The semi-classical saddles in 3d gravity via holography and mini-superspace approach

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Introduction

- One of the important purposes in studying Holography is to understand the correspondence on the **gravity sector**.
- We study the holographic relation between the **semi-classical saddle points** of the (A)dS$_3$ gravity and of the dual CFT$_2$:

$$\mathcal{Z}_{\text{grav}}\left[h_{\mu\nu}\right] \quad = \quad Z_{\text{CFT}}\left[h_{\mu\nu}\right] \qquad (h_{\mu\nu} = g_{\mu\nu}|_{\text{bdy}})$$

semi-classical $\downarrow$ $\qquad\qquad$ $\downarrow$ large c

$$\sum_{n:\,\text{saddles}} \mathcal{Z}_{\text{grav}}\left[g^{(n)}_{\mu\nu}\right] \quad \sim \quad \sum_{n:\,\text{saddles}} Z^{(n)}_{\text{CFT}}$$

- **Q: Which saddle points are relevant in the gravity side?**
  This is a very non-trivial question because **we do not know the correct integration contour** to define the gravitational path integral.

Introduction

- In our previous paper (Yasuaki's talk last year), we have constructed the dual CFT_2 to dS_3 by performing an analytic continuation $\ell_{\text{AdS}} = -i\ell_{\text{dS}}$ from $\text{AdS}_3/\text{CFT}_2$.
- There we observed the **Stokes phenomenon**, in which the contributing saddle points drastically change, under the analytic continuation $\ell_{\text{AdS}} = -i\ell_{\text{dS}}$. As a result, **two** saddle are relevant in the dS_3/CFT_2, compared to **infinite** saddles in $\text{AdS}_3/\text{CFT}_2$.
- In this talk, we discuss the gravity side for both AdS_3 and dS_3, by considering the mini-superspace approach. We then **find the geometries that correspond to CFT saddles**, and **determine the correct contour that reproduces the CFT result**.

Contents

AdS/CFT correspondence

- A dictionary of AdS/CFT is given by so-called GKPW relation

$$\mathcal{Z}_{\mathrm{AdS}}[\phi_0] = \left\langle e^{-\int \mathrm{d}^d x\, \phi_0(x)\mathcal{O}(x)} \right\rangle_{\mathrm{CFT}}, \tag{1}$$

  where ϕ_0 is a boundary condition at the asymptotic boundary in the gravity side.

- In $\mathrm{AdS}_3/\mathrm{CFT}_2$, the central charge of CFT_2 is given by

$$c = \frac{3\ell_{\mathrm{AdS}}}{2G} \tag{2}$$

  Therefore the semi-classical limit $G \to 0$ corresponds to large c limit.

- A heavy primary operator $h \sim \mathcal{O}(c)$ is dual to a black hole or a conical deficit geometry with energy E via

$$\Delta = 2h = \ell_{\mathrm{AdS}} E. \tag{3}$$

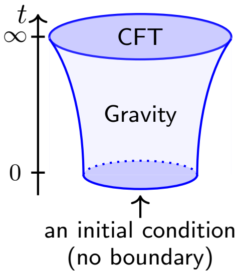
dS/CFT correspondence

- dS/CFT correspondence is an example of de Sitter holography, in which the dual CFT is located at the future boundary [Strominger].

- A dictionary of dS/CFT, analogous to GKPW relation of AdS/CFT, is given by

$$\Psi_{\mathrm{dS}}[\phi_0] = \left\langle e^{-\int \mathrm{d}^d x\, \phi_0(x)\mathcal{O}(x)} \right\rangle_{\mathrm{CFT}},$$

where $\Psi_{\mathrm{dS}}[\phi_0]$ is the wave functional of universe with boundary condition $\phi|_{\mathrm{boundary}} = \phi_0$.



- The dual CFT to dS has many exotic features such as non-unitarity. Nevertheless, this duality has been applied to, say, calculations of cosmological correlators. [Maldacena 2002,...]

dS/CFT correspondence

- dS/CFT can be regarded as analytic continuation from AdS/CFT

$$ds^2_{\rm EAdS} = \ell^2_{\rm AdS}\frac{dz^2 + d\mathbf{x}^2}{z^2} \quad \xrightarrow[z=-it]{\ell_{\rm AdS}=-i\ell_{\rm dS}} \quad ds^2_{\rm dS} = \ell^2_{\rm dS}\frac{-dt^2 + d\mathbf{x}^2}{t^2} \tag{4}$$

- For example, central charge c of the dual CFT_2 to dS_3 is obtained as

$$c = \frac{3\ell_{\rm AdS}}{2G} = -i\frac{3\ell_{\rm dS}}{2G} \equiv -ic^{(g)} \tag{5}$$

- Similarly, the dictionary for a heavy operator with $h \sim \mathcal{O}(c)$ reads

$$2h = \ell_{\rm AdS}E = -i\ell_{\rm dS}E \equiv -i \cdot 2h^{(g)} \tag{6}$$

- As long as E is not too large, the dual geometry is expected to have a conical defect with a deficit angle

$$2\pi(1 - \sqrt{1 - 8G_N E}). \tag{7}$$

Setup

Throughout this talk, we consider the 2d **Liouville theory** as the dual CFT to $EAdS_3$ gravity.

