Giant Graviton Expansions for Orbifolds and Orientifolds

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Yosuke Imamura
Tokyo Institute of Technology

Based on a collaborating work with
S. Fujiwara, T. Mori, S. Murayama, and D. Yokoyama
arXiv:2310.03332

Abstract

We study giant graviton expansions of the superconformal index of 4d orbifold/orientifold theories. In general, a giant graviton expansion is given as a multiple sum over wrapping numbers, but it was found by Gaiotto and Lee that for N=4 SYM it can be reduced to a simple sum. We find in many examples of orbifold and orientifold theories such reduction occurs, and theories defined with different orbifold/orientifold projections are connected by giant graviton expansions.

GG expansion for N=4 U(N) SYM

Superconformal index

[Romelsberger, hep-th/0510060]
[Kinney, Maldacena, Minwalla, Raju, hep-th/0510251]

Superconformal index of $\mathcal{N} = 4$ SYM

U(1) action

$S^3 \times R_t$

U(N)

$$I(q,p,x,y,z) = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F q^{J_1} p^{J_2} x^{R_x} y^{R_y} z^{R_z}\right], \qquad (qp = xyz)$$

H : Hamiltonian (Dilatation)
J_1, J_2 : Angular momenta
R_x, R_y, R_z : R-charges

I is a function of four independent variables.

PSU(2,2|4)

We can calculate the index for an arbitrary rank N by the localization method.

Localization formula

We can calculate the index by the localization formula

$$\int \prod_{i=1}^{N} dz_i$$

$$I_{U(N)} = \int_{U(N)} d\mu \ \mathrm{Pexp}\left(f_{\mathrm{vec}} \chi_{\mathrm{adj}}^{U(N)}\right)$$

Pexp Plethystic exponential.

$$f_{\mathrm{vec}} = 1 - \frac{(1-x)(1-y)(1-z)}{(1-q)(1-p)}$$ Letter index for the N=4 vector mult.

$$\chi_{\mathrm{adj}}^{U(N)}(z_i) = \sum_{i,j=1}^{N} \frac{z_i}{z_j}$$ Character of the U(N) adjoint rep.

$z_i \ (i = 1 \sim N)$

Pexp (Plethystic exponential)

General definition

$$\mathrm{Pexp}\left(\sum_i c_i x_i\right) = \prod_i \frac{1}{(1-x_i)^{c_i}}$$

Letter index

$$\mathrm{Pexp}(x) = \frac{1}{1-x}$$

$$\mathrm{Pexp}(-x) = 1-x$$

Examples

$$\mathrm{Pexp}(2x-3y) = \frac{(1-y)^3}{(1-x)^2}$$

$$\mathrm{Pexp}(2pq^2) = \frac{1}{(1-pq^2)^2}$$

$$\mathrm{Pexp}(t^{-1}) = \frac{1}{1-t^{-1}} = \frac{-t}{1-t}$$

Large N limit

[Kinney, Maldacena, Minwalla, Raju, hep-th/0510251]

We can easily obtain the large N limit by the saddle point method.

$$I_{U(\infty)} = \mathrm{Pexp}\, i_{\mathrm{KK}} = \mathrm{Pexp}\left(\frac{x}{1-x} + \frac{y}{1-y} + \frac{z}{1-z} - \frac{q}{1-q} - \frac{p}{1-p}\right)$$

This simple form suggests the existence of weakly coupled description.

= Holographic description (AdS/CFT correspondence)

$$I_{U(\infty)} = I_{\mathrm{SUGRA}}$$

Finite N corrections

Let us extract the finite N correction I_{GG} by the relation

$$\frac{I_{U(N)}}{I_{U(\infty)}} = 1 + I_{\mathrm{GG}}$$

I_{GG} is the contribution from determinant operators $\sim$ giant gravitons.

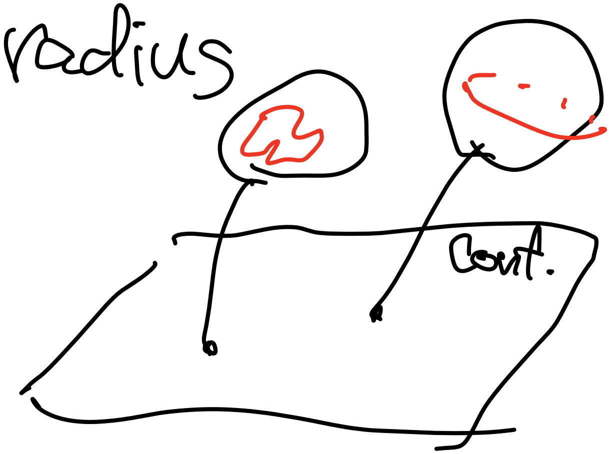
[Arai, YI, arXiv:1904.09776][YI, arXiv:2108.12090]
[Gaiotto, Lee, arXiv:2109.02545][Lee, arXiv:2204.09286]
[Murthy, 2202.06897]

