Exact WKB analysis for PT symmetric quantum mechanics: Study of the Ai-Bender-Sarkar conjecture

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  - PT symmetric QM
  - The Ai-Bender-Sakar (ABS) conjecture
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  - Borel resummation ☆ **All the essences are here!!**
  - Exact WKB analysis (EWKB) ⊏ **Borel resummation theory**
- Application of EWKB to a negative coupling potential
  - Massive case (ω>0) ☆ **Today's goal**
  - Massless case (ω=0) **(If we have time)**

Model and our claim

- PT symmetric QM
- The Ai-Bender-Sakar (ABS) conjecture

PT symmetric QM

[C. M. Bender et al. 1999]

PT symmetric QM (Non Hermitian)

$$V_{\mathcal{PT}}(x) = \omega^2 x^2 + g x^2 (ix)^{\varepsilon}, \qquad (\omega \in \mathbb{R}_{\geq 0},\ g, \varepsilon \in \mathbb{R}_{>0},\ x \in \mathbb{C})$$

PT transform $\quad \mathcal{P}:\ x \to -x, \qquad \mathcal{T}:\ x \to \bar{x},\ i \to -i,$

* The domain of x is deformed from the real axis to satisfy the PT invariance.

Real and bounded energy spectrum for positive g and epsilon

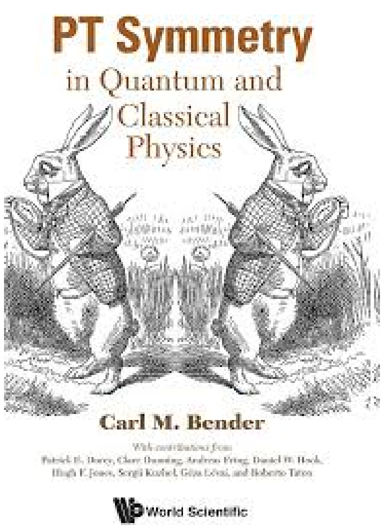

Table 2.1 Numerical values of the first five eigenvalues, E_0, E_1, E_2, E_3, and E_4, of the Hamiltonian $\hat{H} = \hat{p}^2 + \hat{x}^2(i\hat{x})^{\varepsilon}$ for various values of ε.

	E_0	E_1	E_2	E_3	E_4
ε=0	1	3	5	7	9
ε=1/2	1.048956	3.434539	6.051737	8.791012	11.620695
ε=1	1.156267	4.109229	7.562274	11.314422	15.291554
ε=3/2	1.301514	4.969791	9.480030	14.530476	19.997745
ε=2	1.477150	6.003386	11.802434	18.458819	25.791792
ε=5/2	1.679907	7.208428	14.540831	23.134243	32.741996
ε=3	1.908265	8.587221	17.710809	28.595103	40.918891
ε=7/2	2.161511	10.143518	21.328941	34.879469	50.390825
ε=4	2.439346	11.881565	25.411553	42.023722	61.222419

[C. M. Bender et al. 1998,
P. Dorey et al. 2001,
H. F. Jones et al. 2006,
Y. Emery et al. 2020,
etc ...]

The ABS conjecture

[W.-Y. Ai et. al. 2022]

PT symmetric theory vs. Analytic continuation of the Hermitian theory

Conjectured relation with a negative coupling theory ($\varepsilon = 2$)
(motivated by formulating non-Hermitian field theory, e.g. Beyond the SM)

$$V_{\mathcal{H}}(x) = \omega^2 x^2 + \lambda x^4, \qquad (\omega \in \mathbb{R}_{\geq 0}, \lambda \in \mathbb{R})$$

$$\lambda \to \lambda = e^{\pm \pi i} g \quad \text{with} \quad g \in \mathbb{R}_{>0}$$

$$V_{\mathrm{AC}}(x) = \omega^2 x^2 - g x^4, \qquad (\omega \in \mathbb{R}_{\geq 0},\ g, \varepsilon \in \mathbb{R}_{>0},\ x \in \mathbb{C})$$

PT potential form = AC potential form
Domain of x(PT) ≠ Domain of x(AC)

The ABS conjecture

[W.-Y. Ai et. al. 2022]

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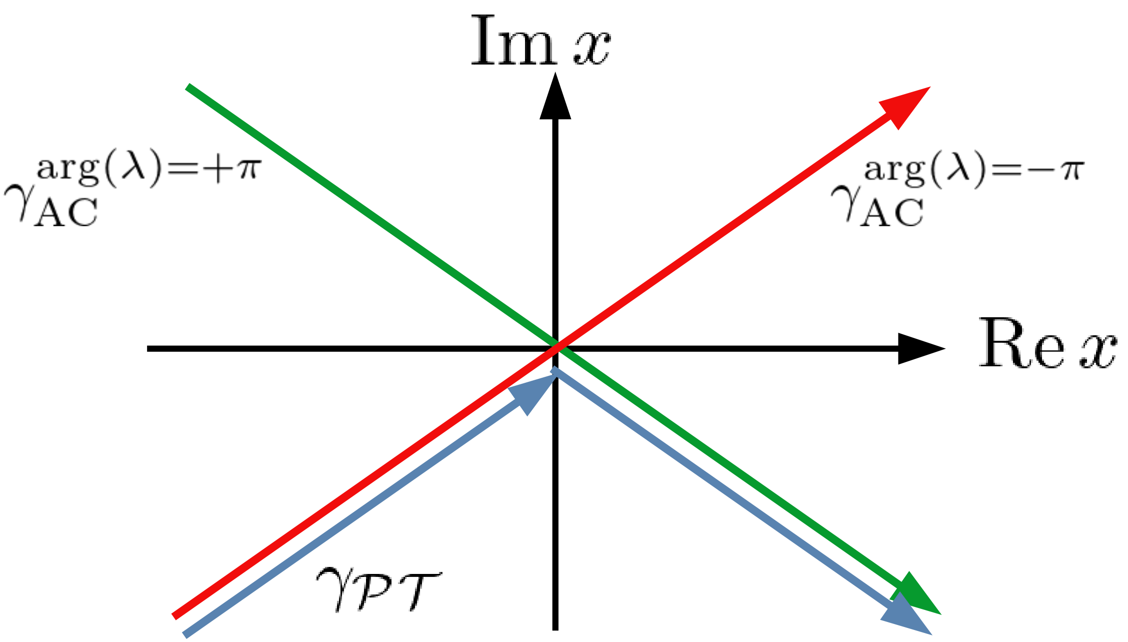
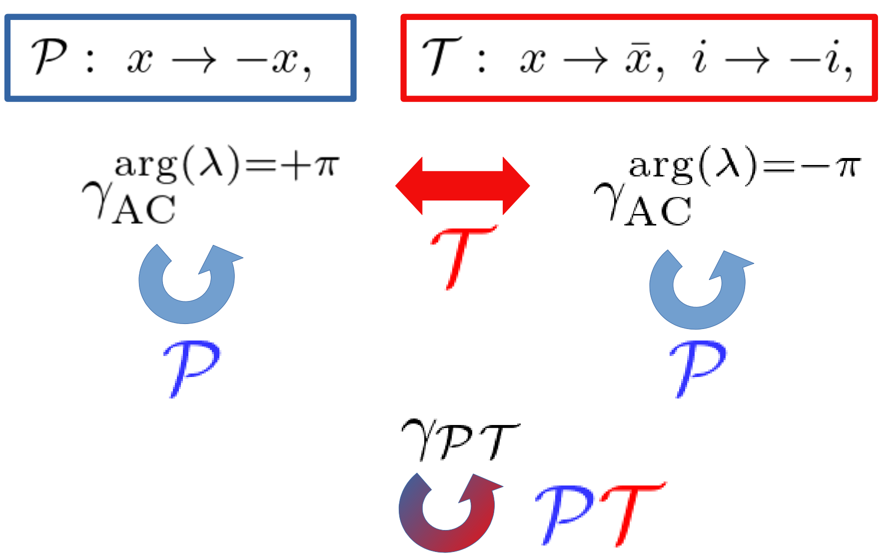




The ABS conjecture

[W.-Y. Ai et. al. 2022]

PT symmetric theory vs. Analytic continuation of the Hermitian theory

Conjectured relation with a negative coupling theory ($\varepsilon = 2$)
(motivated by formulating non-Hermitian field theory, e.g. Beyond the SM)

For D=0, $$Z_{\mathcal{PT}}(g) = \mathrm{Re} Z_{\mathcal{H}}(\lambda = -g + i0_\pm), \qquad g \in \mathbb{R}_{>0}.$$

This correct.

The ABS conjecture

For D>0, $$\log Z_{\mathcal{PT}}(g) = \mathrm{Re} \log Z_{\mathcal{H}}(\lambda = -g + i0_\pm), \qquad g \in \mathbb{R}_{>0}.$$

- **W.-Y. Ai, C. M. Bender, and S. Sarkar [2022]**
  Based on a semiclassical analysis, Lefschetz thimble
- **S. Lawrence, R. Weller, C. Peterson, and P. Romatschke [2023]**
  Contradiction for ω=0 ... 0 dim toy model, QM., N-comp. scalar
- **S.K. [2024]**
  The ABS conjecture is not correct in the QM for all non-negative ω≧0.
  Provided an alternative form for ω>0.

Reformulation of the ABS conjecture

[S.K. 2024]

Exact WKB analysis …. Beyond-semiclassical analysis
(Including all pert. and nonpert. sectors)
A sort of Borel resummation theroy

Claim for QM (D = 1)

The situation is different between ω>0 and ω=0:

- **If ω>0, then the ABS conjecture is violated when exceeding a semiclassical level of the 1st NP order, i.e. from the 2nd NP sectors. However, an alternative form can be formulated by Borel resummation theory.**

- **If ω=0, then the ABS conjecture is completely violated. No alternative form can be constructed by Borel resummation theory.**

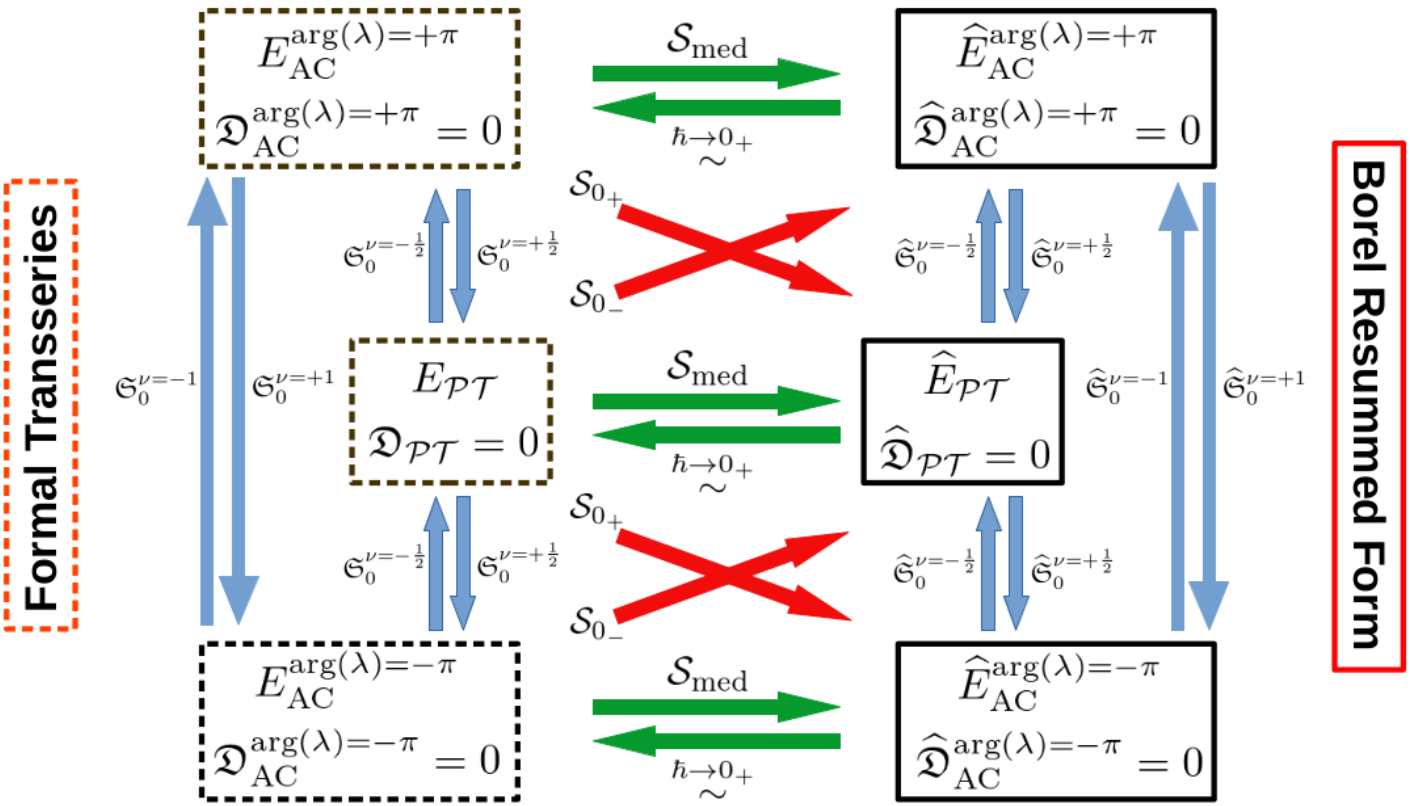
We investigate the energy spectrum.
The partition function holds the same result.

$$Z = \sum_k \exp\left[-\beta E_k\right]$$

Reformulation of the ABS conjecture

[S.K. 2024]

Modified ABS conjecture for ω>0



$\mathcal{S}_{0_\pm}$ … Borel resummation

$\mathcal{S}_{\mathrm{med}} = \mathcal{S}_{0_\pm} \circ \mathfrak{S}_0^{\mp 1/2}$ … Median resummation

$\mathfrak{S}_0^{\nu\in\mathbb{R}}$ … One-parameter Stokes automorphism

$\widehat{\mathfrak{S}}_0^{\nu\in\mathbb{R}}$ … Pushout of $\mathfrak{S}_0^{\nu\in\mathbb{R}}$