- As is well known, Liouville theory itself cannot be the dual CFT to the Einstein gravity. However, assuming a concrete model [Gaberdiel, Gopakumar], the quantities of the dual CFT can be calculated by the correlation functions of Liouville theory. Furthermore, the semi-classical limit is expected to show the universal behavior of the gravity sector.
- An advantage to consider Liouville theory is good analytic properties of correlation functions [DOZZ], which makes it easy to perform the analytic continuation $c \to -ic^{(g)}$ to dS/CFT.

To do

In the remainder of this talk, we discuss the followings in order:

1. We find the relevant semi-classical saddles from Liouville CFT via GKPW relation $\mathcal{Z}_{\rm AdS}, \Psi_{\rm dS} \sim Z_{\rm CFT}$.
2. We discuss the interpretation of the saddles as complex metrics.
3. We determine the integration contour of the gravitational path integral $\mathcal{Z}_{\rm AdS}, \Psi_{\rm dS}$ in the mini-superspace approach.

* In this talk, we mainly focus on the correspondence of the partition functions without any operator insertions. It would be straightforward to include a couple of insertions.

Contents

Liouville theory

- In Liouville theory, the central charge c is often expressed as

$$c = 1 + 6Q^2 \,, \qquad Q = b + \frac{1}{b} \,. \tag{8}$$

The limit $c \to \infty$ corresponds to $b \to 0$, then

$$c = \frac{6}{b^2} + 13 + \mathcal{O}(b^2) \,. \tag{9}$$

- In the DOZZ conventions, the partition function is $(\lambda > 0)$

$$Z_{\text{Liouville}} = \frac{2\pi}{b^2} \left[\lambda \frac{\gamma(b^2)}{b^2} \right]^{Q/b} \gamma(-b^2) \gamma \left(-1 - b^{-2} \right) \,.$$

- We would like to evaluate the partition functions for both $c \to \infty$ (EAdS case) and $c = -ic^{(g)} \to -i\infty$ (dS case).

Liouville theory

- We have to note that the Stirling formula for $\Gamma(z)$ has different forms depending on the sign of $\mathrm{Re}\, z$ (Stokes phenomenon):

$$\Gamma(z) \sim \begin{cases} e^{z \log z - z} & \mathrm{Re}\, z > 0 \\ \frac{1}{e^{i\pi z} - e^{-i\pi z}} e^{z \log(-z) - z} & \mathrm{Re}\, z < 0 \end{cases}$$

- Due to

$$\frac{1}{b^2} = \frac{c}{6} - \frac{13}{6} + \mathcal{O}\left(\frac{1}{c}\right) ,$$

a factor $\gamma(-1 - b^{-2}) = \Gamma(-1 - b^{-2})/\Gamma(2 + b^{-2})$ behaves as

$$\gamma(-1 - b^{-2}) \sim \begin{cases} \left(e^{-\frac{i\pi c}{6}} - e^{\frac{i\pi c}{6}}\right)^{-1} e^{-\frac{c}{3}\log\frac{c}{6} + \frac{c}{3}} & c \to \infty \\ \left(e^{\frac{\pi c^{(g)}}{6}} - e^{-\frac{\pi c^{(g)}}{6}}\right) e^{\frac{i c^{(g)}}{3}\log(\frac{c^{(g)}}{6}) + \frac{i c^{(g)}}{6}} & c^{(g)} \to \infty \end{cases}$$

Liouville theory

- For $c \to \infty$, (We implicitly shift $c \to c + i\epsilon$)

$$Z_{\text{Liouville}} = \frac{1}{e^{-\frac{i\pi c}{6}} - e^{\frac{i\pi c}{6}}} \lambda^{\frac{c}{6}} = \sum_{n=0}^{\infty} e^{\frac{i\pi(2n+1)c}{6}} \lambda^{\frac{c}{6}} \,. \tag{10}$$

  Each summand actually corresponds to the on-shell action on a complex solution $\phi^{(n)}$ of Liouville theory [Harlow, Maltz, Witten].
  $\to$ **infinite** saddle points are expected to contribute in EAdS gravity.