Finite N corrections

L : AdS radius

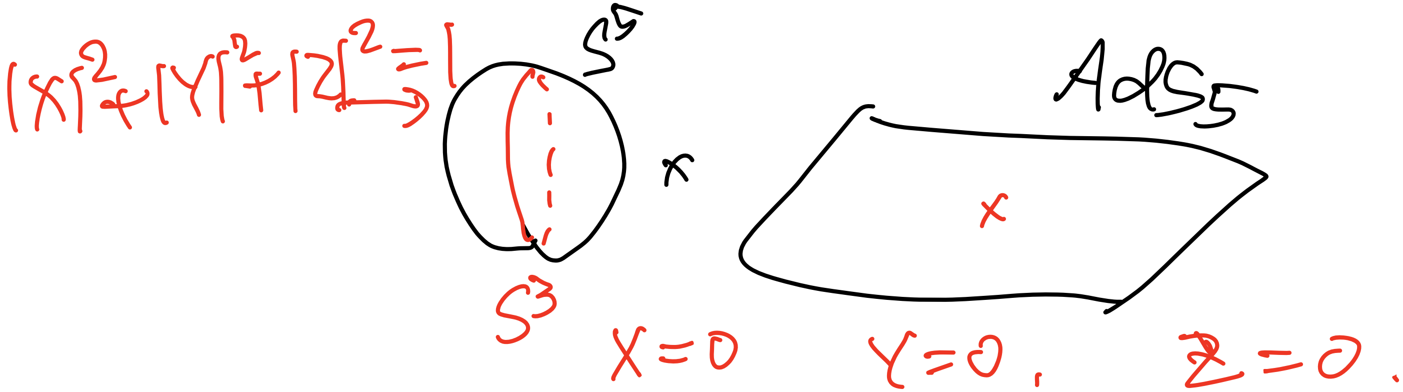
Parameter relations

$$N = \frac{L^4}{l_p^4} = L^4 T_{D3}$$



If localization works, the path integral for D3-branes will be localized at fixed points in the D3-brane configuration space.

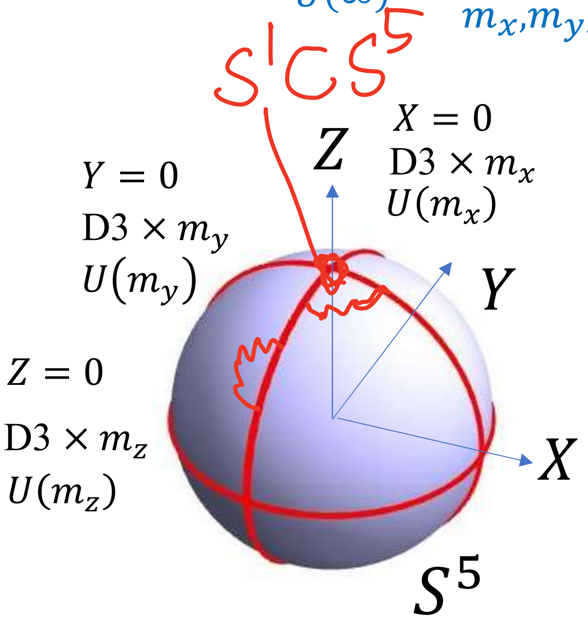
Three (topologically trivial) three-cycles in S^5 will contribute to the index.

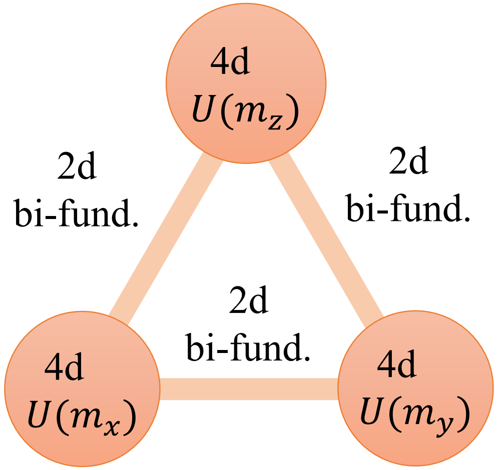


GG expansion (with multiple sum)

GG expansion for N=4 U(N) SYM [YI, arXiv:2108.12090]

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m_x,m_y,m_z=0}^{\infty} x^{m_x N} y^{m_y N} z^{m_z N} F_{m_x,m_y,m_z}(q,p,x,y,z)$$