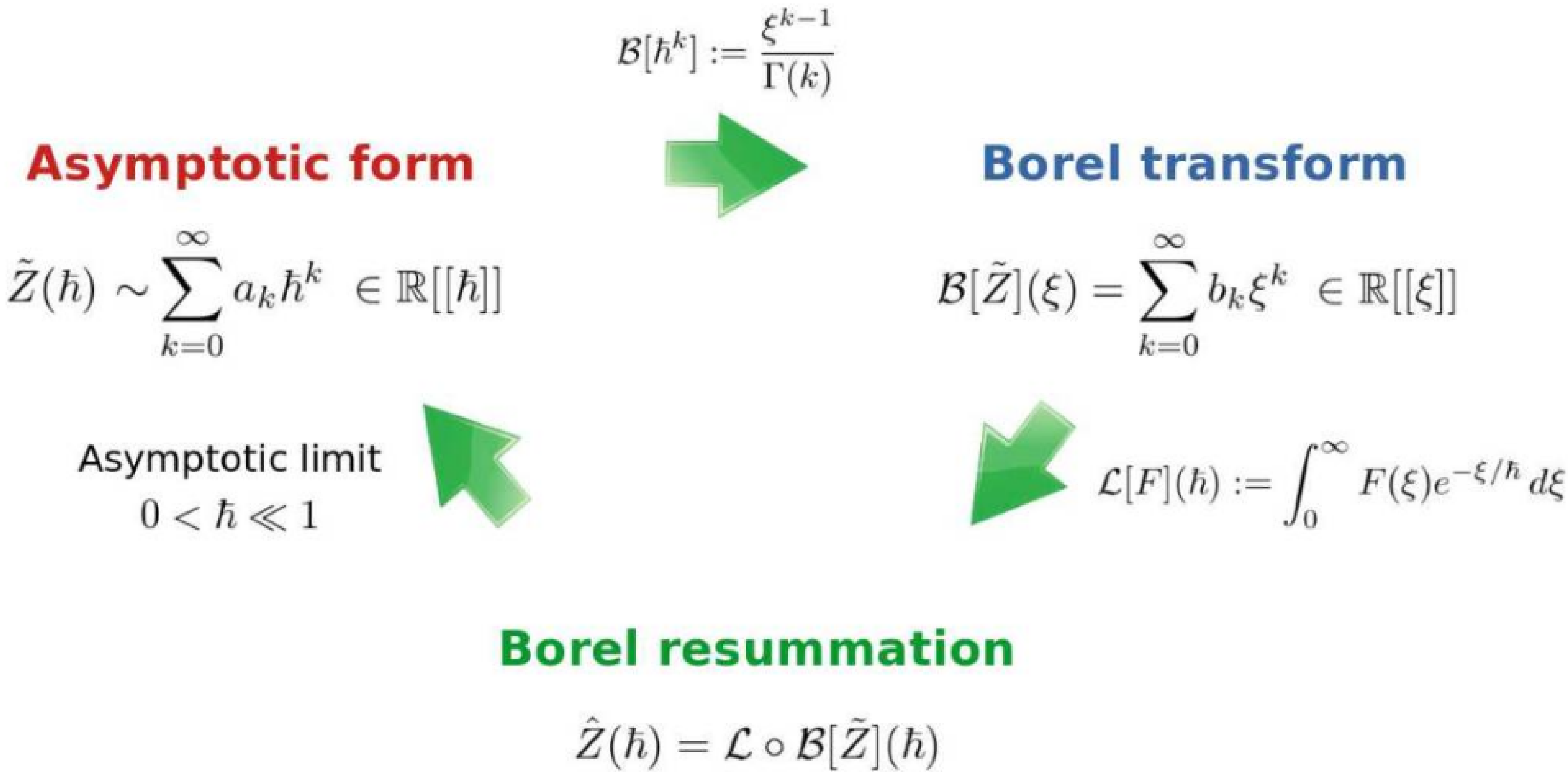
Mathematical tools

- Borel resummation
- Exact WKB analysis (EWKB)

- **Borel resummation**

[J. Ecalle, D. Sauzin, O. Costin, E. Delabaere…]

Reconstruction of an analytic function from a formal power series



- **Borel resummation**

Notation:

$\widehat{f}(\hbar)$ ... analytic function

$f(\hbar)$ ... asymptotic expansion

$$f(\hbar) \sim \sum_{n\in\mathbb{N}} c_n \hbar^n \quad \text{as} \quad \hbar \to 0_+. \qquad (c_n \in \mathbb{C})$$

$$c_n \sim A S^{-n} n! \quad \text{as} \quad n \to \infty, \qquad \lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = 0. \qquad (A, S \in \mathbb{C})$$

Gevrey-1 class

Borel resummation $\mathcal{S}_\theta := \mathcal{L}_\theta \circ \mathcal{B}$

Borel transform

$$\mathcal{B}[f](\xi) := \sum_{n\in\mathbb{N}} \frac{c_n}{\Gamma(n)} \xi^{n-1} = \underline{f_B(\xi)},$$

Analytic function (at least. formally)

Laplace integral

$$\mathcal{L}_\theta[f_B](\hbar) := \int_0^{\infty e^{i\theta}} d\xi \, e^{-\frac{\xi}{\hbar}} f_B(\xi).$$

$$\underline{\mathcal{S}_0[f](\hbar)} \to f(\hbar) \quad \text{as} \quad \hbar \to 0_+$$ **Always True??**

Analytic function (at least, formally)

- **Borel summability**

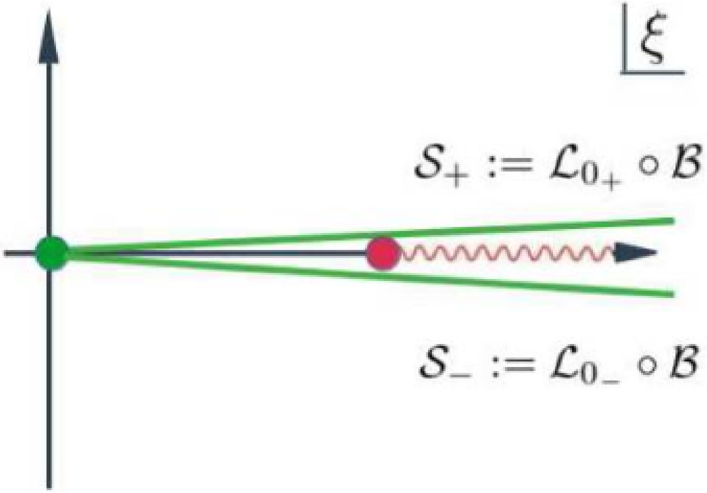
Notation:

$\widehat{f}(\hbar)$ ... analytic function

$f(\hbar)$ ... asymptotic expansion

When **Borel non-summable**,
we can avoid the singularities by introducing a complex phase to the integration ray:

$$\mathcal{L} \rightarrow \mathcal{L}_\theta := \int_0^{+\infty e^{i\theta}} d\xi e^{-\xi/\hbar}$$



However, since the singularities give discontinuity between $\mathcal{S}_+$ and $\mathcal{S}_-$, one obtains a result as

$$\mathcal{S}_+[f](\hbar) \neq \mathcal{S}_-[f](\hbar) \neq \widehat{f}(\hbar).$$

Question :

How to obtain $\widehat{f}(\hbar)$ via $\mathcal{S}_\pm$ when $f(\hbar)$ is Borel non-summable?

※ This is a key point to get a resurgent relation.

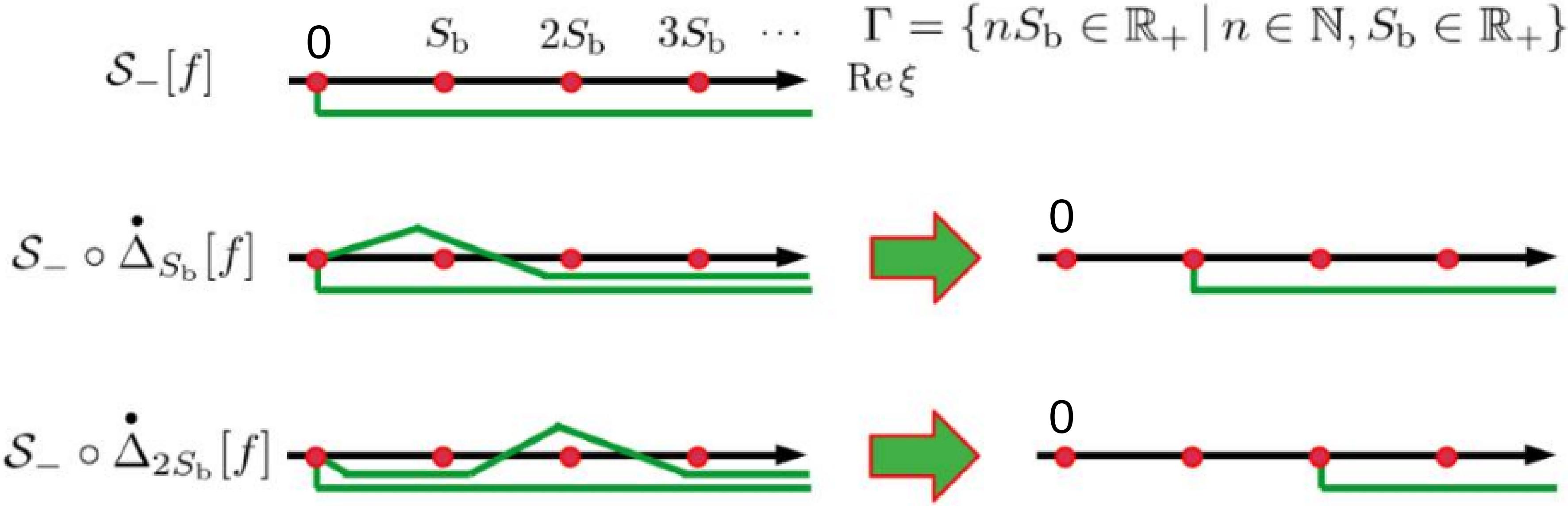
- **Alien derivative**

$$\mathcal{S}_{\theta+0_+} = \mathcal{S}_{\theta+0_-} \circ \mathfrak{S}_\theta.$$

Stokes automorphism

Alien derivative $\dot{\Delta}_w$: Transseries $\to$ Transseries

$\Gamma = \{ n S_{\rm b} \in \mathbb{R}_+ \,|\, n \in \mathbb{N}, S_{\rm b} \in \mathbb{R}_+ \}$



Collection of all of $\dot{\Delta}_{w\in\Gamma}$ $\Rightarrow$ Stokes automorphism $\mathfrak{S} = \exp\left[\sum_{w\in\Gamma} \dot{\Delta}_w\right]$

*** In order to obtain, e.g., the 2nd sector from the 0th sector,**

$$\Big[\underbrace{\dot{\Delta}_{2S_{\rm b}}}_{(1)} + \frac{1}{2}\underbrace{(\dot{\Delta}_{S_{\rm b}})^2}_{(2)}\Big][f]$$

(1) 0th $\Rightarrow$ 2nd
(2) 0th $\Rightarrow$ 1st $\Rightarrow$ 2nd

- **One-parameter Stokes automorphism**

Stokes automorphism can be extended to a one-parameter group

$$\mathfrak{S}_\theta^\nu = \exp\left[\nu \dot{\Delta}_\theta\right], \qquad \dot{\Delta}_\theta = \sum_{w \in \Gamma(\theta)} \dot{\Delta}_w, \qquad (\nu \in \mathbb{R})$$

A set of singular points along θ
(We normally take θ=0.)

Complex conjugation $\mathcal{C}$

$$\mathcal{C} \circ \mathcal{S}_{0_+} = \mathcal{S}_{0_-} \circ \mathcal{C}, \quad \mathcal{C} \circ \mathfrak{S}_0^\nu = \mathfrak{S}_0^{-\nu} \circ \mathcal{C}, \quad \mathcal{C} \circ \dot{\Delta}_0 = -\dot{\Delta}_0 \circ \mathcal{C}$$

Median resummation

$$\mathcal{S}_{\mathrm{med},0} := \mathcal{S}_{0_+} \circ \mathfrak{S}_0^{\nu=-1/2} = \mathcal{S}_{0_-} \circ \mathfrak{S}_0^{\nu=+1/2}$$

$$\mathcal{C} \circ \mathcal{S}_{\mathrm{med},0} = \mathcal{S}_{\mathrm{med},0} \circ \mathcal{C}$$

No dependence on the discontinuity!!

$$\mathcal{S}_0[f](\hbar) \to f(\hbar) \quad \text{as} \quad \hbar \to 0_+$$

Always True?? **No!!**

⇓

$$\mathcal{S}_{\mathrm{med},0}[f](\hbar) \to f(\hbar) \quad \text{as} \quad \hbar \to 0_+$$

Med resum gives "Yes"
(But generally depending on its underlying math structure)

- **Properties of the operations**

Borel resummation … homomorphism

$$\mathcal{S}_\theta[f_1 + f_2] = \mathcal{S}_\theta[f_1] + \mathcal{S}_\theta[f_2], \qquad \mathcal{S}_\theta[f_1 f_2] = \mathcal{S}_\theta[f_1] \cdot \mathcal{S}_\theta[f_2].$$

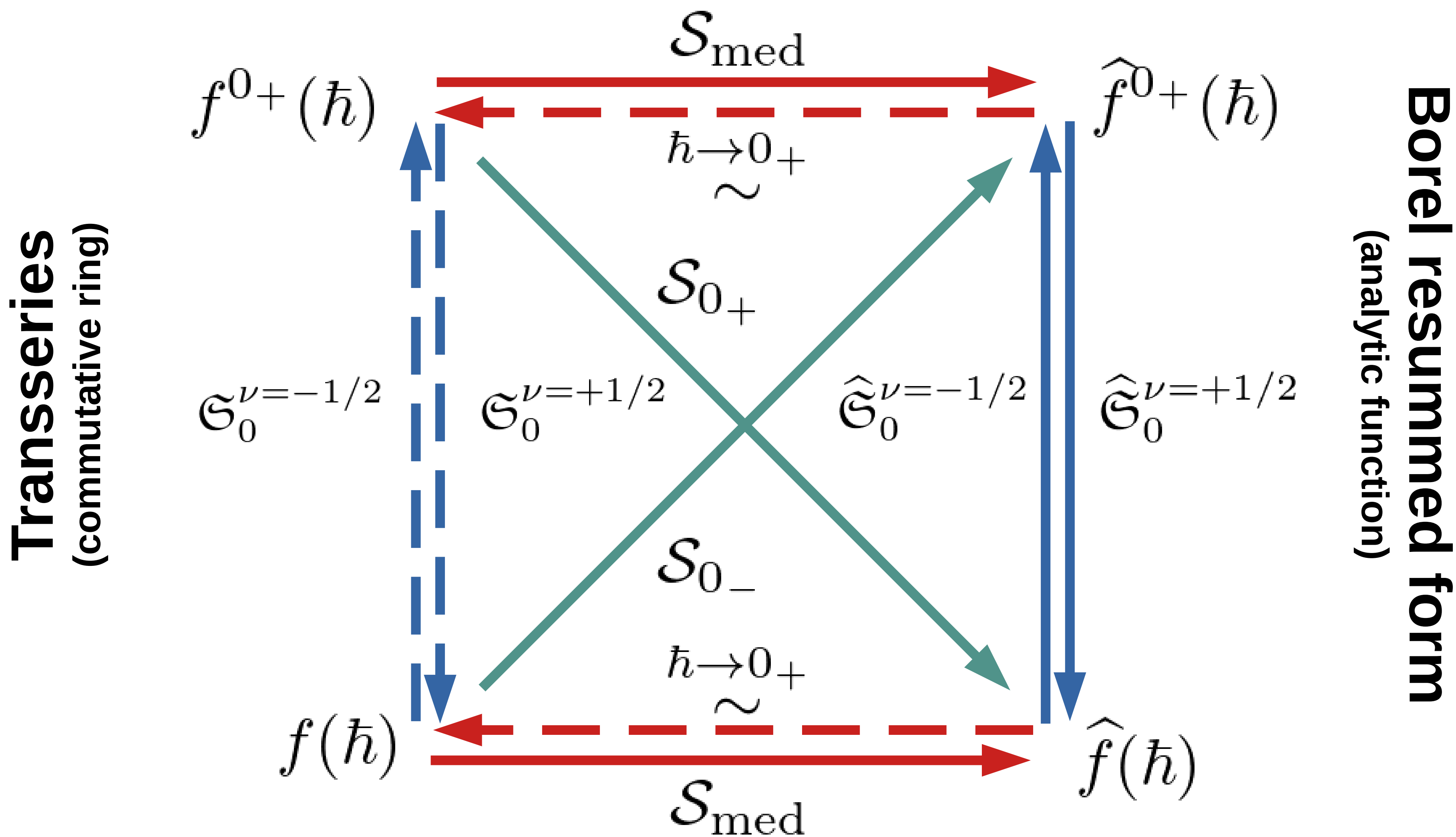
Stokes automorphism … homomorphism

$$\mathfrak{S}_\theta[f_1 + f_2] = \mathfrak{S}_\theta[f_1] + \mathfrak{S}_\theta[f_2], \qquad \mathfrak{S}_\theta[f_1 f_2] = \mathfrak{S}_\theta[f_1] \cdot \mathfrak{S}_\theta[f_2].$$

Alien derivative … additive and Leibniz rule

$$\dot{\Delta}_w[f_1 + f_2] = \dot{\Delta}_w[f_1] + \dot{\Delta}_w[f_2], \qquad \dot{\Delta}_w[f_1 f_2] = \dot{\Delta}_w[f_1] \cdot f_2 + f_1 \cdot \dot{\Delta}_w[f_2].$$

