Krylov complexity, double-scaled SYK, (and holography)

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Outline

A simple lecture on

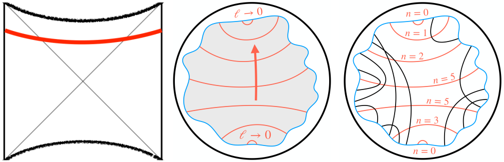
- Krylov complexity
- Double-scaled SYK with vacuum chords

Various complexities

The difficulty for preparing a target state/operator from applying operators on a reference state/operator.

Some notions of complexities

- Nielsen's complexity [Nielsen:'05]
- Computational complexity [Watrous:'08]
- The holographic complexity ? [Susskind:'14]



The Krylov complexity (KC):

- Independent from locality nor the definition of simple/hard operators.
- Could diagnose quantum chaos
  - Operator growth [Parker, Cao, Avdoshkin,et-al:'19][Jian,Swingle,ZYX:'21];
  - Spectrum statistics [Kar,et-al:'20][Rabinovici,et-al:'20][Balasubramanian,et-al:'22][Erdmenger,Jian,ZYX:'23].
- Chords states in DSSYK [Berkooz etal:'18][Lin:'22][Rabinovici:'23]...

Krylov basis

Measure the complexity of $|\Psi_t\rangle = e^{-it\mathcal{L}}|\Psi_0\rangle$ ($\mathcal{L}^\dagger = \mathcal{L}$) on the unit of $\mathcal{L}$. Krylov space

$$\mathcal{K} = \text{span}\left\{|\Psi_0\rangle, \mathcal{L}|\Psi_0\rangle, \mathcal{L}^2|\Psi_0\rangle, \cdots\right\} \xrightarrow[\text{orthogonalization}]{\text{Gram}-\text{Schmidt}} \{|0\rangle, |1\rangle, |2\rangle, \cdots, |L-1\rangle\}.$$

We want a basis

$$\langle n|m\rangle \propto \delta_{mn}$$

$$|n\rangle = \sum_{m=0}^{n} c_{nm}\mathcal{L}^m|\Psi_0\rangle$$

$$|O_n\rangle \equiv |n\rangle / \sqrt{\langle n|n\rangle}$$

$\Leftarrow$

Lanczos algorithm (monic version)

$$|0\rangle = |\Psi_0\rangle, \quad b_0 = 0,$$

$$|n\rangle = \mathcal{L}|n-1\rangle - b_{n-1}^2|n-2\rangle,$$

$$b_n^2 = \frac{\langle n|n\rangle}{\langle n-1|n-1\rangle},$$

$$b_n|O_n\rangle = \mathcal{L}|O_{n-1}\rangle - b_{n-1}|O_{n-2}\rangle.$$

Lanczos coefficients

$$\langle O_m|\mathcal{L}|O_n\rangle = \begin{pmatrix} 0 & b_1 & 0 & \cdots & 0 \\ b_1 & 0 & b_2 & \cdots & 0 \\ 0 & b_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1} \\ 0 & 0 & 0 & b_{L-1} & 0 \end{pmatrix}, \quad \frac{\langle m|\mathcal{L}|n\rangle}{\langle m|m\rangle} = \begin{pmatrix} 0 & b_1^2 & 0 & \cdots & 0 \\ 1 & 0 & b_2^2 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & b_{L-1}^2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Krylov complexity

Krylov wave-function

$$\phi_n(t) = i^n \langle O_n | \Psi_t \rangle$$

Discrete Schrödinger equation

$$\partial_t \phi_n = -b_{n+1}\phi_{n+1} + b_n \phi_{n-1}.$$

Krylov complexity

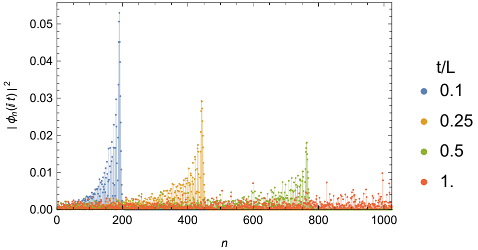
$$\hat{n} = \sum_{n=0}^{L-1} |O_n\rangle \, n \, \langle O_n| \,, \quad n(t) = \sum_{n=0}^{L-1} n \, |\phi_n(t)|^2$$