$\rightarrow F_{m_x,m_y,m_z}$

$G = U(m_x) \times U(m_y) \times U(m_z)$

Functions F_{m_x,m_y,m_z}

The function F_{m_x,m_y,m_z} is given by

$$F_{m_x,m_y,m_z} = \int_G d\mu \ \mathrm{Pexp}\left(i_x[m_x] + i_y[m_y] + i_z[m_z] + \cdots\right)$$

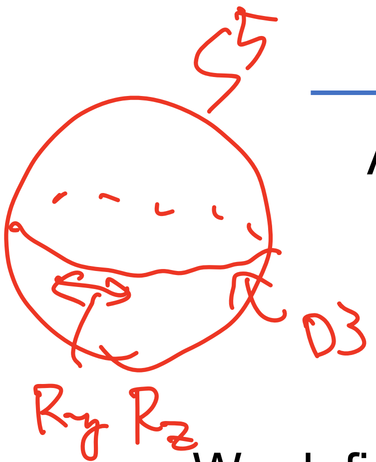
$i_x[m_x]$ ($i_y[m_y], i_z[m_z]$) is the letter index of adjoint fields on $X = 0$, ($Y = 0, Z = 0$)

$$i_x[m_x] = f_{X=0}\chi_{\mathrm{adj}}^{U(m_x)}$$

$f_{X=0}$ is the letter index for the fields in D3-brane wrapped around X=0 (S^3), and is in fact closely related to f_{vec}.

Boundary-GG map

[Arai, YI, arXiv:1904.09776] [Gaiotto, Lee, arXiv:2109.02545]



	spacetime	internal
AdS boundary	H, J_1, J_2	R_x, R_y, R_z
GG on X=0	H, R_y, R_z	R_x, J_1, J_2

We define the involution σ_x exchanging the AdS boundary and the GG worldvolume on X=0.

$$\sigma_x : J_1, J_2 \leftrightarrow R_y, R_z$$

Automorphism

We can extend σ_x to an automorphism of $\mathrm{su}(2|2)^2 \subset psu(2,2|4)$.
Its action on the Cartan generators and fugacities are as follows.

σ_x

$$(J_1, J_2) \leftrightarrow (R_y, R_z) \qquad (q, p) \leftrightarrow (y, z)$$

$$R_x \leftrightarrow -R_x \qquad x \leftrightarrow x^{-1}$$

$H - R_x$: invariant

$$f_{\mathrm{vec}} = 1 - \frac{(1-x)(1-y)(1-z)}{(1-q)(1-p)} \xrightarrow{\sigma_x} f_{x=0} = 1 - \frac{(1-x^{-1})(1-q)(1-p)}{(1-y)(1-z)}$$

By using this variable change we can easily obtain $f_{X=0}$ from f_{vec}.

$$f_{X=0} = \sigma_x f_{\text{vec}}$$

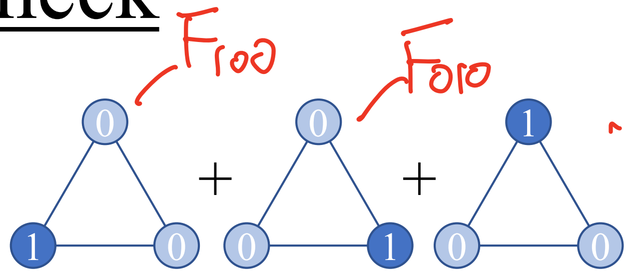
$$\begin{aligned} F_{m_x,m_y,m_z} &= \int_G d\mu\ \text{Pexp}\left(i_x[m_x] + i_y[m_y] + i_z[m_z] + \cdots\right) \\ &= \int_G d\mu\ \text{Pexp}\left(\underbrace{\sigma_x f_{\text{vec}} \chi_{\text{adj}}^{U(m_x)} + \sigma_y f_{\text{vec}} \chi_{\text{adj}}^{U(m_y)} + \sigma_z f_{\text{vec}} \chi_{\text{adj}}^{U(m_z)}}_{\text{vector mult.}} \underbrace{+ \cdots}_{\text{intersection}}\right) \end{aligned}$$

The intersection contribution … have to be determined separately.