- For $c \equiv -ic^{(g)} \to -i\infty$,

$$Z_{\text{Liouville}} = \left(e^{\frac{\pi c^{(g)}}{6}} - e^{-\frac{\pi c^{(g)}}{6}} \right) \lambda^{-\frac{ic^{(g)}}{6}} \,. \tag{11}$$

  $\to$ **Two** saddle points are expected to contribute in dS gravity.

- The above difference can be regarded as the **Stokes phenomenon**.

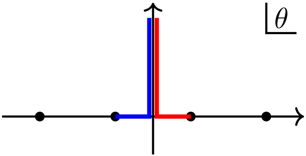
dS saddle points

- In our previous work (Yasuaki's talk last year), we considered the dual geometries to the dS case

$$Z_{\text{Liouville}} \sim \left(e^{\frac{\pi c^{(g)}}{6}} - e^{-\frac{\pi c^{(g)}}{6}} \right) \lambda^{\frac{ic^{(g)}}{6}}$$

- The two saddles correspond to the no-boundary [Hartle, Hawking] and the tunneling [Vilenkin] wave functions, respectively.

$$ds^2 = d\theta^2 + \cos^2\theta d\Omega_2^2$$



- Physical meaning: The data of initial condition (=real factor) of Ψ is obtained from the dual CFT.

AdS saddle points

- **Q: How are the infinite AdS saddles interpreted geometrically?**
- In dS case, the label n of the complex saddles $\phi^{(n)}$ may be understood as the winding number ($\simeq$ Chern-Simons action) of Euclidean section ($\simeq S^3$) of geometry.
- The Euclidean section of geometry is constructed by Wick rotation from

$$-(X^0)^2 + (X^1)^2 + (X^1)^2 + (X^3)^2 = \ell_{\rm dS}^2 \tag{12}$$

with $X^0 = i\tilde{X}^0$ into

$$(\tilde{X}^0)^2 + (X^1)^2 + (X^1)^2 + (X^3)^2 = \ell_{\rm dS}^2 \tag{13}$$

- We would like to construct the EAdS saddles similarly, attaching an analytically continued geometry.

AdS saddle points

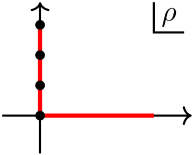
- For the non-trivial winding number to be defined in EAdS, in the embedding coordinate

$$(X^0)^2 + (X^1)^2 + (X^1)^2 - (X^3)^2 = -\ell_{\rm AdS}^2 \,, \tag{14}$$

we continue $X^3 = i\tilde{X}^3$, then

$$(X^0)^2 + (X^1)^2 + (X^1)^2 + (\tilde{X}^3)^2 = -\ell_{\rm AdS}^2 \,. \tag{15}$$

- **We propose that the dual geometries are 3d disks with a imaginary radius sphere attached.**
- We will check this proposal by considering the mini-superspace approach for both dS and AdS.



$ds^2 = d\rho^2 + \sinh^2 \rho d\Omega_2^2$

Contents

Mini-superspace model for dS

- The Einstein-Hilbert action

$$I = -\frac{1}{16\pi G}\int d^3x \left(R - \frac{2}{\ell_{\rm dS}^2}\right) - I_{\rm GH} - I_{\rm ct}\,. \tag{16}$$

- We consider a class of gemetries with the metric ansatz

$$ds^2 = \ell_{\rm dS}^2\left(N(\tau)^2 d\tau^2 + a(\tau)^2 d\Omega_2^2\right)\,. \tag{17}$$

  and two boundaries at $\tau = 0, 1$. We impose the Dirichlet boundary conditions $a(0) = 0$ and $a(1) = a_1$.

- We can fix gauge as $N' = 0$, so that

$$\Psi_{\rm dS} = \int_{\mathcal{C}} dN \int \mathcal{D}a\, e^{-I[a;N] - I_{\rm ct}}\,, \tag{18}$$

$$I[a;N] = -\frac{\ell_{\rm dS}}{2G}\int_0^1 d\tau N\left[\frac{1}{N^2}a'^2 - a^2 + 1\right] \tag{19}$$