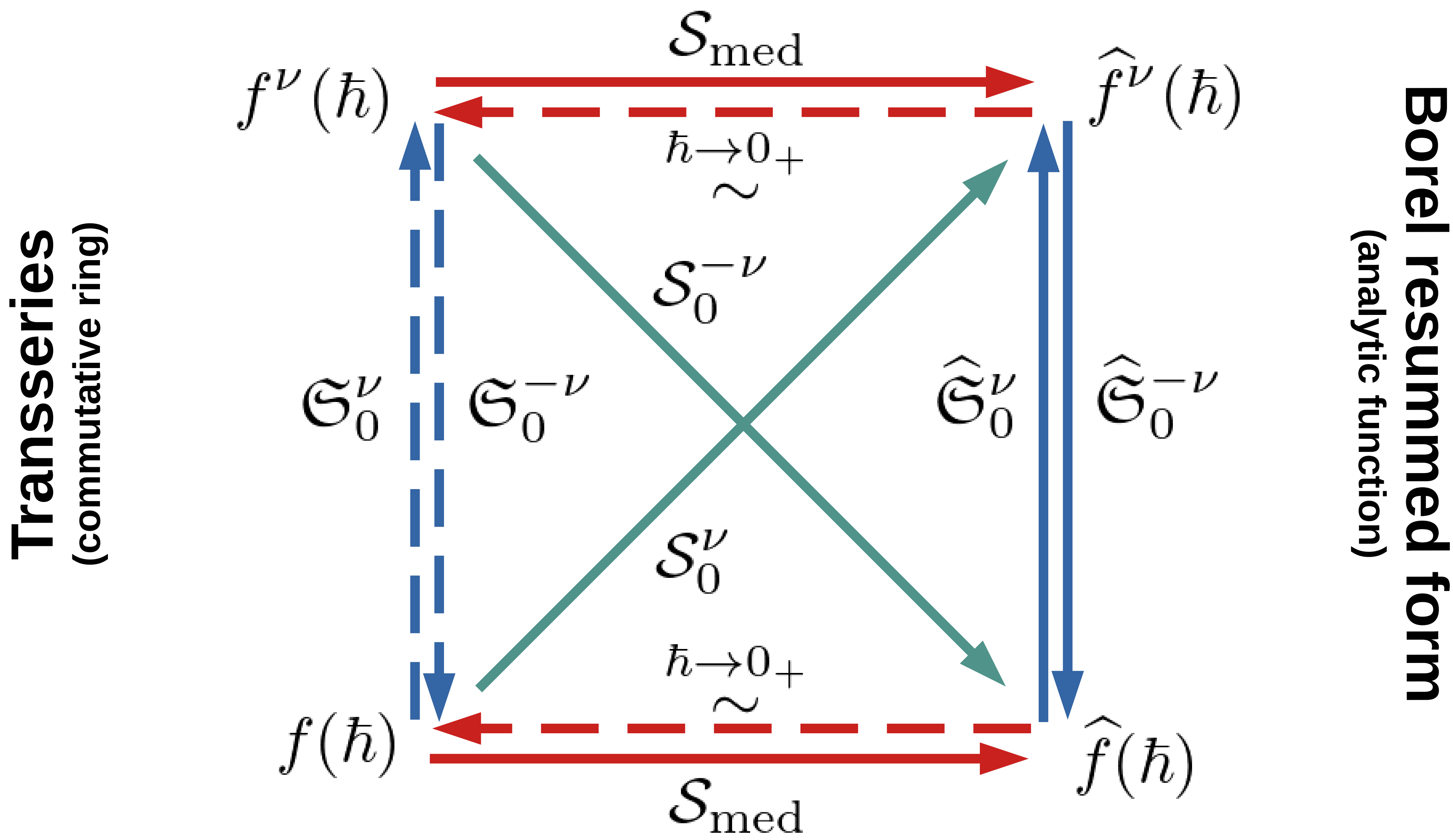
- **Borel resummation as morphism**



$$\mathcal{S}_{\mathrm{med},\theta} \circ \mathfrak{S}_\theta^\nu = \widehat{\mathfrak{S}}_\theta^\nu \circ \mathcal{S}_{\mathrm{med},\theta}.$$

• **Borel resummation as morphism**

$\mathcal{S}_{0_\pm}$ **is a special case**



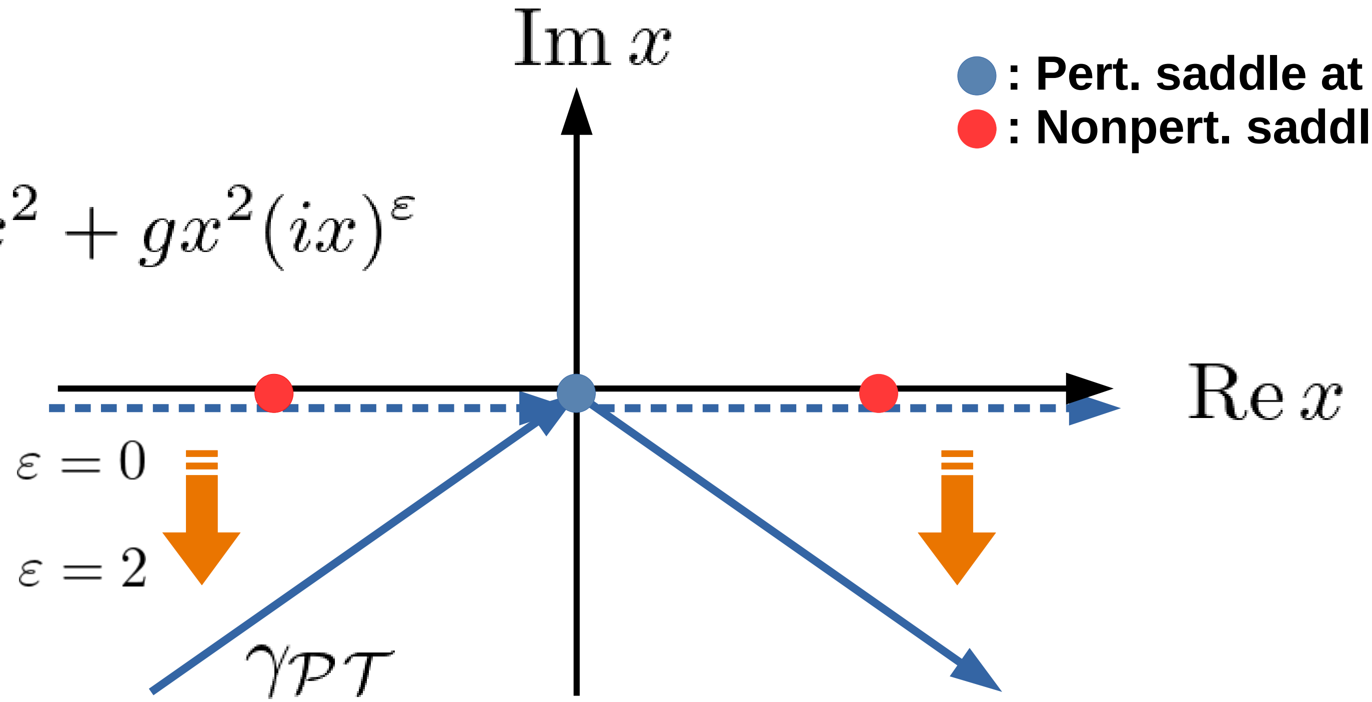
$$\mathfrak{S}_\theta^{\nu=0} = 1, \qquad \mathfrak{S}_\theta^{\nu_1} \circ \mathfrak{S}_\theta^{\nu_2} = \mathfrak{S}_\theta^{\nu_2} \circ \mathfrak{S}_\theta^{\nu_1} = \mathfrak{S}_\theta^{\nu_1+\nu_2}, \qquad (\nu_1, \nu_2 \in \mathbb{R})$$

- **Example 2: 0 dim PT symmetric model** [W.-Y. Ai et. al. 2022]

$$Z_{\mathcal{PT}} = \int_{\gamma_{\mathcal{PT}}} dx \, \exp\left[-x^2 + g x^4\right], \qquad g \in \mathbb{R}_{>0},$$

$$\gamma_{\mathcal{PT}} := s e^{+\frac{\pi}{4}i}\theta(-s) + s e^{-\frac{\pi}{4}i}\theta(+s), \qquad s \in \mathbb{R},$$

$$V(x) = x^2 + g x^2 (ix)^{\varepsilon}$$



: Pert. saddle at 0

: Nonpert. saddles at $\pm\frac{1}{\sqrt{2g}}$

Exact sol.

$$\widehat{Z}_{\mathcal{PT}} = \frac{\pi e^{-\frac{1}{8g}}}{4\sqrt{g}} \left[I_{-\frac{1}{4}}\left(\frac{1}{8g}\right) + I_{+\frac{1}{4}}\left(\frac{1}{8g}\right) \right],$$

- **Example 2: 0 dim PT symmetric model** [W.-Y. Ai et. al. 2022]

$$\widehat{Z}_{\mathcal{PT}} \sim Z_{\mathcal{PT}} = \sqrt{\pi} \sum_{n \in \mathbb{N}_0} \frac{\left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{n!} (4g)^n \quad \text{as} \quad g \to 0_+,$$

$(a)_n := \Gamma(a+n)/\Gamma(a)$ **: Pochhammer symbol**

Borel transform

$$\mathcal{J}_{0,B} := \mathcal{B}[\mathcal{J}_0] = \frac{2K\left(\frac{4\sqrt{\xi}}{2\sqrt{\xi}+1}\right)}{\sqrt{\pi}\sqrt{2\sqrt{\xi}+1}}, \qquad (\mathcal{J}_0 := g Z_{\mathcal{PT}})$$

$K(x)$ **: Elliptic integral**

A set of singular points along θ = 0 from K(1)

$$\Gamma(\theta = 0) = \left\{ \frac{1}{4} \right\}$$

Laplace integral

$$\mathcal{S}_{0_\pm}[\mathcal{J}_0] = \frac{\pi e^{-\frac{1}{8g}}\sqrt{g}}{4} \left[I_{-\frac{1}{4}}\left(\frac{1}{8g}\right) + I_{+\frac{1}{4}}\left(\frac{1}{8g}\right) \pm i\frac{\sqrt{2}}{\pi} K_{\frac{1}{4}}\left(\frac{1}{8g}\right) \right]$$

$$= \widehat{\mathcal{J}}_0 + \widehat{\mathcal{J}}_\pm,$$

$$\widehat{\mathcal{J}}_\pm := \pm i \frac{e^{-\frac{1}{8g}}\sqrt{2g}}{4} K_{\frac{1}{4}}\left(\frac{1}{8g}\right)$$

Modified Bessel function of the 2nd kind

- **Example 2: 0 dim PT symmetric model** [W.-Y. Ai et. al. 2022]

Alien derivative and Stokes automorphism

$$(\dot{\Delta}_{\frac{1}{4}})[\mathcal{J}_0] = \int_{+\infty+i0_-}^{+\infty+i0_+} d\xi\, e^{-\frac{\xi}{g}} \mathcal{J}_{0,B}(\xi) \sim \mathcal{J}_+ - \mathcal{J}_-, \qquad (\dot{\Delta}_{\frac{1}{4}})^{n>1}[\mathcal{J}_0] = 0,$$

$$(\dot{\Delta}_{\theta=0})^{n\in\mathbb{N}}[\mathcal{J}_\pm] = 0, \qquad (\mathcal{J}_+ + \mathcal{J}_- = 0 \quad \text{and} \quad \mathcal{J}_+ - \mathcal{J}_- = \pm 2\mathcal{J}_\pm.)$$

$$\mathfrak{S}_0^\nu[\mathcal{J}_0] = \mathcal{J}_0 + \nu(\mathcal{J}_+ - \mathcal{J}_-), \qquad \mathfrak{S}_0^\nu[\mathcal{J}_\pm] = \mathcal{J}_\pm,$$

$$\widehat{\mathfrak{S}}_0^\nu[\widehat{\mathcal{J}}_0] = \widehat{\mathcal{J}}_0 + \nu(\widehat{\mathcal{J}}_+ - \widehat{\mathcal{J}}_-), \qquad \widehat{\mathfrak{S}}_0^\nu[\widehat{\mathcal{J}}_\pm] = \widehat{\mathcal{J}}_\pm.$$

Median resummation

$$\mathcal{J}_0^{0\pm} = \mathfrak{S}_0^{\mp 1/2}[\mathcal{J}_0] = \mathcal{J}_0 \mp \frac{1}{2}(\mathcal{J}_+ - \mathcal{J}_-) = \mathcal{J}_0 - \mathcal{J}_\pm = \mathcal{J}_0 + \mathcal{J}_\mp,$$

$$\mathcal{S}_{0_\pm}[\mathcal{J}_0^{0\pm}] = \mathcal{S}_{\mathrm{med},0}[\mathcal{J}_0] = \widehat{\mathcal{J}}_0.$$

• **Example 2: 0 dim PT symmetric model** [W.-Y. Ai et. al. 2022]

$$\widehat{Z}_{\mathcal{H}} := \int_{-\infty}^{+\infty} dx \, \exp\left[-x^2 - \lambda x^4\right] = \frac{e^{\frac{1}{8\lambda}}}{2\sqrt{\lambda}} K_{\frac{1}{4}}\left(\frac{1}{8\lambda}\right), \qquad \lambda \in \mathbb{R}_{>0},$$

By taking $\lambda = e^{\pm\pi i} g$ with $g \in \mathbb{R}_{>0}$

$$\mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi} = \mathcal{J}_0 + \mathcal{J}_{\mp} = \mathcal{J}_0 \mp \frac{1}{2}(\mathcal{J}_+ - \mathcal{J}_-), \qquad (\mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi} := g Z_{\mathcal{H}}^{\arg(\lambda)=\pm\pi})$$

$$\widehat{\mathcal{J}}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi} = \frac{\pi e^{-\frac{1}{8g}}\sqrt{g}}{4}\left[I_{\frac{1}{4}}\left(\frac{1}{8g}\right) + I_{-\frac{1}{4}}\left(\frac{1}{8g}\right) \mp i\frac{\sqrt{2}}{\pi}K_{\frac{1}{4}}\left(\frac{1}{8g}\right)\right]$$

$$= \widehat{\mathcal{J}}_0 + \widehat{\mathcal{J}}_{\mp} = \widehat{\mathcal{J}}_0 \mp \frac{1}{2}(\widehat{\mathcal{J}}_+ - \widehat{\mathcal{J}}_-).$$

$$\mathfrak{S}_0^{\pm 1/2}[\mathcal{J}_0] = \mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\mp\pi}, \qquad \mathfrak{S}_0^{\pm 1/2}[\mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi}] = \mathcal{J}_0, \quad \mathfrak{S}_0^{\pm 1}[\mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi}] = \mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\mp\pi},$$

$$\widehat{\mathfrak{S}}_0^{\pm 1/2}[\widehat{\mathcal{J}}_0] = \widehat{\mathcal{J}}_{\mathrm{AC}}^{\arg(\lambda)=\mp\pi}, \qquad \widehat{\mathfrak{S}}_0^{\pm 1/2}[\widehat{\mathcal{J}}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi}] = \widehat{\mathcal{J}}_0, \quad \widehat{\mathfrak{S}}_0^{\pm 1}[\widehat{\mathcal{J}}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi}] = \widehat{\mathcal{J}}_{\mathrm{AC}}^{\arg(\lambda)=\mp\pi},$$

$$\mathcal{S}_{0_\pm}[\mathcal{J}_0] = \widehat{\mathcal{J}}_{\mathrm{AC}}^{\arg(\lambda)=\mp\pi}, \qquad \mathcal{S}_{0_\pm}[\mathcal{J}_{\mathrm{AC}}^{\arg(\lambda)=\pm\pi}] = \widehat{\mathcal{J}}_0.$$