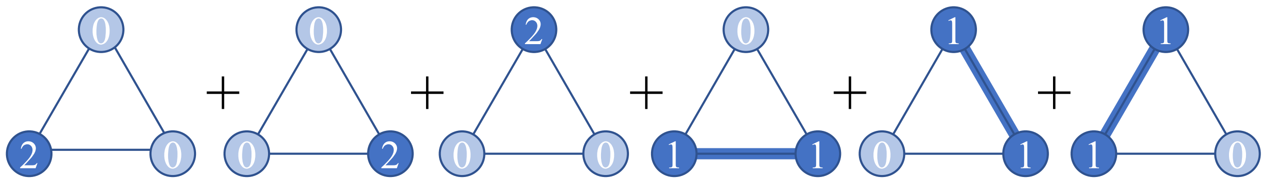
Numerical check

F_{m_x, m_y, m_z}

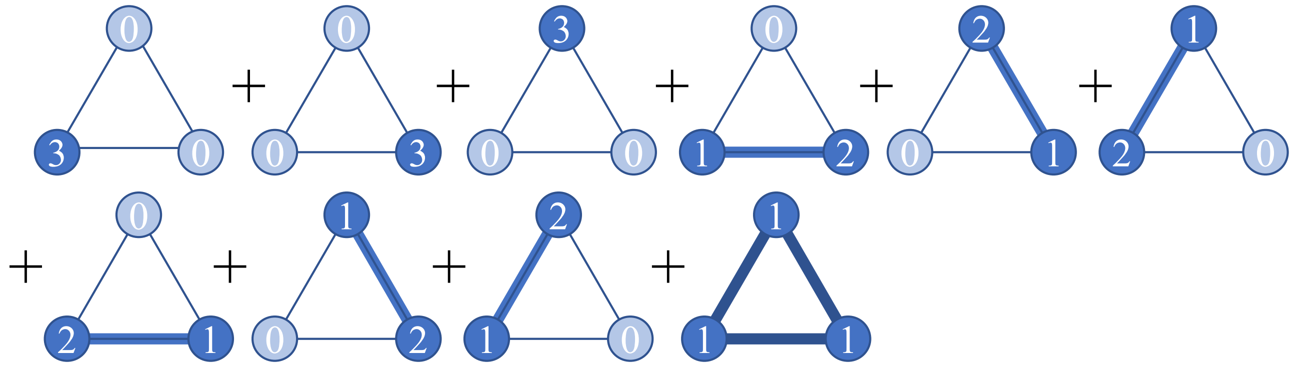
$I_{\text{single}} =$



$I_{\text{double}} =$



$I_{\text{triple}} =$



Numerical check

Inclusion of multiple-wrapping contributions gives

unrefinement:
$(q,p,x,y,z) = (t^{3/2}, t^{3/2}, t, t, t)$

$$I_{U(1)} = I_{\text{SUGRA}} + I_{\text{single}} + I_{\text{double}} + I_{\text{triple}} + \cdots$$

$$= 1 + 0t^{\frac{1}{2}} + 3t - 2t^{\frac{3}{2}} + 3t^2 + 0t^{\frac{5}{2}} + 0t^3 + 6t^{\frac{7}{2}} - 6t^4$$

$$+ 0t^{\frac{9}{2}} + 12t^5 - 18t^{\frac{11}{2}} + 27t^6 - 12t^{\frac{13}{2}} - 27t^7$$

$$+ 60t^{\frac{15}{2}} - 60t^8 + 24t^{\frac{17}{2}} + 76t^9 - 174t^{\frac{19}{2}} + 162t^{10}$$

$$+ 0t^{\frac{21}{2}} - 240t^{11} + 432t^{\frac{23}{2}} - 348t^{12} - 144t^{\frac{25}{2}}$$

$$+ 783t^{13} + \cdots$$

By summing up contributions with $n_1 + n_2 + n_3 \leq 3$ we find complete agreement up to q^{13} for $N = 1$. [YI, arXiv:2108.12090]

Expected error $I_{\text{quadruple}} \sim O(t^{20})$

Application to other systems (multiple-sum GGE)

AdS	CFT	Large N	Single-wrapping	Multiple-wrapping
$AdS_5 \times S^5$	4d N=4 SYM	[Kinney, Maldacena, Minwalla, Raju, 05]	✔[Arai, YI, 19]	✔[Arai, Fujiwara, YI, Mori, 20][YI, 21]
$AdS_5 \times S^5/Z_k^S$	4d N=3	[IY, Yokoyama, 16]	✔[Arai, YI, 19]	
$AdS_5 \times S^5/\Gamma$	4d N=1 quiver	[Nakayama, 05]	✔[Arai, Fujiwara, YI, Mori, 19]	
$AdS_5 \times SE_5$	4d N=1 quiver	[Nakayama, 06][Eager, Schmude, Tachikawa, 12][Agarwal, Amariti, Mariotti 13]	✔[Arai, Fujiwara, YI, Mori, 19]	(✔) [Fujiwara 23] (special limit)
$AdS_5 \times S_\alpha^5$	4d N=2 AD & MN	[Fayyazuddin, Spalinski, 98][Aharony, Fayyazuddin, Maldacena, 98]	✔[YI, Murayama, 21]	
$AdS_4 \times S^7/Z_k$	3d ABJM	[Bhattacharya, Bhattacharyya, Minwalla, Raju, 08][Kim, 09]	✔[Arai, Fujiwara, YI, Mori, Yokoyama, 20]	
$AdS_7 \times S^4$	6d (2,0)	[Bhattacharya, Bhattacharyya, Minwalla, Raju, 08]	✔[Arai, Fujiwara, YI, Mori, Yokoyama, 20]	(✔) [Arai, Fujiwara, YI, Mori, Yokoyama](special limit)
$AdS_7 \times S^4/Z_k$	6d (1,0)	[Ahn, Oh, Tatar, 98]	✔[Fujiwara, YI, Mori, 21]	

As far as we have checked the formula reproduces finite N index correctly!