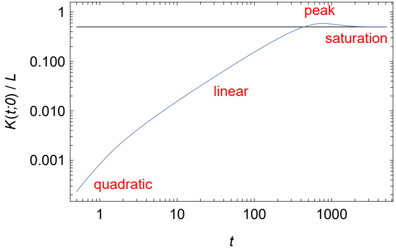
Rewrite

$$\begin{aligned} \mathcal{L} &= b_{\hat{n}} \hat{\alpha}^\dagger + \hat{\alpha} b_{\hat{n}}, \quad \hat{\alpha}^\dagger |O_n\rangle = |O_{n+1}\rangle \\ &= \bar{\alpha}^\dagger + \bar{\alpha} b_{\hat{n}}^2, \quad \bar{\alpha}^\dagger |n\rangle = |n+1\rangle \end{aligned}$$

Ehrenfest theorem

$$\partial_t^2 \langle \hat{n} \rangle = -\langle [[\hat{n}, \mathcal{L}], \mathcal{L}] \rangle = 2 \left\langle (b_{\hat{n}+1}^2 - b_{\hat{n}}^2) \right\rangle$$





$$\mathcal{L} = H_{\mathrm{GUE}} \otimes 1, \; |\Psi_0\rangle = |\mathrm{MES}\rangle \in \mathcal{H}_{LR}$$

Double-scaled SYK

For N Majorana fermions ψ_j ($\{\psi_i, \psi_j\} = 2\delta_{ij}$) with p-body random interaction

$$h = \sqrt{\lambda} H = i^{p/2} \sum_{1 \le j_1 < j_2 < \ldots < j_p \le N} J_{j_1 j_2 \ldots j_p} \psi_{j_1} \psi_{j_2} \ldots \psi_{j_p}, \quad \left\langle J^2_{j_1 j_2 \ldots j_p} \right\rangle = \mathcal{J}^2 \Big/ \binom{N}{p}, \; \mathcal{J} = 1$$

Double-scaled limit

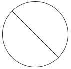
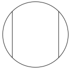
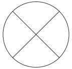
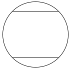
$$\lambda = \frac{2p^2}{N} = \text{fixed}, \quad N \to \infty$$

Moments of H with normalized trace $\mathrm{Tr} 1 = 1$ [Berkooz+:'18]

$$\langle \mathrm{Tr}[h^n] \rangle = i^{np/2} \sum_{I_1, I_2, \cdots, I_n} \langle J_{I_1} J_{I_2} \cdots J_{I_n} \rangle \, \mathrm{Tr}[\psi_{I_1} \psi_{I_2} \cdots \psi_{I_n}], \quad I = j_1 j_2 \ldots j_p, \; \psi_I = \psi_{j_1} \ldots \psi_{j_p}$$

$$\psi_{I_1} \psi_{I_2} = (-1)^{|I_1 \cap I_2|} \psi_{I_2} \psi_{I_1} \xrightarrow[\text{mean}=\lambda/2]{|I_1 \cap I_2| = \text{Poisson}} q \psi_{I_2} \psi_{I_1}, \quad q = e^{-\lambda}$$

Chord diagrams

$\mathrm{Tr}[h^2] =$  $= 1, \quad \mathrm{Tr}[h^4] =$  +  +  $= 1 + q + 1, \quad$ penalty $q < 1$

Chord states

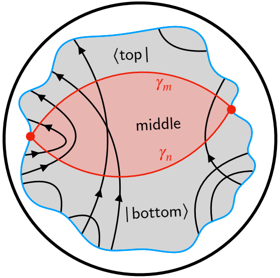
Define the maximally entangled state $|\text{MES}\rangle = \frac{1}{\sqrt{L}}\sum_p^L |E_p\rangle\,|E_p\rangle$ and $|O\rangle = O\otimes 1\,|\text{MES}\rangle$

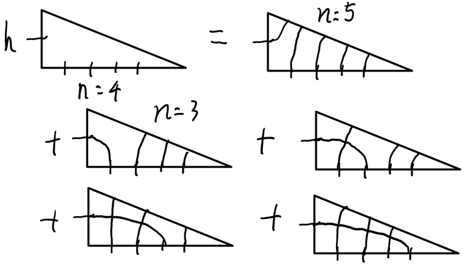
$$\text{Tr}[h^{20}] = \langle h^7|h^{13}\rangle = \sum_{mn}\langle h^7|m^*\rangle\,\langle m|n\rangle\,\langle n^*|h^{13}\rangle\,,\quad \langle n^*|m\rangle = \delta_{mn},\quad |n^*\rangle = |n\rangle\,/\,\langle n|n\rangle$$