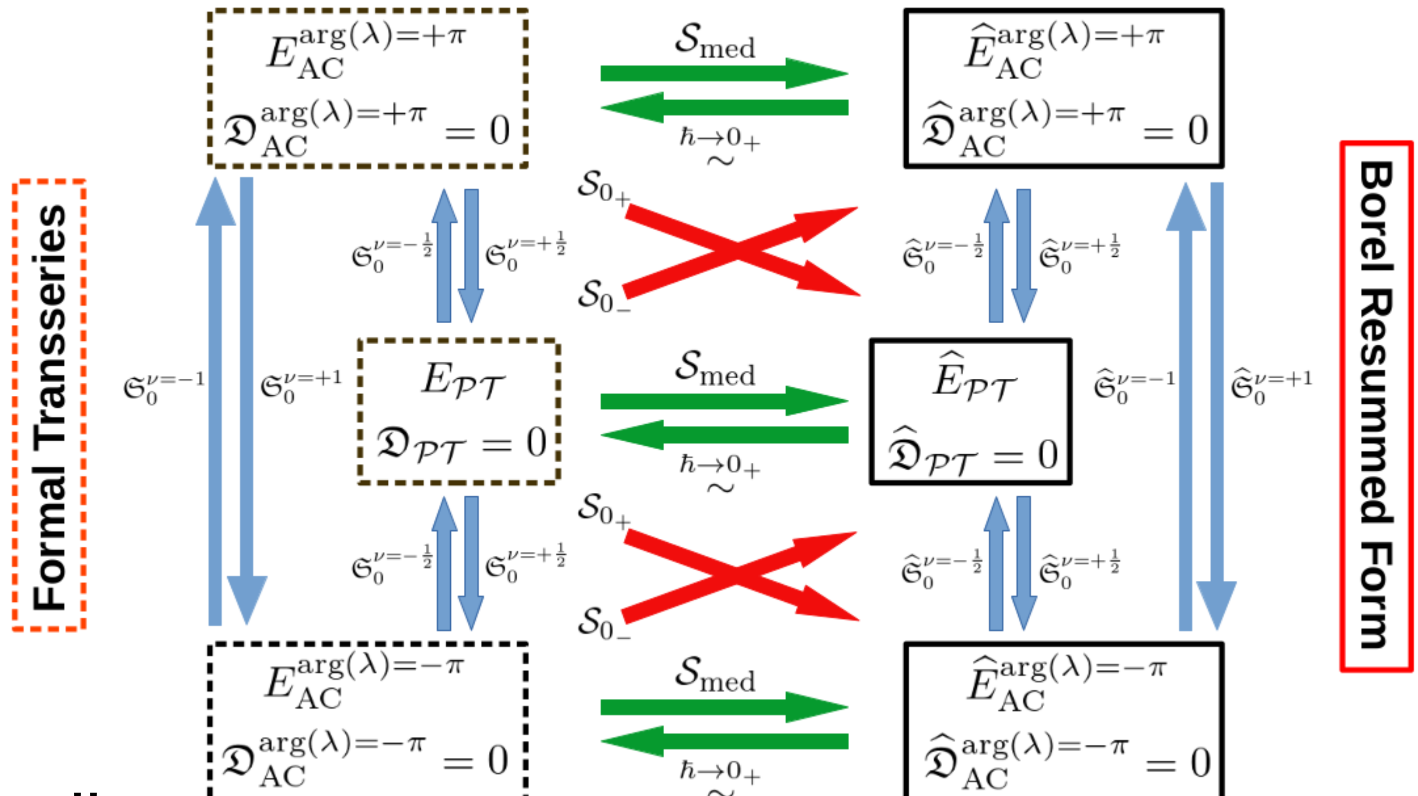
The 0 dim ABS conjecture is reproduced.
[W.-Y. Ai, et. al., 2022]

$$Z_{\mathcal{PT}}(g) = \mathrm{Re}[Z_{\mathcal{H}}(\lambda = -g + i0_\pm)]$$

Reformulation of the ABS conjecture

[S.K. 2024]

Modified ABS conjecture for ω>0



0 dim PT sym. model satisfies the same relations!!

$\mathcal{S}_{0_\pm}$ … Borel resummation

$\mathcal{S}_{\rm med}=\mathcal{S}_{0_\pm}\circ\mathfrak{S}_0^{\mp 1/2}$ … Median resummation

$\mathfrak{S}_0^{\nu\in\mathbb{R}}$ … One-parameter Stokes automorphism

$\widehat{\mathfrak{S}}_0^{\nu\in\mathbb{R}}$ … Pushout of $\mathfrak{S}_0^{\nu\in\mathbb{R}}$

Exact WKB analysis

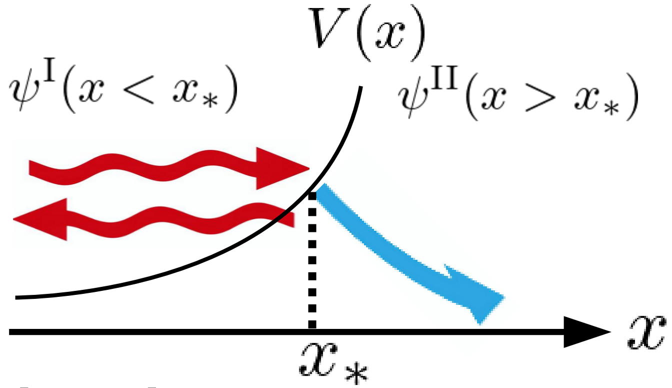
[A. Voros, E. Delabaere, H. Dillinger, H. J. Silverstone, F. Pham, T. Aoki, T. Kawai, Y. Takei, K. Iwaki, T. Nakanishi, ...]

A basic methodology is the same to a (standard) WKB.
But the important differences are ...

1) A wavefunction contains all order of $\hbar$

2) Gluing wavefunctions in each the domain is performed through Borel resummation

$$\mathcal{S}_\theta[\psi^{\mathrm{I}}(x_* + 0_-)] = \mathcal{S}_\theta[\psi^{\mathrm{I}}(x_* + 0_+)],$$
$$\psi^{\mathrm{I}}(x_* + 0_+) := M^{\mathrm{I}\to\mathrm{II}}\psi^{\mathrm{II}}(x_* + 0_+),$$



3) The resulting quantization condition (QC) and E spectrum contains all pert. and nonpert. corrections (but as Borel resummed form).

4) Voros symbol (Cycle)-representation
… QCs are expressed by periodic cycles

In order to consider the ABS conjecture, we find E spectrum as transseries.
(Finding the Borel resummed form is too tough in practice.)

Exact WKB analysis

[A. Voros, E. Delabaere, H. Dillinger, H. J. Silverstone, F. Pham, T. Aoki, T. Kawai, Y. Takei, K. Iwaki, T. Nakanishi, ….]

Procedure

1) Prepare ansatz for the wavefunction and draw its Stokes graph.

2) Analytic continuation on the complex x-plane as taking care of the connection formula to obtain the quantization condition.

$$\mathcal{S}_{0\pm}[\psi_{a_1}^{0\pm\mathrm{I}}] = \mathcal{S}_{0\pm}[\mathcal{M}^{0\pm}\psi_{a_1}^{0\pm\mathrm{VI}}] \qquad \mathcal{M}_{12}^{0\pm} = 0 \quad \Rightarrow \quad \mathfrak{D}^{0\pm}(E) = 0$$

3) Eliminate the discontinuity in the transseries by the DDP formula.

$$\mathcal{S}_{0_+}[\mathfrak{D}^{0+}] = \mathcal{S}_{0_-}[\mathfrak{D}^{0-}] \quad \Rightarrow \quad \mathfrak{D}^{0} = \mathfrak{S}_0^{\nu=\pm 1/2}[\mathfrak{D}^{0\pm}]$$

4) Solve the quantization condition to find the energy spectrum

Go to the details...

- **Ansatz of the wavefunction**

A standard WKB ansatz ➡ All order quantum corrections

$$\mathcal{L} = -\hbar^2 \partial_x^2 + V(x) - E, \qquad \mathcal{L}\psi(x) = 0, \qquad (x \in \mathbb{C}, \quad E, \hbar \in \mathbb{R}_{>0})$$

$$\psi_a(x,\hbar) = \sigma(\hbar) \exp\left[\int_a^x dx' \, S(x',\hbar)\right], \qquad (x \in \mathbb{C})$$

$$S(x,\hbar) = \sum_{n \in \mathbb{N}_0} S_{n-1}(x)\hbar^{n-1} \quad \text{as} \quad \hbar \to 0_+,$$

⬅ Include all order

Riccati eq.

$$S(x,\hbar)^2 + \partial_x S(x,\hbar) = \hbar^{-2} Q(x), \qquad Q(x) := V(x) - E.$$

Solving order-by-order...

➡

$$S_{-1}(x) = \pm\sqrt{Q(x)}, \qquad S_0(x) = -\frac{\partial_x \log Q(x)}{4},$$

$$S_{+1}(x) = \pm\frac{1}{8\sqrt{Q(x)}}\left[\partial_x^2 \log Q(x) - \frac{(\partial_x \log Q(x))^2}{4}\right], \qquad \cdots.$$

- **Ansatz of the wavefunction**

A standard WKB ansatz ➡ All order quantum corrections

$$\mathcal{L} = -\hbar^2 \partial_x^2 + V(x) - E, \qquad \mathcal{L}\psi(x) = 0, \qquad (x \in \mathbb{C}, \quad E, \hbar \in \mathbb{R}_{>0})$$

$$\psi_a(x,\hbar) = \sigma(\hbar) \exp\left[\int_a^x dx'\, S(x',\hbar)\right], \qquad (x \in \mathbb{C})$$

$$S(x,\hbar) = \sum_{n\in\mathbb{N}_0} S_{n-1}(x)\hbar^{n-1} \quad \text{as} \quad \hbar \to 0_+,$$

⬅ Include all order

Define Sod and Sev, which contain only odd- and even-power expansion, respectively, as

$$S_{\rm od}(x,\hbar) = \sum_{n\in\mathbb{N}_0} S_{2n-1}(x)\hbar^{2n-1}, \qquad S_{-1}(x) = \sqrt{Q(x)},$$

$$S_{\rm ev}(x,\hbar) = \sum_{n\in\mathbb{N}_0} S_{2n}(x)\hbar^{2n} = -\frac{1}{2}\partial_x \log S_{\rm od}(x,\hbar).$$

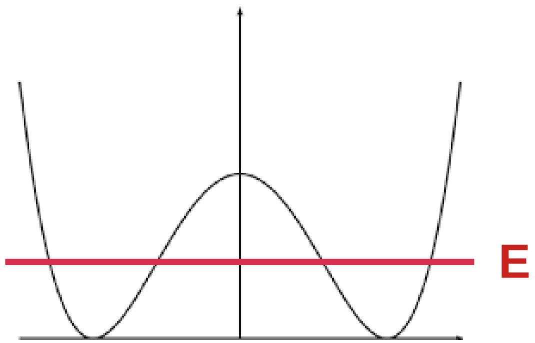
Sev can be expressed by Sod

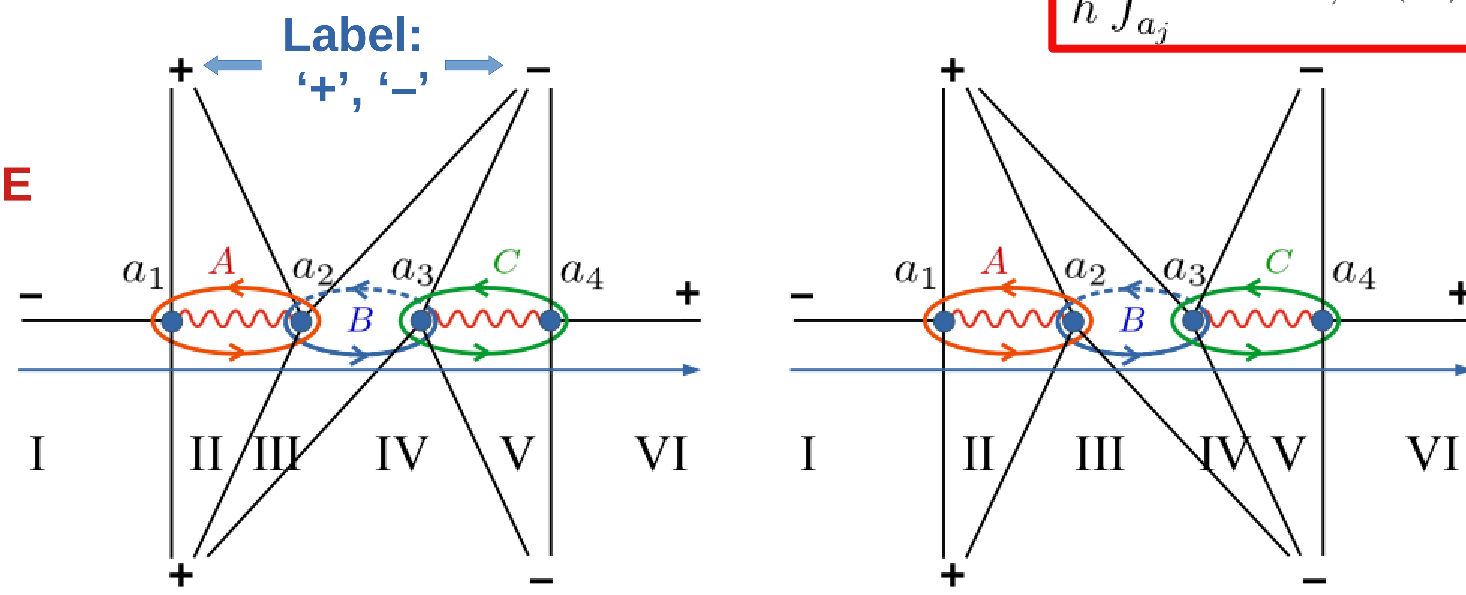
➡ $$\psi_{a\pm}(x,\hbar) = \frac{\sigma_\pm(\hbar)}{\sqrt{S_{\rm od}(x,\hbar)}} \exp\left[\pm \int_a^x dx'\, S_{\rm od}(x',\hbar)\right].$$

- **Stokes graph**

A graph that contains all information of Borel summability of the wavefunction and is drawn by its classical part

$$\frac{1}{\hbar}\int_{a_j}^{x} dx'\, S_{\mathrm{od},-1}(x')$$





(a) $\arg(\hbar) = 0_+$ (b) $\arg(\hbar) = 0_-$

Turning points … kinetic energy = 0 for a fixed E in the Sch eq.

$$\mathrm{TP} := \{x \in \mathbb{C} \,|\, Q(x) = 0\}.$$

Stokes lines … on which the wavefunction becomes Borel nonsummable

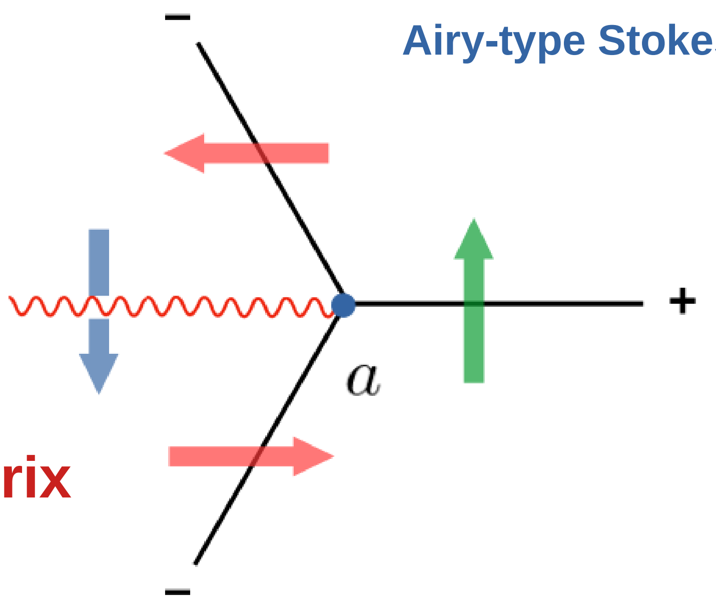
$$\mathrm{Im}\left[\frac{1}{\hbar}\int_{a_j}^{x} dx'\, S_{\mathrm{od},-1}(x')\right] = 0, \qquad a_j \in \mathrm{TP}.$$