Technical difficulty for $m \geq 2$ contributions

Although the GG expansion (with multiple sum) seems to work for many examples (including toric SE5, orientifolds, and S-folds), calculation of contribution from intersecting cycles is technically difficult.

Even the wrapping number is larger than 1, contribution from single-cycle (like $F_{m_x,0,0}$) can be relatively easily calculated.

Surprisingly, GG expansion with simple sum was proposed [Gaiotto, Lee, arXiv:2109.02545]

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m=0}^{\infty} x^{mN} F_{m,0,0} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{U(m)}$$

This can be true due to the "wall-crossing behavior" of F_{m_x,m_y,m_z}.

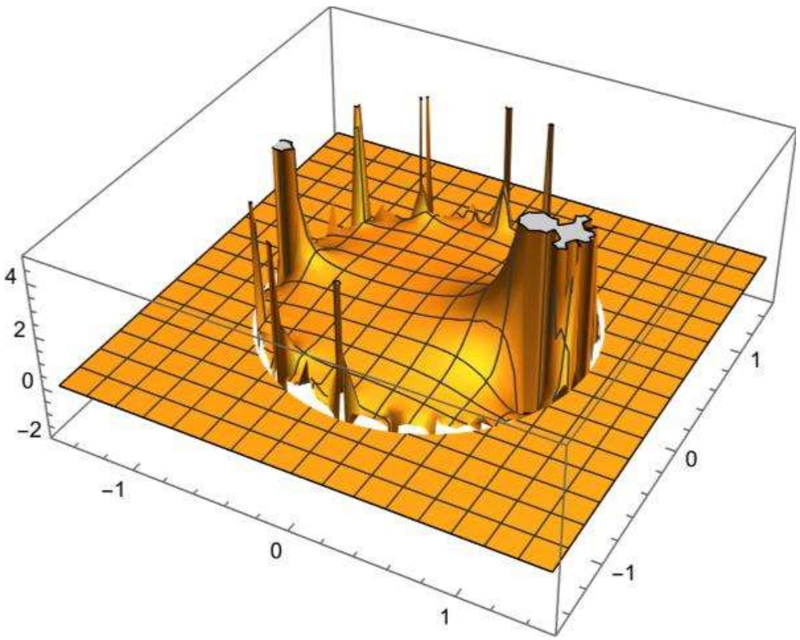
By choosing an appropriate "chamber" some contributions vanish.

Toy model with "wall crossing" behavior

$q + q^2 + q^3 + \cdots$

Toy model.

$$f(q) = \mathrm{Pexp}\left(\frac{1}{1-q}\right) = \frac{1}{(q;q)_\infty} = \prod_{k=1}^{\infty} \frac{1}{1-q^k}$$



q-expansion (Expansion around q = 0)

$$f(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots$$

q^{-1}(=:s)-expansion (Expansion around q = ∞)

$$f(q) = \prod_{k=1}^{\infty} \frac{1}{1-s^{-k}} = \prod_{k=1}^{\infty} \frac{-s^k}{1-s^k} = s^\infty + \cdots = 0$$

"$\mathrm{Pexp}(s^{-1} + s^{-2} + s^{-3} + \cdots)$

Degree assignment

To consider different expansion variables, it is convenient to introduce an auxiliary variable t, which has “degree” +1.

For a function of multiple variables x_i, we assign degree d_i for each variable, and replace x_i by $t^{d_i} x_i$, and then we carry out t-expansion (around $t = 0$).

$q \to tq \to t$-expansion

$\deg(q) = +1$

$$f(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + \cdots$$

$$f(q) = \mathrm{Pexp}\left(\frac{1}{1-q}\right)$$

$$f(q) = (q^{-1})^{\infty} + \cdots = 0$$

$\deg(q) = -1$

$q \to t^{-1} q \to t$-expansion

Criterion:

If there are infinitely many negative degree terms in the letter index, then its plethystic exponential vanishes.

Decoupling

For N=4 SYM the following degree assignment is convenient.

$$\deg(q,p,x,y,z)=(1,1,0,1,1)$$

This means we consider t-expansion after (q,p,x,y,z)→(tq,tp,x,ty,tz).

With this degree assignment we can show that $F_{m_x,m_y,m_z}=0$ for $m_y+m_z\geq 1$.

