - m l^0 = 1, \quad n_1 w_t k^{t+1} - m l^{t+1} = 1 \tag{5}$$

This means the moduli group is congruence subgroup

$$\Gamma_0(w_t) = \begin{pmatrix} m & n_1 \\ -k^{t+1} w_t & -l^{t+1} \end{pmatrix}$$

A selection rule in the presence of nontrivial p.

Some future directions

- Modularity in other dimension?
- Non-supersymmetric cases?
- Modular bootstrap
- SL$(3, \mathbb{Z})$ Farey tail summation?
- Search the gravitational solution! Black holes and black lens