Label: '+', '–' $\quad \mathrm{Re}\left[\frac{1}{\hbar}\int_{a_j}^{x} dx'\, S_{\mathrm{od},-1}(x')\right] \to \pm\infty$

Branch cuts

- **Stokes graph**

$$\psi_a = \begin{pmatrix} \psi_{a+} \\ \psi_{a-} \end{pmatrix}.$$



Guiding principle for connection matrix

$$\mathcal{S}_\theta[\psi^{\rm I}(x_* + 0_-)] = \mathcal{S}_\theta[\psi^{\rm I}(x_* + 0_+)],$$
$$\psi^{\rm I}(x_* + 0_+) := M^{\rm I\to II}\psi^{\rm II}(x_* + 0_+),$$

$${\rm TP} := \{x \in \mathbb{C} \,|\, Q(x) = 0\}.$$

$${\rm Im}\left[\frac{1}{\hbar}\int_{a_j}^{x} dx'\, S_{{\rm od},-1}(x')\right] = 0, \qquad a_j \in {\rm TP}.$$

Connection formula (Airy-type)

$$M_+ = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}, \qquad M_- = \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix},$$

Normalization matrix

$$N_{a_j,a_k} := \begin{pmatrix} e^{+\int_{a_k}^{a_j} dx\, S_{\rm od}(x,\hbar)} & \\ & e^{-\int_{a_k}^{a_j} dx\, S_{\rm od}(x,\hbar)} \end{pmatrix} = N^{-1}_{a_k,a_j}, \qquad a_j, a_k \in {\rm TP}.$$

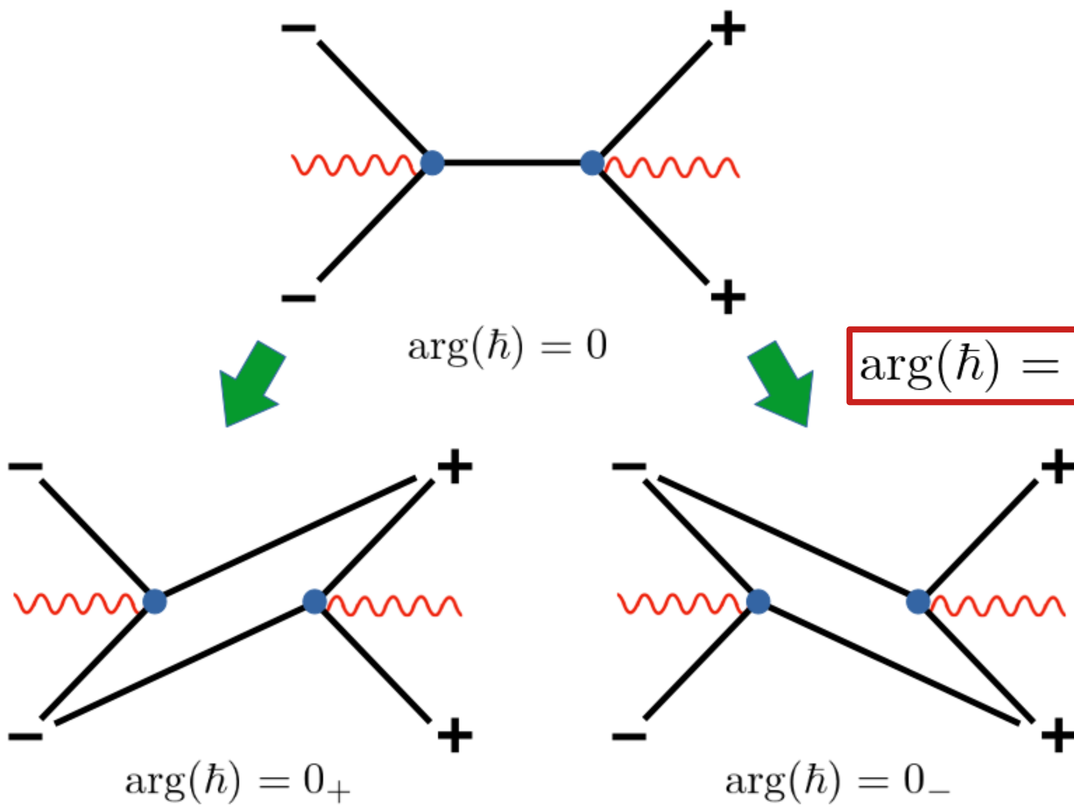
• Resolving degeneracy

When degeneracy between Stokes lines occur, the wavefunction becomes Borel nonsummable for any complex x.

Introduce an infinitesimal phase

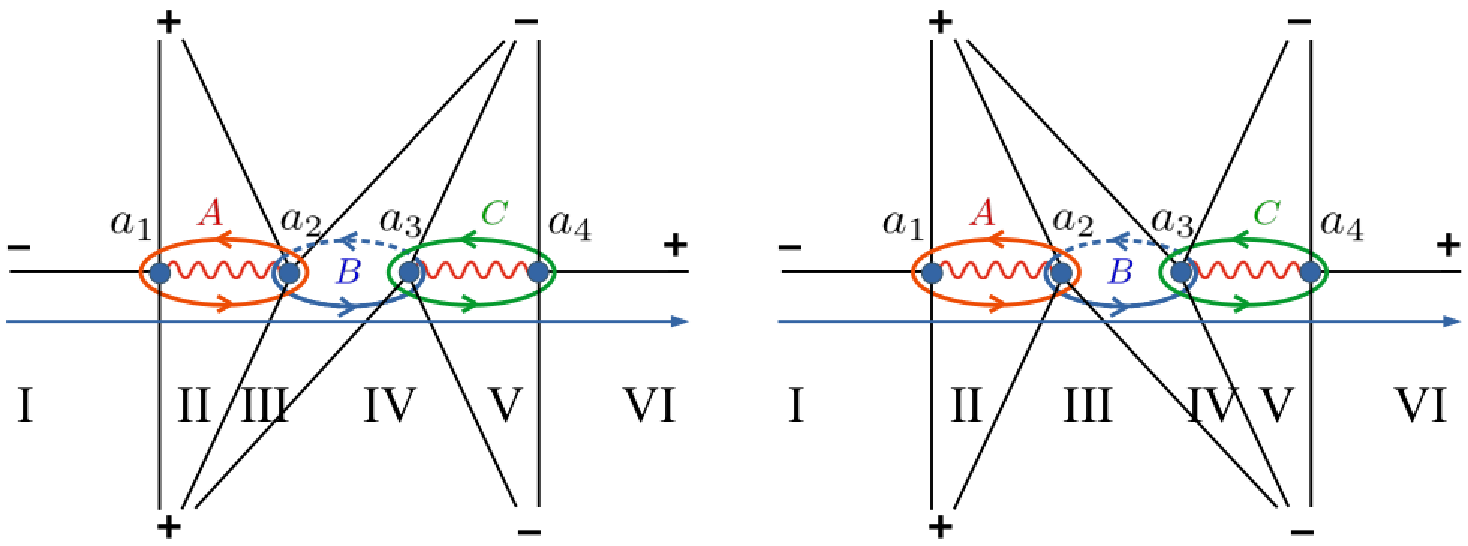
Borel nonsummable

Borel summable



$$\arg(\hbar) = 0 \to \arg(\hbar) \neq 0$$

- **Example: Double-well potential**



(a) $\arg(\hbar) = 0_+$ (b) $\arg(\hbar) = 0_-$

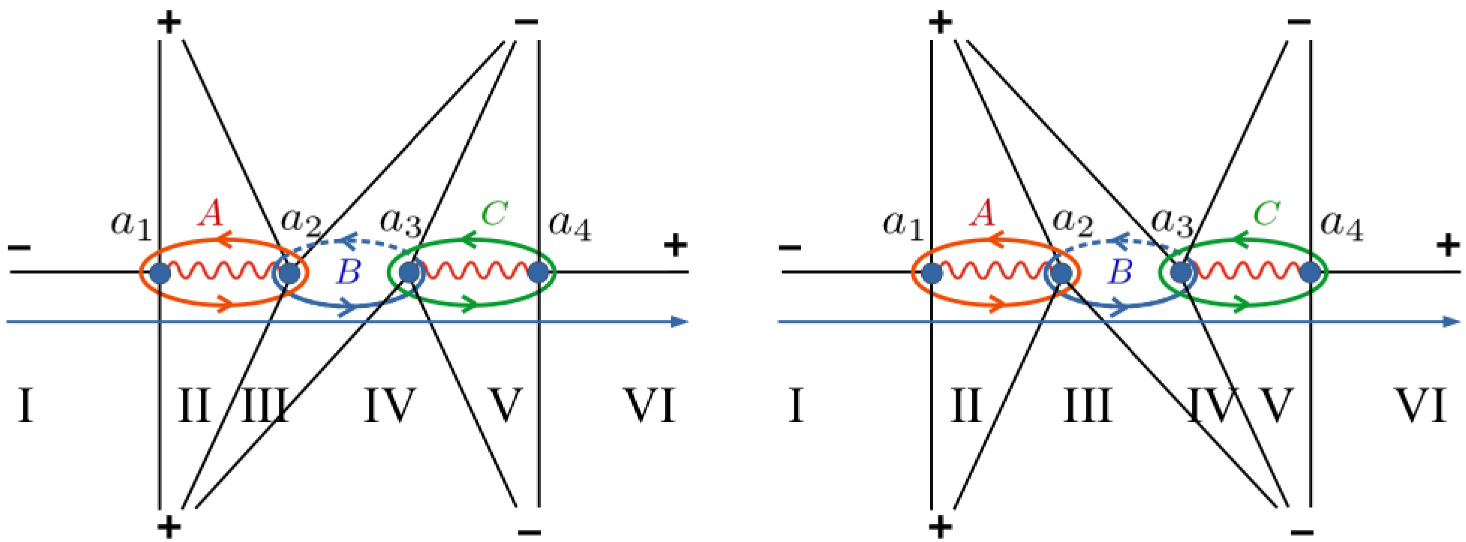
Analytic continuation along →

$$\mathcal{S}_{0_\pm}[\psi_{a_1}^{0\pm \mathrm{I}}] = \mathcal{S}_{0_\pm}[\mathcal{M}^{0\pm}\psi_{a_1}^{0\pm \mathrm{VI}}],$$

$$\mathcal{M}^{0+} = M_+ N_{a_1,a_2} M_+ N_{a_2,a_3} M_+ M_- N_{a_3,a_4} M_- N_{a_4,a_1}$$

$$\mathcal{M}^{0-} = M_+ N_{a_1,a_2} M_+ M_- N_{a_2,a_3} M_- N_{a_3,a_4} M_- N_{a_4,a_1}$$

- **Example: Double-well potential**



(a) $\arg(\hbar) = 0_+$ (b) $\arg(\hbar) = 0_-$

Quantization Condition (QC)

$$\mathcal{M}^{0\pm}_{jk} = \mathfrak{D}^{0\pm} = 0,$$

$$\mathcal{S}_{0_+}[\mathfrak{D}^{0+}] = \mathcal{S}_{0_-}[\mathfrak{D}^{0-}]$$

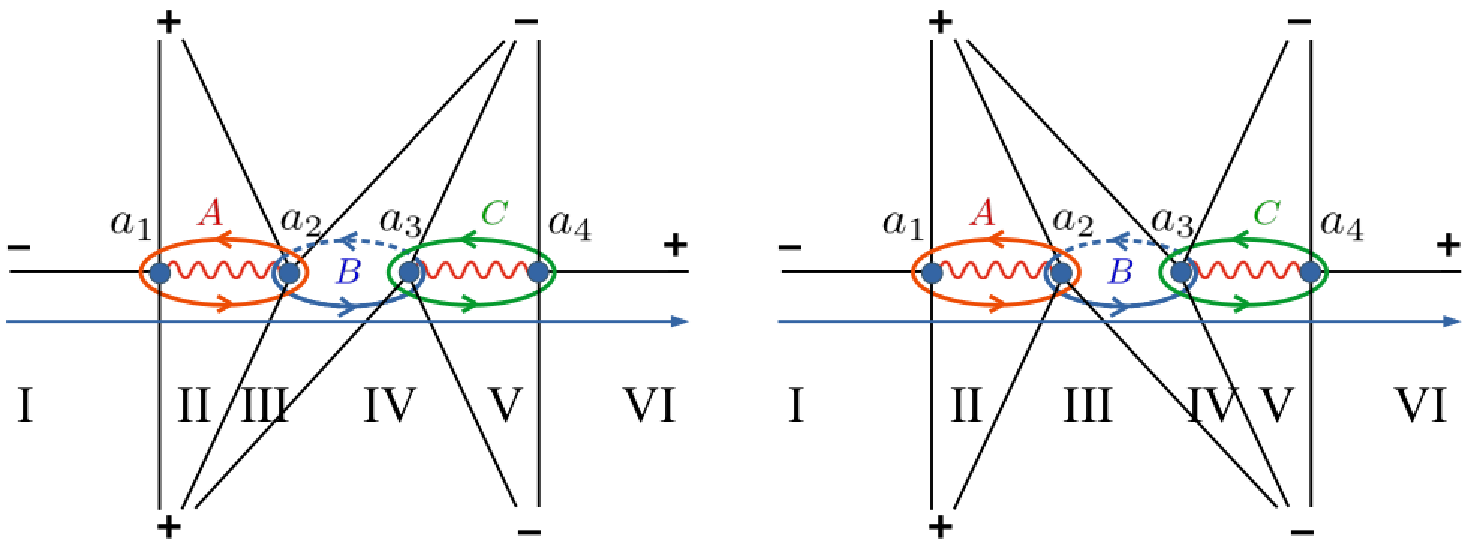
Boundary condition (Normalizability)

$$(j,k) = \begin{cases} (1,1) & \text{if } \psi_-^{0\pm\mathrm{I}}(-\infty) = \psi_+^{0\pm\mathrm{I}}(+\infty) = 0 \\ (1,2) & \text{if } \psi_-^{0\pm\mathrm{I}}(-\infty) = \psi_-^{0\pm\mathrm{I}}(+\infty) = 0 \\ (2,1) & \text{if } \psi_+^{0\pm\mathrm{I}}(-\infty) = \psi_+^{0\pm\mathrm{I}}(+\infty) = 0 \\ (2,2) & \text{if } \psi_+^{0\pm\mathrm{I}}(-\infty) = \psi_-^{0\pm\mathrm{I}}(+\infty) = 0 \end{cases}.$$

In the above cases

$$\mathfrak{D}^{0+} \propto (1+A)(1+C) + AB, \quad \mathfrak{D}^{0-} \propto (1+A)(1+C) + CB$$

- **Example: Double-well potential**



(a) $\arg(\hbar) = 0_+$ (b) $\arg(\hbar) = 0_-$

DDP formula … Pert/Nonpert relations among cycles **[E. Delabaere et. al. 1997]**

$$\mathcal{S}_{\theta+0_+}[A_j] = \mathcal{S}_{\theta+0_-}[A_j] \prod_{B_k \in \mathrm{C}_{\mathrm{NP},\theta}} (1 + \mathcal{S}_{\theta+0_-}[B_k])^{\langle A_j, B_k \rangle},$$

$$\mathcal{S}_{\theta+0_+}[B_k] = \mathcal{S}_{\theta+0_-}[B_k], \qquad B_k \in \mathrm{C}_{\mathrm{NP},\theta},$$

$$\mathfrak{S}_\theta^{\nu=1}[A_j] = A_j \prod_{B_k \in \mathrm{C}_{\mathrm{NP},\theta}} (1 + B_k)^{\langle A_j, B_k \rangle},$$

$$\mathfrak{S}_\theta^{\nu=1}[B_k] = B_k, \qquad B_k \in \mathrm{C}_{\mathrm{NP},\theta},$$