For example, $F_{0,1,0}=\text{Pexp}\left(\sigma_y f_{\text{vec}}\right)$

$$\sigma_y f_{\text{vec}}=1-\frac{(1-p)(1-y^{-1})(1-q)}{(1-z)(1-x)}=y^{-1}+xy^{-1}+x^2y^{-1}+\cdots$$

The letter index $\sigma_y f$ includes infinitely many negative-degree terms. → $F_{0,1,0}$ decouples

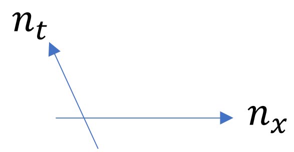
GG expansion with simple sum

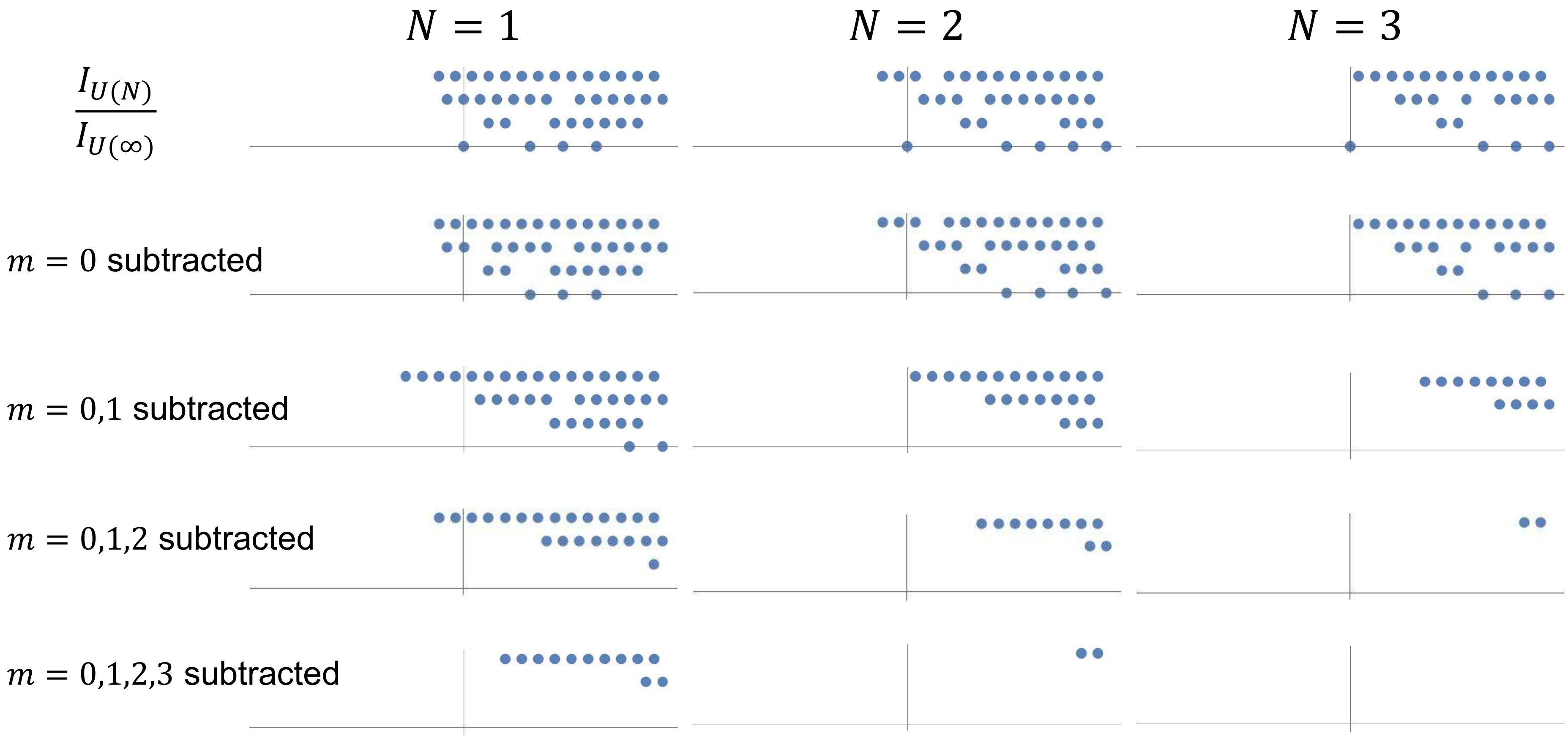
As the result of the decoupling, the multiple-sum GG expansion reduces to the simple-sum GG expansion. [Gaiotto, Lee, arXiv:2109.02545][YI, arXiv:2205.14615]

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m=0}^{\infty} x^{mN} F_{m,0,0} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{U(m)}$$

Numerical test

$$\frac{I_{U(N)}}{I_{U(\infty)}} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{U(m)}$$





The expansion correctly reproduces the index.