Mini-superspace model for dS

- The EOM for $a(\tau)$ is $a'' + N^2 a = 0$, then the solutions are

$$a^{(N)}(\tau) = \frac{a_1}{\sin N} \sin(N\tau)\,. \tag{20}$$

- Substituting the solution and including also the one-loop corrections,

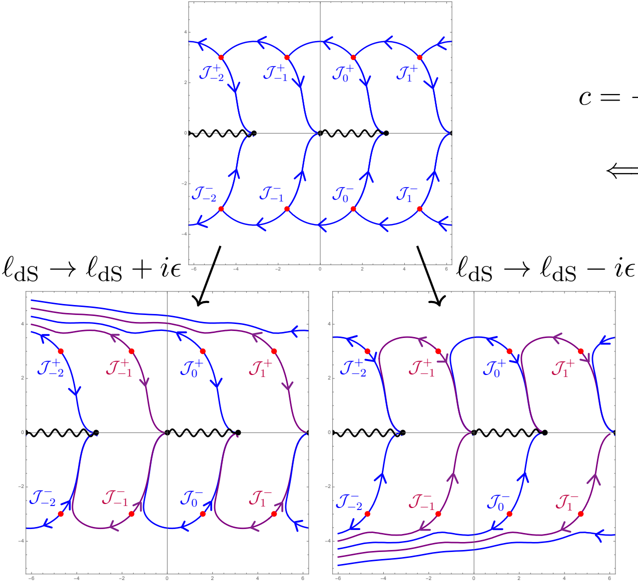
$$\Psi_{\rm dS} = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N}\sin N} \right)^{\frac{1}{2}} e^{\frac{\ell_{\rm dS}}{2G}\left(N + a_1^2 \cot N\right) - I_{\rm ct}} \tag{21}$$

- The saddle points are

$$N_n^{\pm} = \left(n + \frac{1}{2}\right)\pi \pm i \ln\left(a_1 + \sqrt{a_1^2 - 1}\right) \qquad n \in \mathbb{Z}\,. \tag{22}$$

- Actually our model is "critical" in the sense that the theory lies on a point where the Stokes phenomenon happens.

Mini-superspace model for dS



$$c = -i\frac{3\ell_{\rm dS}}{2G} + 13 + \mathcal{O}(G)$$

$$\Longleftrightarrow \quad \ell_{\rm dS} \to \ell_{\rm dS} + i\epsilon$$

Mini-superspace model for dS

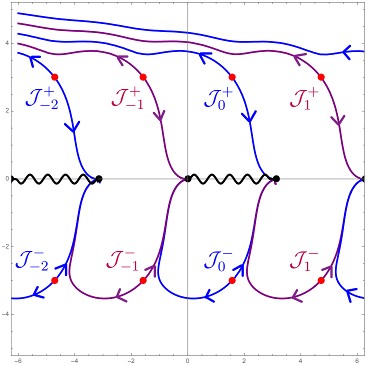
- We consider a contour $\mathcal{C}$ that surrounds two branch points $N = 0, \pi$ and goes into the large imaginary region.
- Such a contour can be deformed to

$$\mathcal{C} = -\mathcal{J}_{-1}^{+} + \mathcal{J}_{0}^{-} + \mathcal{J}_{0}^{+}$$

- We then obtain in $a_1 \gg 1$

$$\Psi_{\mathrm{dS}} \sim \left(e^{\frac{\pi \ell_{\mathrm{dS}}}{4G}} - e^{-\frac{\pi \ell_{\mathrm{dS}}}{4G}} \right) (2a_1)^{i \frac{\ell_{\mathrm{dS}}}{2G}}$$

This reproduces the CFT result!

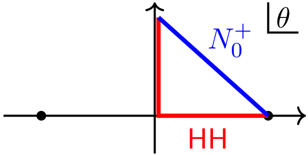


Mini-superspace model for dS

- Let us compare the saddles with HH saddle and Vilenkin's saddle.
- The metric for a saddle $N = N_n^{\pm}$ is

$$ds^2 = (N_n^{\pm})^2 d\tau^2 + \sin^2(N_n^{\pm}\tau) d\Omega_2^2 \,. \quad (23)$$

- For example, the metric for N_0^+ is represented as the blue line ($\theta = N_0^+\tau$).
- N_0^+ saddle and HH saddle can be regarded as equivalent due to the Cauchy's theorem for $a(\tau)$-integral.



- Therefore we have obtained the consistent result with the Liouville calculation.

Mini-superspace model for AdS

- The Einstein-Hilbert action

$$I = -\frac{1}{16\pi G}\int d^3x\left(R + \frac{2}{\ell_{\mathrm{AdS}}^2}\right) - I_{\mathrm{GH}} - I_{\mathrm{ct}}\,. \tag{24}$$

- We consider a class of gemetries with the metric ansatz

$$ds^2 = \ell_{\mathrm{AdS}}^2\left(N(r)^2dr^2 + a(r)^2d\Omega_2^2\right)\,. \tag{25}$$

  and two boundaries at $r = 0, 1$. We impose the Dirichlet boundary conditions $a(0) = 0$ and $a(1) = a_1$.