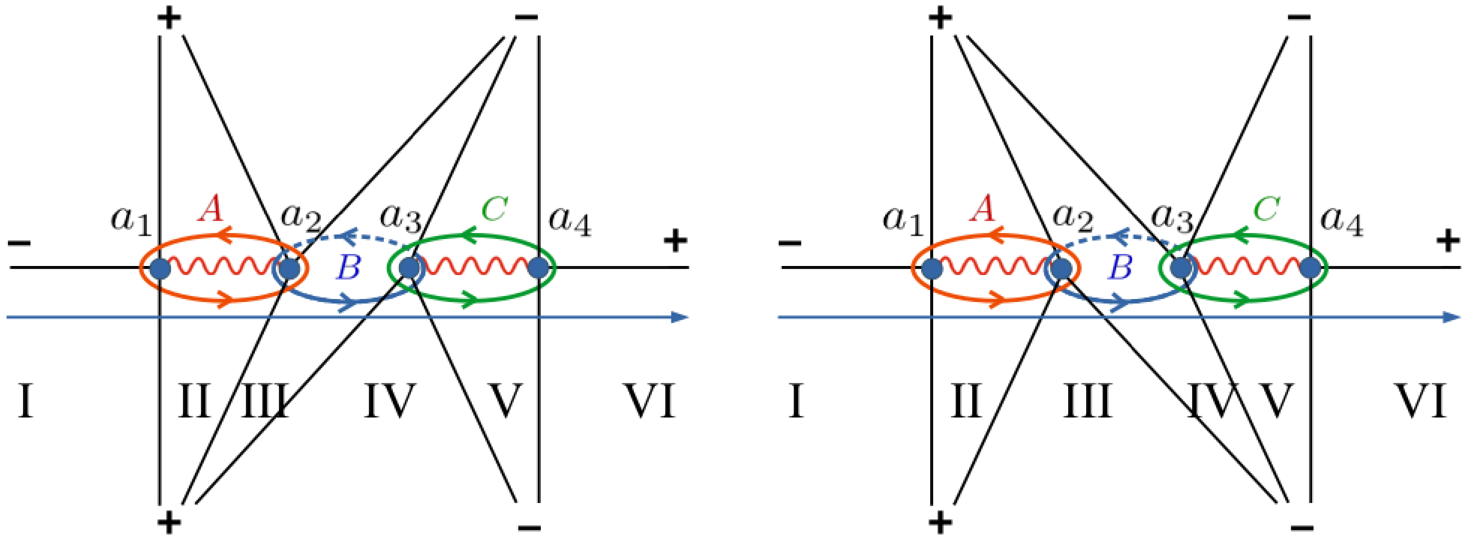
A set of NP cycles

Intersection number

$$\langle \rightarrow, \uparrow \rangle = \langle \leftarrow, \downarrow \rangle = +1, \qquad \langle \rightarrow, \downarrow \rangle = \langle \leftarrow, \uparrow \rangle = -1. \qquad \langle A, B \rangle = -\langle C, B \rangle = -1$$

One-parameter Stokes automorphism $\langle A_j, B_k \rangle \rightarrow \langle A_j, B_k \rangle \times \nu$

- **Example: Double-well potential**



(a) $\arg(\hbar) = 0_+$ (b) $\arg(\hbar) = 0_-$

$$\mathcal{S}_{0_+}[\mathfrak{D}^{0+}] = \mathcal{S}_{0_-}[\mathfrak{D}^{0-}]$$

Median resummation

$$\mathcal{S}_{\mathrm{med},0} := \mathcal{S}_{0_+} \circ \mathfrak{S}_0^{\nu=-1/2} = \mathcal{S}_{0_-} \circ \mathfrak{S}_0^{\nu=+1/2}$$

$$\mathfrak{D}^{0+} \propto (1+A)(1+C) + AB, \quad \mathfrak{D}^{0-} \propto (1+A)(1+C) + CB$$

$$\mathfrak{S}_0^{\nu}[A] = A(1+B)^{-\nu}, \quad \mathfrak{S}_0^{\nu}[C] = C(1+B)^{+\nu}, \quad \mathfrak{S}_0^{\nu}[B] = B.$$

$$\mathfrak{D}^0 := \mathfrak{S}_0^{\pm 1/2}[\mathfrak{D}^{0\pm}] \propto 1 + AC + (A+C)\sqrt{1+B}$$

$$\mathcal{S}_{\mathrm{med},0}[\mathfrak{D}^0] = \mathcal{S}_{0_\pm}[\mathfrak{D}^{0\pm}]$$

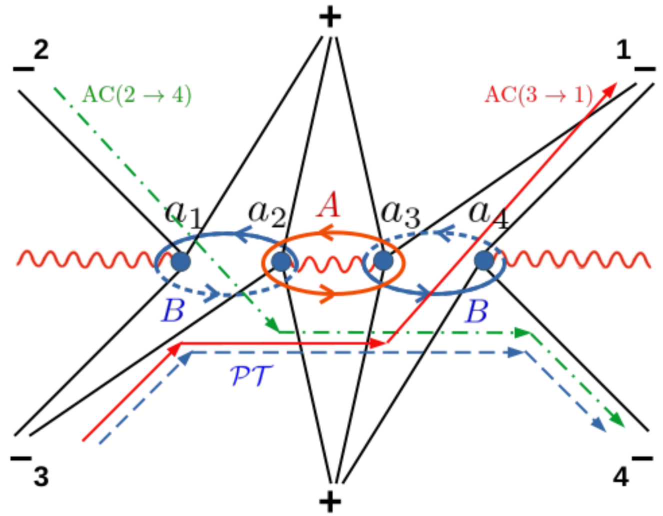
Application of EWKB

to a negative coupling potential

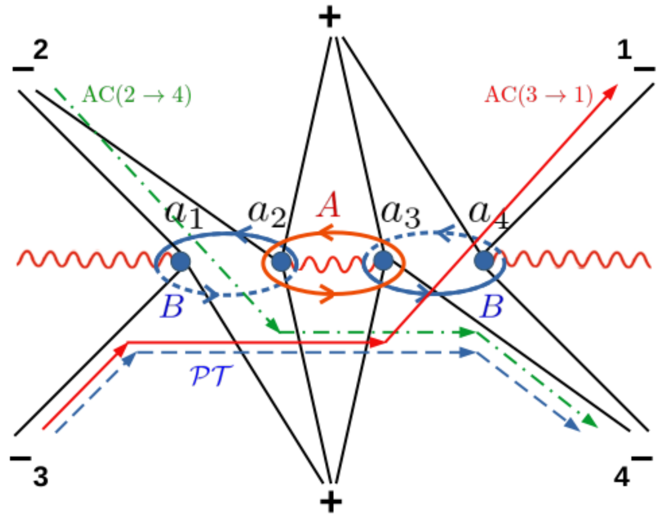
- Massive case (ω>0)
- Massless case (ω=0)

$$V(x) = \omega^2 x^2 - g x^4$$

• **Analytic continuation**



(a) $\arg(g) = 0_+$

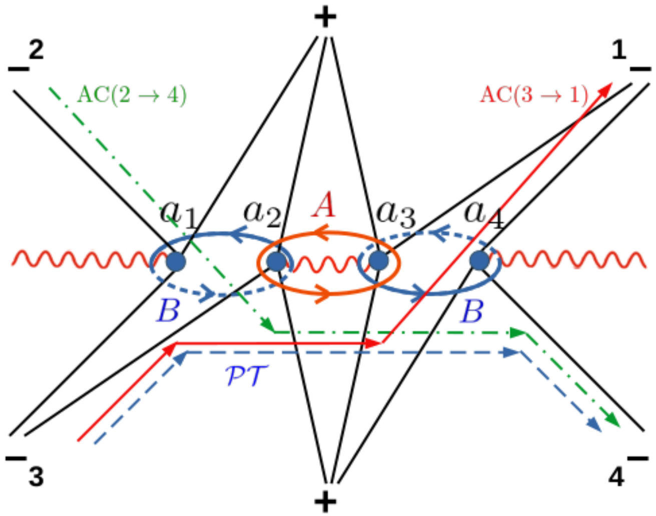


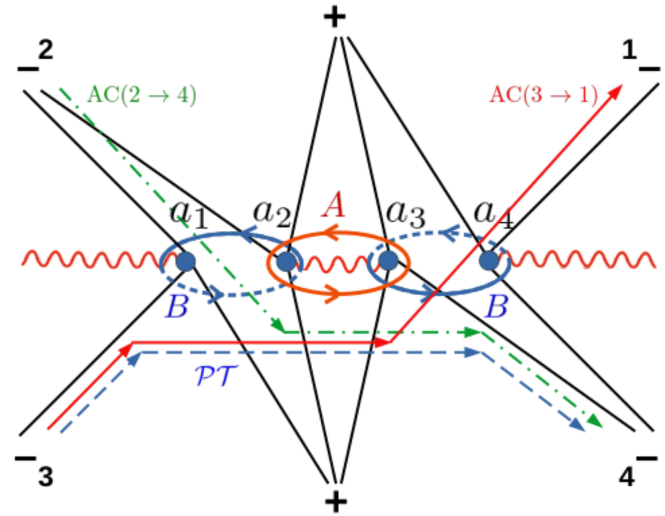
(b) $\arg(g) = 0_-$

Figure 8: Stokes graph of the $\mathcal{PT}$ symmetric potential with a quadratic term. The paths for the analytic continuation are denoted by colored lines, $\gamma_{3\to1}$ (red), $\gamma_{2\to4}$ (green), and $\gamma_{3\to4}$ (blue).

$$\mathcal{M}^{0+}_{\mathrm{AC}(3\to1)} = N_{a_1,a_2} M_+ N_{a_2,a_3} M_+ M_- N_{a_3,a_1},$$
$$\mathcal{M}^{0-}_{\mathrm{AC}(3\to1)} = M_+ N_{a_1,a_2} M_+ N_{a_2,a_3} M_+ M_- N_{a_3,a_4} M_+^{-1} N_{a_4,a_1},$$
$$\mathcal{M}^{0+}_{\mathrm{AC}(2\to4)} = M_+^{-1} N_{a_1,a_2} M_- M_+ N_{a_2,a_3} M_+ N_{a_3,a_4} M_+ N_{a_4,a_1},$$
$$\mathcal{M}^{0-}_{\mathrm{AC}(2\to4)} = N_{a_1,a_2} M_- M_+ N_{a_2,a_3} M_+ N_{a_3,a_1},$$
$$\mathcal{M}^{0+}_{\mathcal{PT}} = N_{a_1,a_2} M_+ N_{a_2,a_3} M_+ N_{a_3,a_4} M_+ N_{a_4,a_1},$$
$$\mathcal{M}^{0-}_{\mathcal{PT}} = M_+ N_{a_1,a_2} M_+ N_{a_2,a_3} M_+ N_{a_3,a_1}.$$

- **Analytic continuation**





(a) $\arg(g) = 0_+$ (b) $\arg(g) = 0_-$

Figure 8: Stokes graph of the $\mathcal{PT}$ symmetric potential with a quadratic term. The paths for the analytic continuation are denoted by colored lines, $\gamma_{3\to1}$ (red), $\gamma_{2\to4}$ (green), and $\gamma_{3\to4}$ (blue).

$$\mathfrak{D}^{0_+}_{\mathrm{AC}(3\to1)} \propto 1 + A, \qquad \mathfrak{D}^{0_-}_{\mathrm{AC}(3\to1)} \propto 1 + \frac{A}{(1+B)^2},$$

$$\mathfrak{D}^{0_+}_{\mathrm{AC}(2\to4)} \propto 1 + A(1+B)^2, \qquad \mathfrak{D}^{0_-}_{\mathrm{AC}(2\to4)} \propto 1 + A,$$

$$\mathfrak{D}^{0_+}_{\mathcal{PT}} \propto 1 + A(1+B), \qquad \mathfrak{D}^{0_-}_{\mathcal{PT}} \propto 1 + \frac{A}{1+B},$$

- **DDP formula**

$$\mathfrak{D}^{0+}_{\mathrm{AC}(3\to1)} \propto 1 + A, \qquad \mathfrak{D}^{0-}_{\mathrm{AC}(3\to1)} \propto 1 + \frac{A}{(1+B)^2},$$
$$\mathfrak{D}^{0+}_{\mathrm{AC}(2\to4)} \propto 1 + A(1+B)^2, \qquad \mathfrak{D}^{0-}_{\mathrm{AC}(2\to4)} \propto 1 + A,$$
$$\mathfrak{D}^{0+}_{\mathcal{PT}} \propto 1 + A(1+B), \qquad \mathfrak{D}^{0-}_{\mathcal{PT}} \propto 1 + \frac{A}{1+B},$$

DDP formula $\mathfrak{S}^{\nu}_{0}[A] = A(1+B)^{-2\nu}, \qquad \mathfrak{S}^{\nu}_{0}[B] = B.$

⇩ **Eliminate a discontinuity**

$$\mathfrak{D}^{0}_{\mathrm{AC}(3\to1)} \propto 1 + \frac{A}{1+B}, \qquad \mathfrak{D}^{0}_{\mathrm{AC}(2\to4)} \propto 1 + A(1+B),$$
$$\mathfrak{D}^{0}_{\mathcal{PT}} \propto 1 + A.$$

A **… Perturbative part** ***B*** **… Non-perturbative part**

⇒ **PT and AC have the same perturbative part, but Borel nonsummable (the DDP formula is nontrivial)**

- **Energy solutions** $\widetilde{E} := E/\hbar \quad (\omega = g = 1)$

Solving the PT QC … $\mathfrak{D}_{\mathcal{PT}} \propto 1 + \mathcal{A} = 0$

$$\widetilde{E}^{(0)} = q - \frac{3\left(q^2+1\right)}{8}\hbar - \frac{q\left(17q^2+67\right)}{64}\hbar^2 - \frac{3\left(125q^4+1138q^2+513\right)}{1024}\hbar^3 + O(\hbar^4), \tag{4.58}$$

with $q \in 2\mathbb{N}_0 + 1$. Since $\mathfrak{D}_{\mathcal{PT}}$ does not contain non-perturbative contributions, Eq.(4.58) is the transseries solution of the $\mathcal{PT}$ energy. Hence, the perturbative part of the AC energy equals to the $\mathcal{PT}$ energy:

$$\widetilde{E}_{\mathcal{PT}} = \widetilde{E}^{(0)}_{\mathrm{AC}(3\to1)} = \widetilde{E}^{(0)}_{\mathrm{AC}(2\to4)} \in \mathbb{R}_{>0}.$$

- **Perturbative parts are the same**
- **Borel nonsummable**
- **Real value**

- **Energy solutions** $\widetilde{E} := E/\hbar \quad (\omega = g = 1)$

$$\widetilde{E}^{(0)} = q - \frac{3\left(q^2+1\right)}{8}\hbar - \frac{q\left(17q^2+67\right)}{64}\hbar^2 - \frac{3\left(125q^4+1138q^2+513\right)}{1024}\hbar^3 + O(\hbar^4),$$

$$\widetilde{E}^{(1)}_{\mathrm{AC}(3\to1)} = -i\sigma\left[1 - \frac{q(q+6)}{8}\hbar + \frac{q^4+q^3-102q^2-43q-134}{128}\hbar^2 - \frac{q\left(q^5-15q^4-184q^3+4371q^2+2400q+20484\right)}{3072}\hbar^3 + O(\hbar^4)\right],$$

Pure imaginary

$$\widetilde{E}^{(2)}_{\mathrm{AC}(3\to1)} = \sigma^2\left[\frac{\zeta_+}{2} + \frac{2q+3}{8}\hbar - \frac{q(q+3)}{8}\zeta_+\hbar - \frac{8q^3+3q^2-102q-43}{128}\hbar^2 + \frac{2q^4+q^3-51q^2-43q-67}{128}\zeta_+\hbar^2 + \frac{12q^5-75q^4-368q^3+2988q^2+2400q+5121}{1536}\hbar^3 - \frac{q\left(2q^5-15q^4-92q^3+996q^2+1200q+5121\right)}{1536}\zeta_+\hbar^3 + O(\hbar^4)\right],$$

Including both Re and Im parts,
i.e.violation of the ABS conjecture.