From chord diagrams, $\langle n^*|h^m\rangle = 0$ if $n > m$ and $\langle m|n\rangle \propto \delta_{mn}$.
Vacuum chord states = Krylov basis with $|\Psi_0\rangle = |\text{MES}\rangle\,,\ \mathcal{L} = h\otimes 1$.
(Ensemble average before Gram-Schmidt orthogonalization $\{\langle\text{Tr}[h^n]\rangle\} \Rightarrow \{b_n\}$)

$$\mathcal{L}\,|n\rangle = |n+1\rangle + b_n^2\,|n-1\rangle\,,\quad b_n^2 = \frac{1-q^n}{1-q},\quad \hat{n} = \sum_n |n\rangle\,n\,\langle n^*| = \frac{1}{2p}\sum_j\left(1 + i\psi_j^L\psi_j^R\right)$$

$$|n\rangle = (1-q)^{-n/2}\,H_n\left(\sqrt{1-q}\mathcal{L}/2\middle|\,q\right)|0\rangle\,,\quad H_n(x|q) = \text{q-Hermtie Poly.},\quad E = 2\cos\theta/\sqrt{1-q}$$





[Lin:'22]

Liouville quantum mechanic

$$\mathcal{L} = \frac{1}{\sqrt{1-q}} \left(e^{i\lambda k} \sqrt{1-e^{-l}} + \sqrt{1-e^{-l}} e^{-i\lambda k} \right), \quad l = \lambda \hat{n}, \quad k = -i\partial_l$$

Triple-scaling limit

$$\lambda \to 0, \quad l \to \infty, \quad e^{-l}/\lambda^2 = e^{-\tilde{l}} = \text{fixed}$$

Liouville quantum mechanic

[Bagrets,Altland,Kamenev:'16][Harlow,Jafferis:'18]

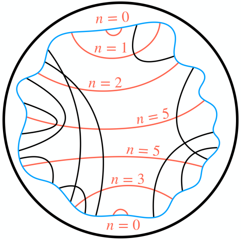
$$\tilde{H} = -\frac{1}{\sqrt{\lambda}}\mathcal{L} + \frac{2}{\lambda} = \lambda \left(k^2 + e^{-\tilde{l}} \right).$$

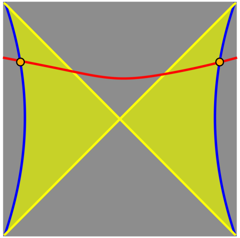
Liouville equation for $e^{-i\tilde{H}t}$ [Rabinovici et-al:'23]

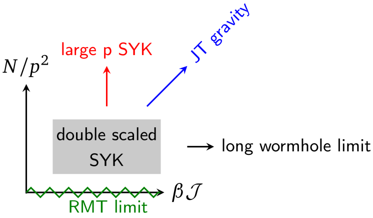
$$\partial_t^2 \tilde{l} = 2\lambda^2 e^{-\tilde{l}} = 2e^{-l} \xrightarrow[E=\lambda e^{-\tilde{l}(0)}]{\tilde{l}'(0)=0} \tilde{l} = 2\log\left(\sqrt{\frac{\lambda}{E}} \cosh \sqrt{\lambda E} t \right)$$

Ehrenfest theorem of Krylov complexity

$$\lambda \partial_t^2 \langle \hat{n} \rangle = 2 \left\langle (b_{\hat{n}+1}^2 - b_{\hat{n}}^2) \right\rangle = 2 \left\langle e^{-\lambda \hat{n}} \right\rangle \geq 2 e^{-\lambda \langle \hat{n} \rangle}$$







[Lin, Stanford:'23]

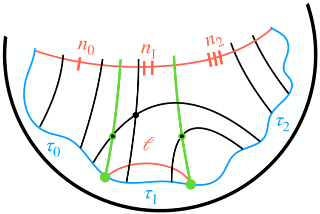
Recall

$$\mathcal{L} = b_{\hat{n}}\hat{\alpha}^\dagger + \hat{\alpha} b_{\hat{n}} = \bar{\alpha}^\dagger + \bar{\alpha} b_{\hat{n}}^2 = a^\dagger + a,$$

Chord algebra

$$aa^\dagger - qa^\dagger a = 1, \quad [a, a^\dagger] = q^{\hat{n}}, \quad [\hat{n}, a^\dagger] = a^\dagger, \quad [\hat{n}, a] = -a$$

Matter chords

$$|n_0, n_1, n_2\rangle =$$ 

- multi-point function [Lin,Stanford:'23]
- quantum group $SL_q(2)$ and generator algebra $U_q(SL(2))$
- The holographic dual of DSSYK at finite λ is unknown. Candidate: quantum disk [Almheiri,Popov:'24]