- We can fix gauge as $N' = 0$, so that

$$\mathcal{Z}_{\mathrm{AdS}} = \int_{\mathcal{C}} dN \int \mathcal{D}a\, e^{-I[a;N]-I_{\mathrm{ct}}}\,, \tag{26}$$

$$I[a;N] = -\frac{\ell_{\mathrm{AdS}}}{2G}\int_0^1 dr N\left[\frac{1}{N^2}a'^2 + a^2 + 1\right] \tag{27}$$

Mini-superspace model for AdS

- The EOM for $a(r)$ is $a'' - N^2 a = 0$, then the solutions are

$$a^{(N)}(r) = \frac{a_1}{\sinh N} \sinh(Nr) \,. \tag{28}$$

- Substituting the solution and including also the one-loop corrections,

$$\mathcal{Z}_{\text{AdS}} = \int_{\mathcal{C}} dN \left(\frac{1}{\sqrt{N} \sinh N} \right)^{\frac{1}{2}} e^{\frac{\ell_{\text{AdS}}}{2G}\left(N + a_1^2 \coth N\right) - I_{\text{ct}}} \tag{29}$$

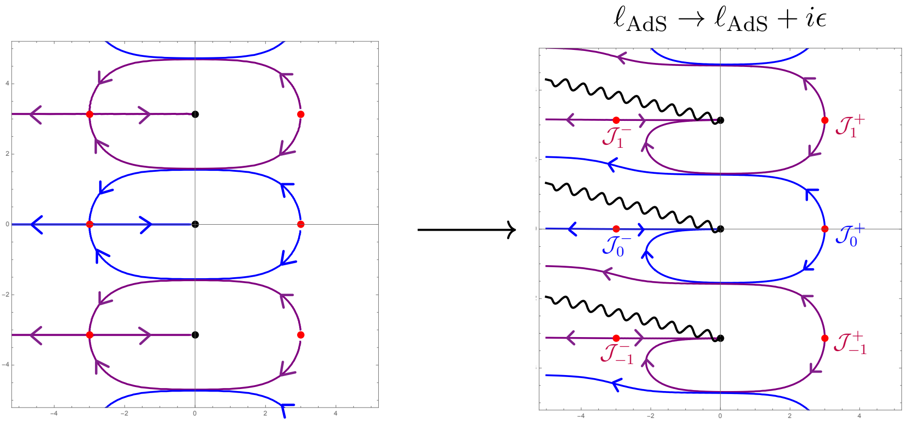
- The saddle points are

$$N_n^{\pm} = n\pi i \pm \ln\left(a_1 + \sqrt{a_1^2 - 1}\right) \qquad n \in \mathbb{Z} \,. \tag{30}$$

- The contribution from each saddle $\mathcal{J}_n^{\pm}$ is

$$\mathcal{Z}_n \sim e^{\frac{n\pi i \ell_{\text{AdS}}}{2G}} (2a_1)^{\pm\frac{\ell_{\text{AdS}}}{2G}} \,. \tag{31}$$

Mini-superspace model for AdS



- Again our model is "critical," but the final result does not depend on the choice of $\ell_{\mathrm{AdS}} \to \ell_{\mathrm{AdS}} \pm i\epsilon$, so we choose $+$.

Mini-superspace model for AdS

- We consider a natural contour $\mathcal{C} = \mathbb{R}_+$
- Such a contour can be deformed to

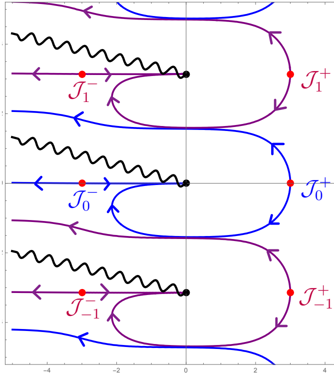
$$\mathcal{C} = \sum_{n=0}^{\infty} \mathcal{J}_n^+ + \sum_{n=1}^{\infty} \mathcal{J}_n^-$$

- We then obtain in $a_1 \gg 1$

$$\mathcal{Z}_{\mathrm{AdS}} \sim e^{-\frac{i\pi\ell_{\mathrm{AdS}}}{4G}} \sum_{n=0}^{\infty} e^{\frac{i(2n+1)\pi\ell_{\mathrm{AdS}}}{4G}} (2a_1)^{\frac{\ell_{\mathrm{AdS}}}{2G}}$$

Up to the overall factor, this reproduces the CFT result!

- The label n for $\mathcal{J}_n^+$ can be regarded as the winding number.

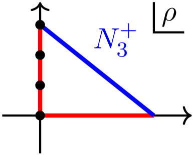


Mini-superspace model for AdS

- Our proposal was that the dual geometries are realized as 3d disk with an imaginary sphere attached.
- We can check this proposal in the similar discussion to dS case.
- The metric for a saddle $N = N_n^{\pm}$ is

$$ds^2 = (N_n^{\pm})^2 dr^2 + \sinh^2(N_n^{\pm} r) d\Omega_2^2 \,. \quad (32)$$

- For example, the metric for N_3^+ is represented as the blue line ($\rho = N_3^+ r$).
- The two saddles are equivalent due to the Cauchy's theorem.