Generalizations

Example 1 : Orbifolds

Orbifolds

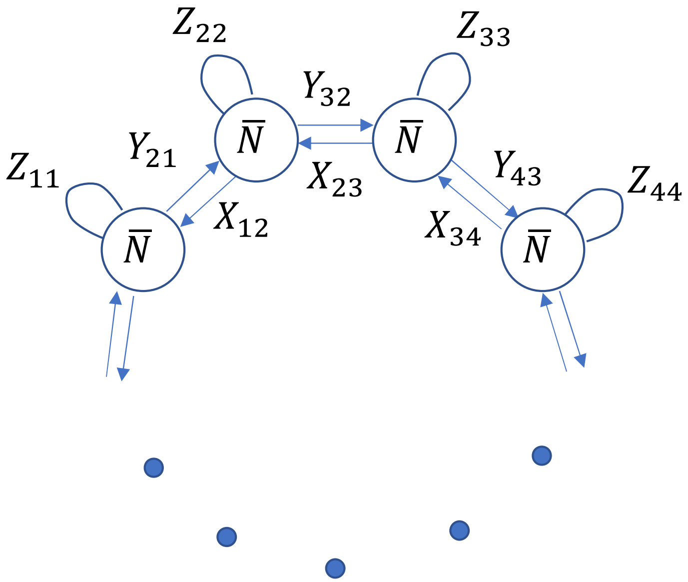
[Douglas and Moore, arXiv:hep-th/9603167]

$\mathbb{Z}_k$ orbifold

Z_k orbifold of N=4 U(N) SYM with

$$U_k = \exp\left(\frac{2\pi i}{k}(R_x - R_y)\right)$$

The boundary theory is the N=2 gauge theory with a circular quiver diagram.



The gauge group is $U(\bar{N})^k$ $\quad (N = k\bar{N})$

We denote this theory by $T_{\bar{N}}$

Insertion of U_k $\leftrightarrow$ $(q,p,x,y,z) \to (q,p,\omega_k x,\omega_k^{-1} y,z)$ $\left(\omega_k = \exp\frac{2\pi i}{k}\right)$

The letter index of the orbifold theory is given by the projection.

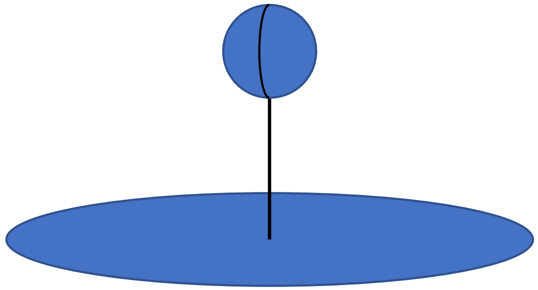
$$P_k\left[f_{\rm vec}\chi_{\rm adj}^{U(N)}\right] = \frac{1}{k}\sum_{\omega\in Z_k} f_{\rm vec}(\omega x,\omega^{-1}y,z,q,p)P_k\chi_{\rm adj}^{U(N)}$$

P_k also non-trivially acts on $\chi_{\rm adj}^{U(N)}$, and it breaks $U(N)$ to $U(\bar{N})^k$

The index is given by

$$I_{T_{\bar{N}}} = \int_G d\mu\ \mathrm{Pexp}\left(P_k\left[f_{\rm vec}\chi_{adj}^{U(N)}\right]\right)$$

Dual geometry



The large N index

$$I_{T_\infty} = \mathrm{Pexp}\left(\frac{x^k}{1-x^k} + \frac{y^k}{1-y^k} + \frac{kz}{1-k} - \frac{kq}{1-q} - \frac{kp}{1-p}\right)$$

(Gravity multiplet) + (k tensor multiplets on $AdS_5 \times S^1$)

Functions F_{m_x,m_y,m_z}

The contribution from GG system is given by

$$F_{m_x,m_y,m_z} = \int d\mu \, \mathrm{Pexp}(i_x[m_x] + i_y[m_y] + i_z[m_z] + \cdots)$$

where $i_x[m_x]$ is the letter index for m_x GGs wrapped around X=0, and given by

$$i_x[m_x] = P_k\left[\sigma_x f_{\mathrm{vec}} \chi_{adj}^{U(m_x)}\right]$$

Although the projection P_k removes some terms from the letter index, it still includes negative degree terms.

Decoupling

If we take the degree assignment $\deg(q,p,x,y,z)=(1{,}1{,}0{,}1{,}1)$

$i_y[m_y(\geq 1)]$ and $i_z[m_z(\geq 1)]$ satisfy the decoupling criterion.