where σ and $\zeta_\pm$ are defined as

$$\sigma := \sqrt{\frac{2}{\pi}}\frac{e^{-\frac{2}{3\hbar}}}{\Gamma(\frac{q+1}{2})}\left(\frac{\hbar}{2}\right)^{-\frac{q}{2}}, \qquad \zeta_\pm := \psi^{(0)}\left(\frac{q+1}{2}\right) + \log\left(\frac{\hbar}{2}\right) \pm \pi i.$$

- **Reformulation of the ABS conjecture**

The ABS conjecture is violated, but there is an alternative form…

One-parameter Stokes automorphism

$$\mathfrak{S}_0^{\nu}[A] = A(1+B)^{-2\nu}, \qquad \mathfrak{S}_0^{\nu}[B] = B.$$

The QCs are continuously connected to each other by one-parameter!!

$$\mathfrak{D}^0_{\mathrm{AC}(3\to1)} \propto 1 + \frac{A}{1+B}, \qquad \mathfrak{D}^0_{\mathrm{AC}(2\to4)} \propto 1 + A(1+B),$$
$$\mathfrak{D}^0_{\mathcal{PT}} \propto 1 + A.$$

➡

$$\mathfrak{D}_{\mathcal{PT}} = \mathfrak{S}_0^{-1/2}[\mathfrak{D}_{\mathrm{AC}(3\to1)}] = \mathfrak{S}_0^{+1/2}[\mathfrak{D}_{\mathrm{AC}(2\to4)}].$$

In the QCs, E is a free parameter, but the above Eqs. implies that the energy solutions have the same relations such that:

$$E_{\mathcal{PT}} = \mathfrak{S}_0^{-1/2}[E_{\mathrm{AC}(3\to1)}] = \mathfrak{S}_0^{+1/2}[E_{\mathrm{AC}(2\to4)}].$$

$$E_{\mathrm{AC}(3\to1)} = \mathfrak{S}_0^{+1}[E_{\mathrm{AC}(2\to4)}].$$

- **Reformulation of the ABS conjecture**

Keep … = 0 under the ν-evolution

$$\mathfrak{S}_0^{\nu}[\mathfrak{D}(E)] = 0, \qquad \mathfrak{S}_0^{\nu=0}[\mathfrak{D}(E)] = \mathfrak{D}_{\mathcal{PT}}(E),$$

$$\mathfrak{S}^{\nu} = 1 + \nu\dot{\Delta} + \frac{\nu^2}{2}(\dot{\Delta})^2 + O(\nu^3),$$

$$\dot{\Delta}[\widetilde{E}_{\mathcal{PT}}] = -2i\sigma\left[1 - \frac{q(q+6)}{8}\hbar + \frac{q^4+q^3-102q^2-43q-134}{128}\hbar^2 - \frac{q\left(q^5-15q^4-184q^3+4371q^2+2400q+20484\right)}{3072}\hbar^3 + O(\hbar^4)\right],$$
$$+\pi i\sigma^2\left[1 - \frac{q(q+3)}{4}\hbar + \frac{2q^4+q^3-51q^2-43q-67}{64}\hbar^2 - \frac{q\left(2q^5-15q^4-92q^3+996q^2+1200q+5121\right)}{768}\hbar^3 + O(\hbar^4)\right] + O(\sigma^3),$$

Pure imaginary

$$(\dot{\Delta})^2[\widetilde{E}_{\mathcal{PT}}] = \sigma^2\left[4\zeta + (2q+3)\hbar - q(q+3)\zeta\hbar - \frac{8q^3+3q^2-102q-43}{16}\hbar^2 + \frac{2q^4+q^3-51q^2-43q-67}{16}\zeta\hbar^2 + \frac{12q^5-75q^4-368q^3+2988q^2+2400q+5121}{192}\hbar^3 - \frac{q\left(2q^5-15q^4-92q^3+996q^2+1200q+5121\right)}{192}\zeta\hbar^3 + O(\hbar^4)\right] + O(\sigma^3),$$

Pure real

$$\zeta := \mathrm{Re}[\zeta_{\pm}]$$

- **Reformulation of the ABS conjecture**

$$\mathfrak{S}^{\nu} = 1 + \nu\dot{\Delta} + \frac{\nu^2}{2}(\dot{\Delta})^2 + O(\nu^3),$$

Using the previous results, one finds that

$$\mathfrak{S}_0^{\nu}[\widetilde{E}_{\mathcal{PT}}] = \left(1 + \nu\dot{\Delta}_0 + \frac{(\nu\dot{\Delta}_0)^2}{2} + O(\nu^3)\right)[\widetilde{E}_{\mathcal{PT}}]$$
$$= \begin{cases} \widetilde{E}_{\mathrm{AC}(3\to1)} & \text{for} \quad \nu = +\frac{1}{2} \\ \widetilde{E}_{\mathrm{AC}(2\to4)} & \text{for} \quad \nu = -\frac{1}{2} \end{cases}.$$

The QCs and the energy solutions are continuously connected as

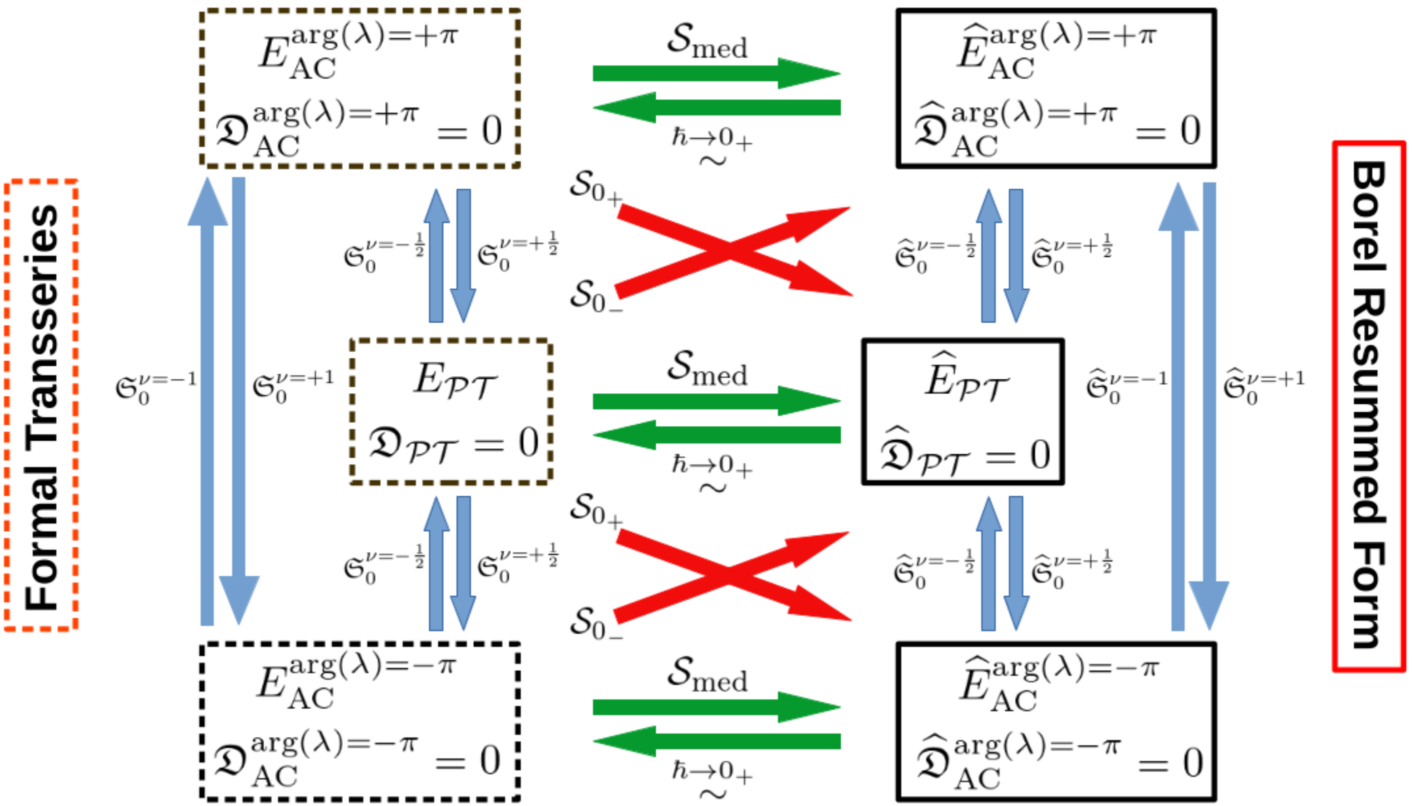
$$\mathfrak{D}^{\nu\in\mathbb{R}} := 1 + A(1+B)^{-2\nu} \propto \begin{cases} \mathfrak{D}_{\mathcal{PT}} & \text{if} \quad \nu = 0 \\ \mathfrak{D}_{\mathrm{AC}(3\to1)} & \text{if} \quad \nu = +\frac{1}{2} \\ \mathfrak{D}_{\mathrm{AC}(2\to4)} & \text{if} \quad \nu = -\frac{1}{2} \end{cases},$$

$$\mathfrak{S}_0^{\nu\in\mathbb{R}}[\mathfrak{D}^{\nu_0}] = \mathfrak{D}^{\nu_0+\nu}.$$

- **Reformulation of the ABS conjecture**

Modified ABS conjecture for ω>0

[S.K. 2024]



$\mathcal{S}_{0_\pm}\ldots$ Borel resummation

$\mathcal{S}_{\rm med}=\mathcal{S}_{0_\pm}\circ\mathfrak{S}_0^{\mp 1/2}\ldots$ Median resummation

$\mathfrak{S}_0^{\nu\in\mathbb{R}}\ldots$ One-parameter Stokes automorphism

$\widehat{\mathfrak{S}}_0^{\nu\in\mathbb{R}}\ldots$ Pushout of $\mathfrak{S}_0^{\nu\in\mathbb{R}}$

- **Massless case (outline)**

The hbar-expansion does not work because the energy is a monomial in terms of hbar.

$$E = c(k)(\lambda\hbar^4)^{1/3}, \qquad c(k) \in \mathbb{R}_{>0},$$

Indeed, Sod has the following form:

$$\int dx\, S_{\rm od}(x,\hbar) = \Phi_{-1}(x)\eta^{-1} + \Phi_{+1}(x)\eta^{+1} + \cdots, \qquad \eta := \frac{\lambda^{1/4}\hbar}{E^{3/4}},$$

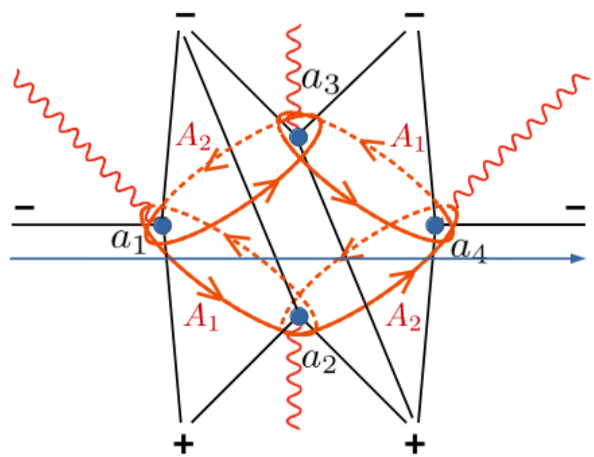
But, c(k) can be determined by the large k-expansion using EWKB.

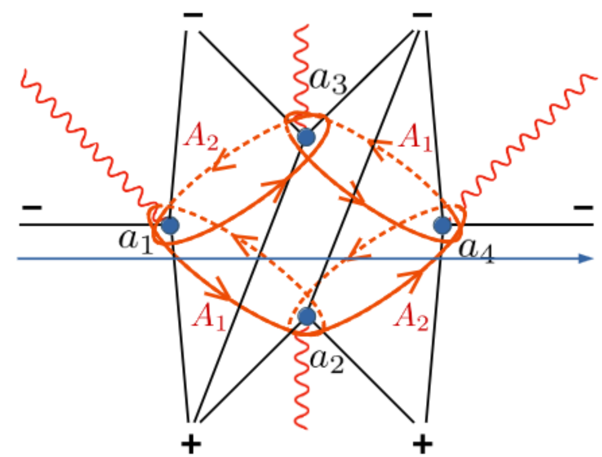
$$\eta^{-1} = \frac{E^{3/4}}{\lambda^{1/4}\hbar} \sim \sum_{n\in\mathbb{N}_0} e^{(0)}_{2n-1}\kappa^{1-2n} + \sum_{\ell\in\mathbb{N}}\sum_{n\in\mathbb{N}_0} e^{(\ell)}_n \sigma^\ell \kappa^{-n} \quad \text{as} \quad \kappa \to +\infty,$$

$$\kappa = \kappa(k) = \pi\left(k + \frac{1}{2}\right), \qquad \sigma := e^{-\kappa}, \qquad k \in \mathbb{N}_0, \qquad e^{(\ell)}_n \in \mathbb{R}.$$

- **Massless case (outline)**

The Hermitian case. $V_{\mathcal{H}}(x) = \lambda x^4$





$$\frac{E_{\mathcal{H}}^{(0)}}{(\lambda\hbar^4)^{1/3}} = \kappa^{4/3}\left[\frac{(3/4)^{4/3}}{[K(-1)]^{4/3}} - \frac{2^{10/3}\sqrt{\pi}\Gamma\left(7/4\right)}{3^{8/3}[K(-1)]^{1/3}\Gamma\left(-3/4\right)}\kappa^{-2} + O(\kappa^{-4})\right],$$

$$\frac{E_{\mathcal{H}}^{(1)}}{(\lambda\hbar^4)^{1/3}} = (-1)^k\sigma\kappa^{1/3}\left[\frac{(3/4)^{1/3}}{[K(-1)]^{4/3}} + \frac{2^{13/3}\sqrt{\pi}\Gamma\left(7/4\right)}{3^{8/3}[K(-1)]^{1/3}\Gamma\left(-3/4\right)}\kappa^{-1}\right.$$
$$\left. - \frac{4\sqrt{\pi}\left(2^{5/6}\pi(6-\pi) + 2^{4/3}\Gamma\left(7/4\right)\Gamma\left(-3/4\right)\right)}{3^{11/3}[K(-1)]^{1/3}[\Gamma\left(-3/4\right)]^2}\kappa^{-2} + O(\kappa^{-3})\right],$$

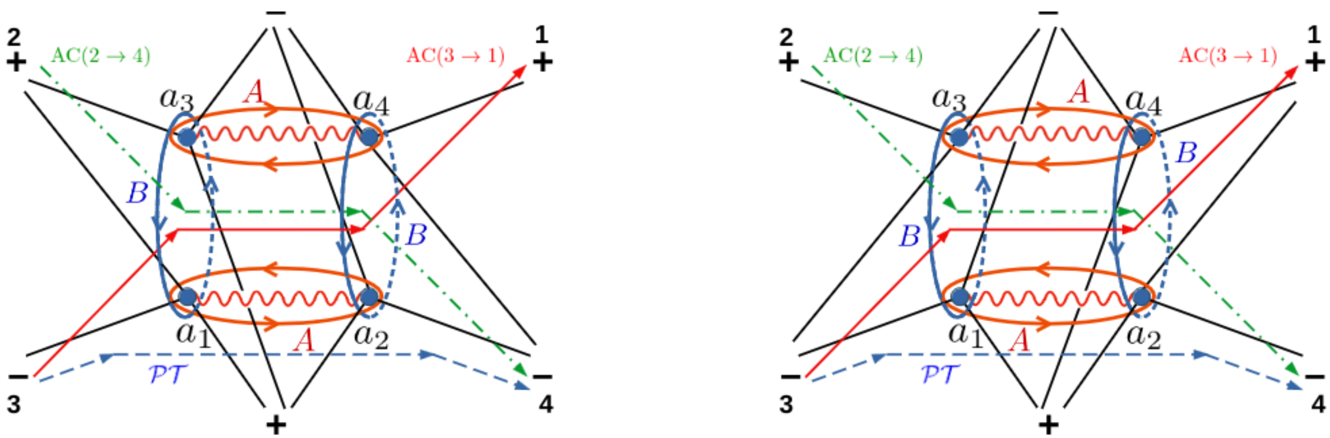
$$\frac{E_{\mathcal{H}}^{(2)}}{(\lambda\hbar^4)^{1/3}} = \sigma^2\kappa^{1/3}\left[-\frac{(3/4)^{1/3}}{[K(-1)]^{4/3}} - \frac{128\sqrt{\pi}[K(-1)]\Gamma\left(7/4\right) - 9\Gamma\left(-3/4\right)}{3\cdot 6^{5/3}[K(-1)]^{4/3}\Gamma\left(-3/4\right)}\kappa^{-1}\right.$$
$$\left. + \frac{2^{10/3}\sqrt{\pi}\left(\sqrt{2}\pi(9-2\pi) + 3\Gamma\left(7/4\right)\Gamma\left(-3/4\right)\right)}{3^{11/3}[K(-1)]^{1/3}[\Gamma\left(-3/4\right)]^2}\kappa^{-2} + O(\kappa^{-3})\right].$$