- Thus we have checked the gravity saddles correspond to what we constructed in the previous section.

Summary and future problems

Summary

- We determined the bulk semi-classical saddles from CFT calculations.
- Under the analytic continuation $c \to -ic^{(g)}$, Stokes phenomenon occurs and the relevant saddles drastically change.
- We propose that the dual geometries are 3d disks with a imaginary radius sphere attached.
- We checked the proposal by the mini-superspace approach.

Future problems

- Other boundaries than S^2 (e.g. torus)
- Higher dimensions
- Relation of "allowable metric" [Witten]

Backup slides

Chern-Simons gravity for AdS_3

- The Einstein gravity with negative cosmological constant can be formulated by $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Chern-Simons theory.

$$S = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\tilde{A}] \,, \quad S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)$$

with $k = \ell_{\mathrm{AdS}}/4G_N$. $A, \tilde{A}$ are $sl(2)$-valued one forms.

- The general solutions to EOM can be put into the forms

$$A = e^{-\rho L_0} a(x^+) e^{\rho L_0} dx^+ + L_0 d\rho \,, \quad \tilde{A} = e^{\rho L_0} \tilde{a}(x^-) e^{-\rho L_0} dx^- - L_0 d\rho$$

- The metric is reproduced as

$$g_{\mu\nu} = \frac{\ell_{\mathrm{AdS}}^2}{2} \, \mathrm{tr}(A_\mu - \tilde{A}_\mu)(A_\nu - \tilde{A}_\nu)$$

Chern-Simons gravity for dS_3

- The Einstein gravity with positive cosmological constant can be formulated by $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$ Chern-Simons theory.

$$S = S_{\mathrm{CS}}[A] - S_{\mathrm{CS}}[\bar{A}] , \quad S_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int \mathrm{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) ,$$

where $k = -i\ell_{\mathrm{dS}}/4G_N$ and $\bar{A}$ is the complex conjugate of A.

- The general solutions to EOM can be put into the forms

$$A = e^{-i(\theta+\pi/2)L_0} a(z) e^{i(\theta+\pi/2)L_0} dz + iL_0 d\theta ,$$
$$\bar{A} = e^{i(\theta+\pi/2)L_0} \bar{a}(\bar{z}) e^{-i(\theta+\pi/2)L_0} d\bar{z} - iL_0 d\theta$$

- The metric is reproduced as

$$g_{\mu\nu} = -\frac{\ell_{\mathrm{dS}}^2}{2} \mathrm{tr}(A_\mu - \bar{A}_\mu)(A_\nu - \bar{A}_\nu) \quad (33)$$

Classification of solutions

- First consider the configuration

$$a = L_1 + \frac{2\pi\mathcal{L}}{\kappa} L_{-1} \,, \quad \bar{a} = -L_{-1} - \frac{2\pi\mathcal{L}}{\kappa} L_1 \,, \tag{34}$$

which leads to

$$\ell^{-2} ds^2 = d\theta^2 - \frac{8\pi\mathcal{L}}{\kappa} \sin^2\theta dt^2 + \frac{8\pi\mathcal{L}}{\kappa} \cos^2\theta d\phi^2 \tag{35}$$

- This solution has the trivial holonomy $\mathcal{P}e^{\oint A} \sim \mathbf{1}$.
- Note that large gauge transformations are not symmetry of CS theory since we are considering $SL(2,\mathbb{C})$ and complex k. Therefore a large gauge transformation with n windings gives another metric

$$\ell^{-2} ds^2 = d\theta^2 + \frac{8\pi(2n+1)^2\mathcal{L}}{\kappa} \sin^2\theta dt_E^2 + \frac{8\pi\mathcal{L}}{\kappa} \cos^2\theta d\phi^2 \tag{36}$$

2-point function

- Consider 2-pt function with insertions of primary operators with

$$\alpha = \frac{\eta}{b}, \quad \eta = \frac{1 - \sqrt{1 - 8G_N E}}{2}. \tag{37}$$

- In our limit, the 2-pt functions of Liouville theory are given by

$$\begin{aligned}\langle V_\alpha(z_1) V_\alpha(z_2)\rangle &\sim \delta(0)|z_{12}|^{-\frac{4\eta(1-\eta)}{b^2}} \lambda^{\frac{1-2\eta}{b^2}} \left[\frac{\gamma(b^2)}{b^2}\right]^{\frac{1-2\eta}{b^2}} \gamma\left(\frac{2\eta - 1}{b^2}\right) \\ &\sim \left(e^{\frac{\pi c^{(g)}}{6}\sqrt{1-8G_N E}} - e^{-\frac{\pi c^{(g)}}{6}\sqrt{1-8G_N E}}\right) \times (\text{phase})\end{aligned}$$

- We have observed again that the Stokes phenomenon occurs and two saddles are relevent. $\rightarrow$ Another example of Witten's proposal!
- The dual geometry involves a conical defect with a deficit angle

$$2\pi(1 - \sqrt{1 - 8G_N E}) \tag{38}$$

3-point functions

- The 3-pt function of Liouville theory is well known as DOZZ formula:

$$C(\alpha_1, \alpha_2, \alpha_3) = \left[\lambda\gamma(b^2)b^{-2b^2}\right]^{(Q-\sum_i \alpha_i)/b}$$
$$\times \frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum_i \alpha_i - Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}.$$

- $b \to 0$ limit of $\Upsilon_b(x/b)$ is given by

$$\Upsilon_b\left(\frac{x}{b}\right) \sim \exp\left(\frac{1}{b^2}\left[-(x-1/2)^2\log b + F(x)\right]\right).$$

for $0 < x < 1$.

- If an argument of Υ_b is not in the range, we use recursion relations:

$$\Upsilon_b(x+b) = \gamma(bx)b^{1-2bx}\Upsilon_b(x),$$
$$\Upsilon_b\left(x+\frac{1}{b}\right) = \gamma\left(\frac{x}{b}\right)b^{\frac{2x}{b}-1}\Upsilon_b(x),$$

3-point functions

- We assume the regions of η_1, η_2, η_3 as

$$\begin{cases} \sum_i \eta_i < 1\,, \\ \eta_i + \eta_j - \eta_k > 0\,. \end{cases} \tag{39}$$

  In fact, these conditions correspond to the condition for the dual geometry with conical deficits to exist.

- For this region, $\Upsilon_b(\sum_i \alpha_i - Q)$ gives $\gamma\left((1 - \sum_i \eta_i)/b^2\right)$, then

$$C(\alpha_1, \alpha_2, \alpha_3) \sim \left(e^{-\pi i \frac{1-\sum_i \eta_i}{b^2}} - e^{\pi i \frac{1-\sum_i \eta_i}{b^2}}\right) \lambda^{(1-\sum_i \eta_i)/b^2} e^{\frac{1}{b^2}[\cdots]}$$

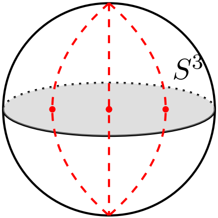
  so in $b \sim ic^{(g)}/6$,

$$|\langle V_{\alpha_1}(z_1, \bar{z}_1) V_{\alpha_2}(z_2, \bar{z}_2) V_{\alpha_3}(z_3, \bar{z}_3)\rangle|^2 \sim \exp\left[\frac{\pi c^{(g)}}{3}\left(1 - \sum_i \eta_i\right)\right].$$

3-point functions

- The dual geometry to $|\Psi|^2$ is given by 3-sphere with three conical defects with deficit angles $4\pi\eta_i$.
- We can derive the above conditions (39) from the existence of this geometry.
- Indeed, the saddle point approximation for this solution is

$$|\Psi|^2 \sim \exp\left[\frac{\pi c^{(g)}}{3}\left(1 - \sum_i \eta_i\right)\right] \tag{40}$$



4-point functions

- We can also consider 4-pt functions with $\alpha_i = \eta_i/b,\ i = 1, \ldots, 4.$
- From the condition for the existence of dual geometry, we can restrict the parameters as

$$\sum_i \eta_i < 1,$$

$$\eta_i + \eta_j + \eta_k - \eta_l > 0, \quad (i \neq j \neq k \neq l),$$

$$-1 < \eta_i + \eta_j - \eta_k - \eta_l < 1, \quad (i \neq j \neq k \neq l),$$

- In these regions of η_i, the conformal block decomposition becomes

$$\langle V_1(1) V_2(\infty) V_3(0) V_4(z,\bar{z}) \rangle$$
$$= i \sum_{\text{poles crossing } \mathbb{R}} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) \operatorname{Res} C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}^{12}_{34}(h_P|z) \mathcal{F}^{12}_{34}(h_P|\bar{z})$$
$$+ \frac{1}{2} \int_{-\infty}^{\infty} \frac{dP}{2\pi} C\left(\alpha_1, \alpha_2, \frac{Q}{2} - iP\right) C\left(\alpha_3, \alpha_4, \frac{Q}{2} + iP\right) \mathcal{F}^{12}_{34}(h_P|z) \mathcal{F}^{12}_{34}(h_P|\bar{z}) \,.$$