E_{AC} can be obtained from E_H because it is a monomial of λ.

$$\boxed{\frac{E_{\rm AC}(g)}{(g\hbar^4)^{1/3}} = e^{\pm\frac{\pi}{3}i}\frac{E_{\mathcal{H}}(\lambda)}{(\lambda\hbar^4)^{1/3}}}$$

• Massless case (outline)

The PT case. $V_{\mathcal{PT}}(x) = -gx^4$



(a) $\arg(g) = 0_+$ (b) $\arg(g) = 0_-$

$$\frac{E_{\mathcal{PT}}}{(g\hbar^4)^{1/3}} = \kappa^{4/3}\left[\frac{3^{4/3}}{4[K(-1)]^{4/3}} - \frac{8\sqrt{\pi}\Gamma(7/4)}{3^{8/3}[K(-1)]^{1/3}\Gamma(-3/4)}\kappa^{-2} + O(\kappa^{-4})\right].$$

$$\frac{E^{(0)}_{\mathcal{H}}}{(\lambda\hbar^4)^{1/3}} = \kappa^{4/3}\left[\frac{(3/4)^{4/3}}{[K(-1)]^{4/3}} - \frac{2^{10/3}\sqrt{\pi}\Gamma(7/4)}{3^{8/3}[K(-1)]^{1/3}\Gamma(-3/4)}\kappa^{-2} + O(\kappa^{-4})\right],$$

$$\boxed{\frac{E_{\rm AC}(g)}{(g\hbar^4)^{1/3}} = e^{\pm\frac{\pi}{3}i}\frac{E_{\mathcal{H}}(\lambda)}{(\lambda\hbar^4)^{1/3}}}$$

- EPT contains only the pert. part.
- Not only the nonpert. part, the pert. part does not match with EAC.

➡ **The ABS conjecture is violated. No alternative form exists.**

Summary

- **Verification of the ABS conjecture for QM.**

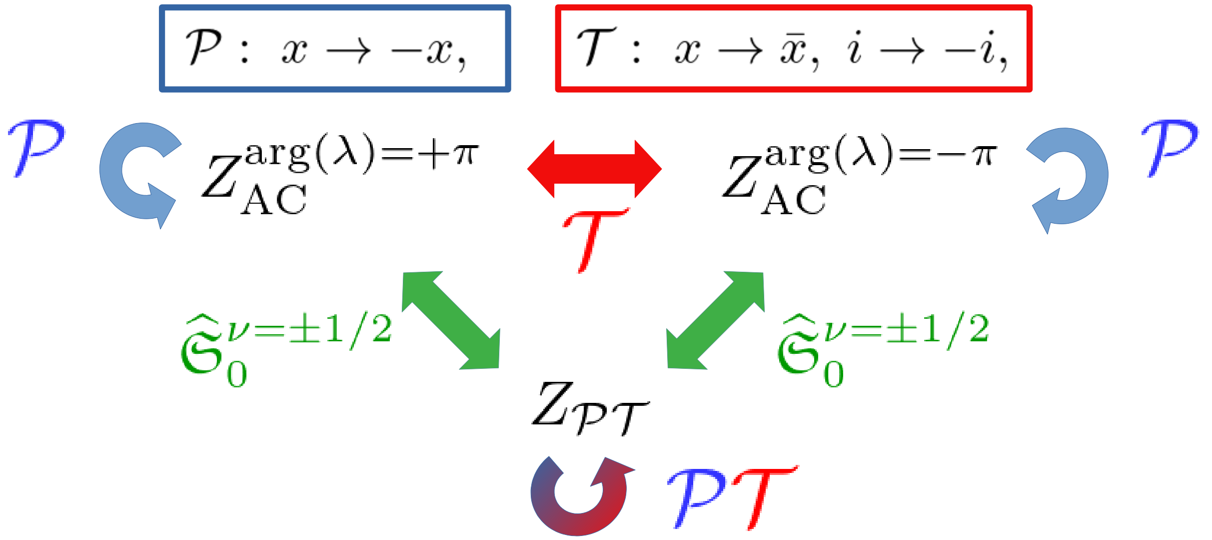
  … The original ABS conjecture is not correct.

  ⇒ **The modified ABS conjecture is formulated in the massive case.**

  ⇒ **No alternative form exists in the massless case.**

- **Does the modified conjecture work in field theories (D>1)???**

- **Any other applications of one-parameter Stokes automorphism???**



Backup slides

- **Example 1:** $$f \sim \sum_{n\in\mathbb{N}} c_n \hbar, \quad c_n = AS^{-n} n! \;\; \text{with} \;\; S \in \mathbb{R}_{>0}$$

Borel transform

$$f_B := \mathcal{B}[f] = \frac{AS}{(S-\xi)^2},$$

A set of singular points along θ = 0

$$\Gamma(\theta = 0) = \{S\}$$

Laplace integral

$$\mathcal{S}_{0_\pm}[f] = \mathcal{L}_{0_\pm}[f_B] = \frac{ASe^{-\frac{S}{\hbar}}}{\hbar}\left[\mathrm{Ei}\left(\frac{S}{\hbar}\right) \pm \pi i\right] - A$$

$$\overset{\hbar\to 0_+}{\sim} \sum_{n\in\mathbb{N}} AS^{-n} n! \hbar^n \pm \pi i \frac{ASe^{-\frac{S}{\hbar}}}{\hbar},$$

Alien derivative and Stokes automorphism

$$\dot{\Delta}_S[f] = -\oint_{\xi=S} d\xi \, e^{-\frac{\xi}{\hbar}} f_B(\xi) = 2\pi i \frac{ASe^{-\frac{S}{\hbar}}}{\hbar}, \qquad (\dot{\Delta}_S)^{n>1}[f] = 0.$$

$$\mathfrak{S}_0^\nu[f] = f + 2\pi\nu i \frac{ASe^{-\frac{S}{\hbar}}}{\hbar},$$

- **Example 1:** $$f \sim \sum_{n\in\mathbb{N}} c_n \hbar, \quad c_n = AS^{-n} n! \;\; \text{with} \;\; S \in \mathbb{R}_{>0}$$

Median resummation

$$\mathfrak{S}_0^{\nu}[f] = f + 2\pi\nu i \frac{ASe^{-\frac{S}{\hbar}}}{\hbar} \;\Rightarrow\; f^{0\pm} = \mathfrak{S}_0^{\mp 1/2}[f] = f \mp \pi i \frac{ASe^{-\frac{S}{\hbar}}}{\hbar}.$$

$$\widehat{f} = \mathcal{S}_{0_\pm}[f^{0\pm}] = \frac{ASe^{-\frac{S}{\hbar}}}{\hbar} \mathrm{Ei}\left(\frac{S}{\hbar}\right) - A$$

$$\overset{\hbar\to 0+}{\sim} \sum_{n\in\mathbb{N}} AS^{-n} n! \hbar^n. \;\Leftarrow$$

Returns the original f

Generalized (push-out) Stokes automorphism

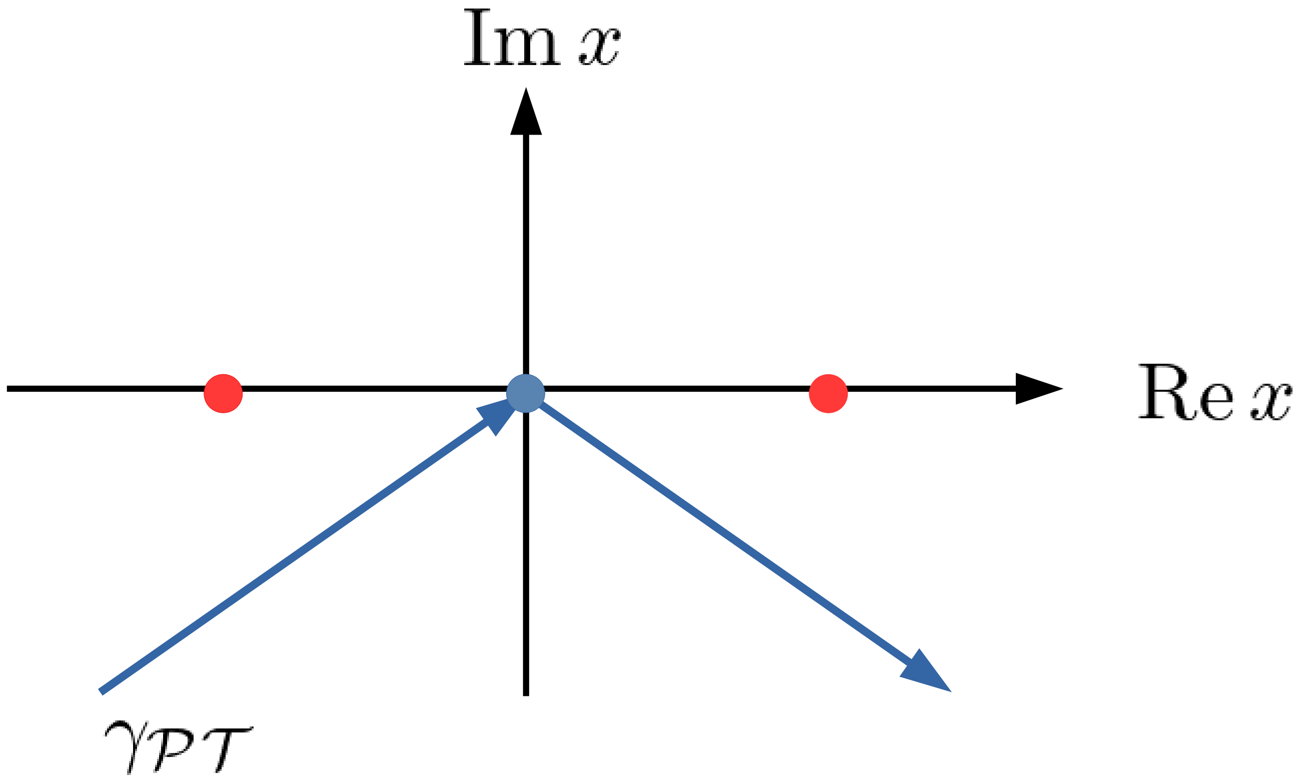
$$\widehat{\mathfrak{S}}_0^{\nu}[\widehat{f}] = \widehat{f} + 2\pi\nu i \frac{ASe^{-\frac{S}{\hbar}}}{\hbar},$$

- **Example 2: 0 dim PT symmetric model** [W.-Y. Ai et. al. 2022]

(Lefschetz thimble picture)

● : **Pert. saddle at 0**

● : **Nonpert. saddles at** $\pm\frac{1}{\sqrt{2g}}$

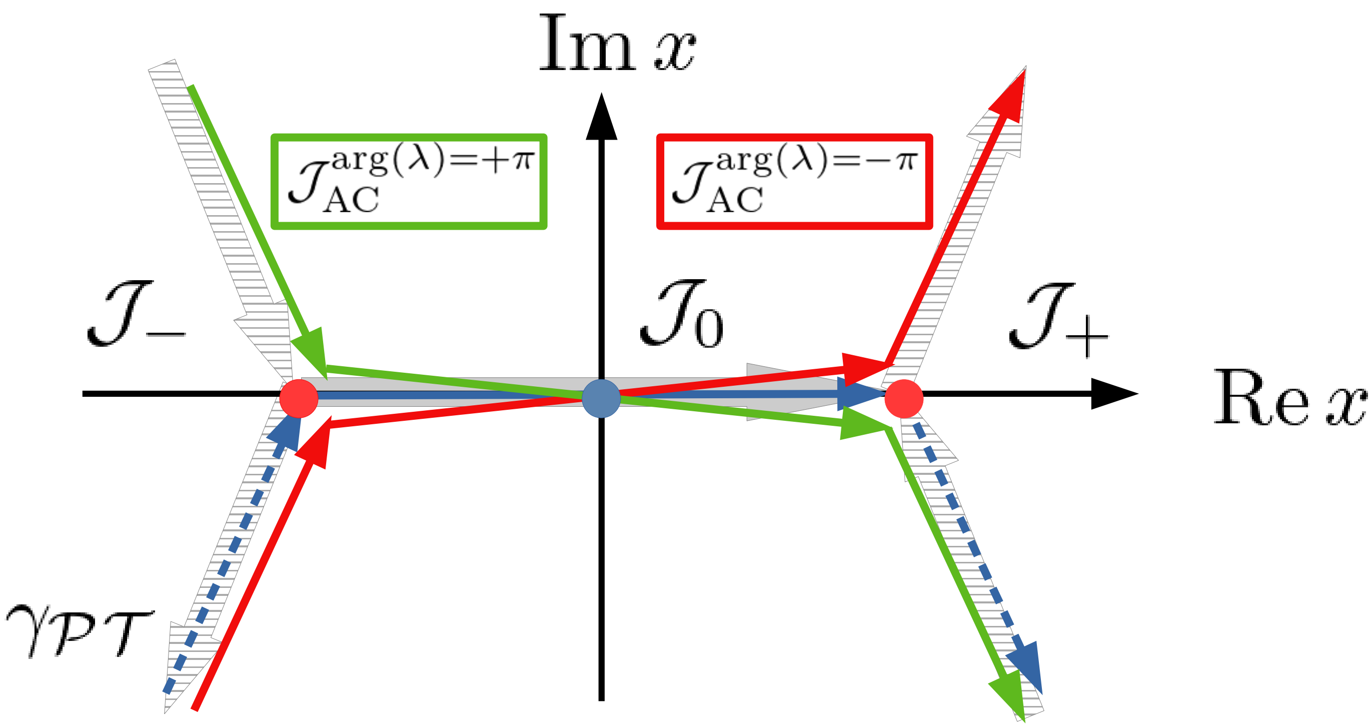


• **Example 2: 0 dim PT symmetric model** [W.-Y. Ai et. al. 2022]

(Lefschetz thimble picture)

● : Pert. saddle at 0

● : Nonpert. saddles at $\pm\frac{1}{\sqrt{2g}}$



$$\mathfrak{S}_0^{\pm 1/2}[\mathcal{J}_0] = \mathcal{J}_{\rm AC}^{\arg(\lambda)=\mp\pi}$$

$$\mathcal{S}_{0_\pm}[\mathcal{J}_0] = \widehat{\mathcal{J}}_{\rm AC}^{\arg(\lambda)=\mp\pi}$$

$$\mathcal{S}_{0_\pm}[\mathcal{J}_{\rm AC}^{\arg(\lambda)=\pm\pi}] = \widehat{\mathcal{J}}_0.$$

$$\mathcal{S}_{\mathrm{med},0} := \mathcal{S}_{0_+} \circ \underline{\mathfrak{S}_0^{\nu=-1/2}} = \mathcal{S}_{0_-} \circ \underline{\mathfrak{S}_0^{\nu=+1/2}}$$

The same procedure to adding complex saddles
to reproduce the original integration path.