4-point functions

- We focus on the small $|z|$ limit, where

$$\mathcal{F}^{12}_{34}(h_P|z) \sim z^{h_P - h_1 - h_2}. \tag{41}$$

- Carefully analyzing the pole structure, we can obtain

$$\langle V_1(1)V_2(\infty)V_3(0)V_4(z,\bar{z})\rangle \sim \left(e^{-i\pi(1-\sum_i \eta_i)/b^2} - e^{i\pi(1-\sum_i \eta_i)/b^2}\right) \times (\text{phase}),$$

so in $b \sim ic^{(g)}/6$,

$$|\langle V_1(1)V_2(\infty)V_3(0)V_4(z,\bar{z})\rangle|^2 \sim \exp\left[\frac{\pi c^{(g)}}{3}\left(1 - \sum_i \eta_i\right)\right].$$

- In the same way as 3-pt functions, we can interpret in the gravity side as 3-sphere with four conical defects with deficit angles $4\pi\eta_i$.

Comments on higher-point functions

- We can extend the above calculation to higher-point functions.
- For 3-pt functions, we use the DOZZ formula

$$C(\alpha_1,\alpha_2,\alpha_3) = \left[\lambda\gamma(b^2)b^{-2b^2}\right]^{(Q-\sum_i \alpha_i)/b}$$
$$\times \frac{\Upsilon_b'(0)\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\sum_i \alpha_i - Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}.$$

- For 4-pt and higher-pt functions, we compute by using the conformal block decomposition.
- For all cases, we can construct the dual geometry with conical deficits.

Stirling's formula

- Here we review the derivation of Stirling's formula of Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t = z^z \int_{-\infty}^{\infty} e^{z(\phi - e^\phi)} \mathrm{d}\phi, \quad \mathrm{Re}\, z > 0. \tag{42}$$

  We can analytically continue $\Gamma(z)$ to $z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$.

- Stirling's formula at large $|z|$ is given by as follows:

$$\Gamma(z) \sim \begin{cases} e^{z \log z - z} & \mathrm{Re}\, z > 0 \\ \frac{1}{e^{i\pi z} - e^{-i\pi z}} e^{z \log(-z) - z} & \mathrm{Re}\, z < 0 \end{cases} \tag{43}$$

- A sketch of the derivation will be explained below, following [Harlow, Maltz, Witten].

A sketch of derivation

- Stirling's formula can be derived by saddle-point approximation.
- The procedure of saddle-point approximation for $\int e^{\mathcal{I}[\phi]} \mathrm{d}\phi$:
  1. Find the stationary points $\{\phi_n\}$ of $\mathcal{I}[\phi]$ in $\mathbb{C}$-valued ϕ, and the steepest descepnt $\mathcal{C}_n$ for each ϕ_n in the $\mathbb{C}$-plane. $\Rightarrow \int_{\mathcal{C}_n} \mathrm{d}\phi \, e^{\mathcal{I}[\phi]} = e^{\mathcal{I}[\phi_n]}$.
     (* A steepest descent $\mathcal{C}_n$ is defined as the gradient flow from ϕ_n)
  2. Deform the contour of integral (by Cauchy's theorem) and express it as sum of the steepest descents $\mathcal{C}_n$ of ϕ_n.
  3. Evaluate the integral as

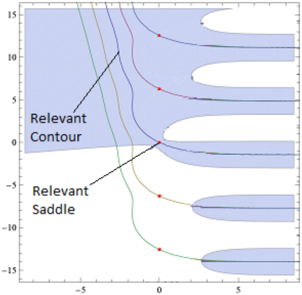
$$\sum_{\{C_n\}} e^{\mathcal{I}[\phi_n]} \tag{44}$$

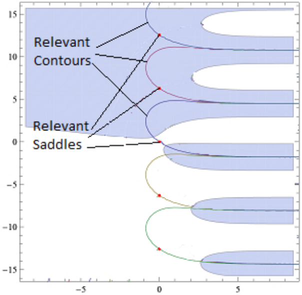
- The stationary points of $\mathcal{I} = z(\phi - e^{\phi})$ are classified as

$$\phi_n = 2\pi i n, \quad n \in \mathbb{Z} \tag{45}$$

A sketch of derivation

The steepest descents $\mathcal{C}_n$ for ϕ_n (cited from [Harlow, Maltz, Witten]):





$$\mathrm{Re}\, z > 0 \quad \xleftrightarrow{\text{Stokes phenomenon}} \quad \mathrm{Re}\, z < 0$$

$$\Gamma(z) \sim z^z e^{\mathcal{I}[\phi_0]} = e^{z \log z - z}, \quad \mathrm{Re}\, z > 0, \tag{46}$$

$$\Gamma(z) \sim z^z \sum_{n=0}^{\infty} e^{\mathcal{I}[\phi_n]} = \frac{1}{e^{i\pi z} - e^{-i\pi z}} e^{z \log(-z) - z}, \quad \mathrm{Re}\, z < 0 \tag{47}$$