⬇

$F_{m_x,m_y,m_z}=0$ for $m_y+m_z\geq 1$, and the GG expansion reduces to the simple sum associated with the cycle X=0.

$$\frac{I_{T_{\bar{N}}}}{I_{T_\infty}}=\sum_{\bar{m}=0}^{\infty} x^{k\bar{m}\bar{N}}\sigma_x I_{T^*_{\bar{m}}}$$

$T^*_{\bar{m}}$ is the theory realized on GG $\neq T_{\bar{m}}$

Theory on GG

The theory on GG wrapped around X=0 is Z_k orbifold defined by

$$U_k = \exp\left(\frac{2\pi i}{k}(R_x - R_y)\right) \quad \Rightarrow \quad U_k^* = \sigma_x U_k \sigma_x = \exp\left(\frac{2\pi i}{k}(-R_x - J_1)\right)$$

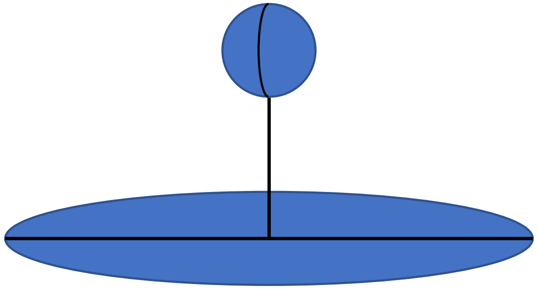
The index is given by the corresponding projection

$$I_{T_{\bar{m}}^*} = \int_G d\mu \ \mathrm{Pexp}\left(P_k^*\left[f_{\mathrm{vec}}\chi_{adj}^{U(m)}\right]\right)$$

This theory is N=4 U(N) SYM in S^3/Z_2, and the orbifolding breaks the gauge symmetry to $U(\bar{m})^k$. ($m = k\bar{m}$)

Dual geometry

S^5/Z_k with fixed S^3



AdS_5/Z_k with fixed AdS_3

$$U_k^* = \exp\left(\frac{2\pi i}{k}(-R_x - J_1)\right)$$

The large $\bar{m}$ limit of the index is

$$I_{T_\infty^*} = \mathrm{Pexp}\left(\frac{x^k}{1-x^k} + \frac{ky}{1-y} + \frac{kz}{1-k} - \frac{q^k}{1-q^k} - \frac{kp}{1-p}\right)$$

(Gravity multiplet) + (k tensor multiplets on $AdS_3 \times S^3$)

GG expansion for $T^*_{\bar{m}}$

We can consider giant graviton expansion of $T^*_{\bar{m}}$.

We can show the decoupling of two cycles for the degree assignment $\deg(q,p,x,y,z) = (1,1,0,1,1) \rightarrow$ GG expansion with simple sum.

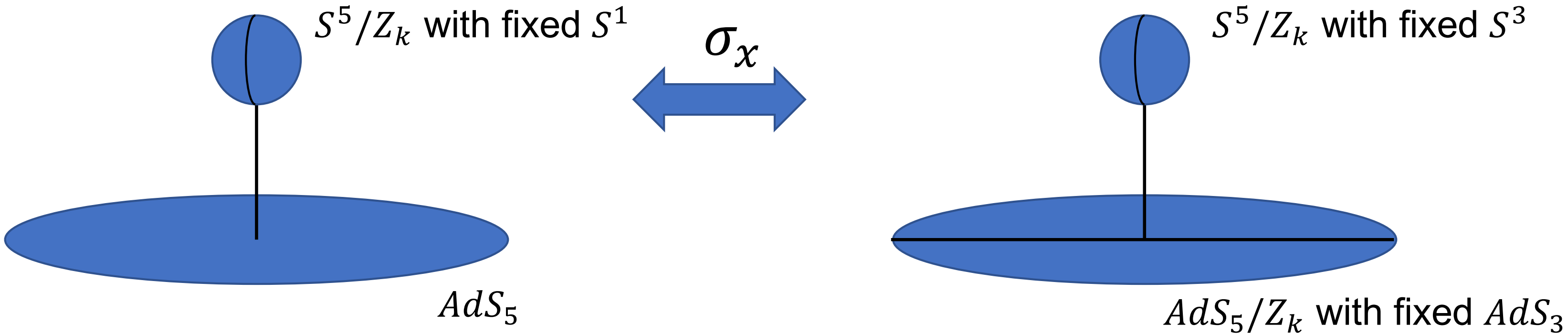
$$\frac{I_{T^*_{\bar{m}}}}{I_{T^*_{\infty}}} = \sum_{\bar{N}=0}^{\infty} x^{k\bar{m}\bar{N}} \sigma_x I_{T_{\bar{N}}}$$

The theory on GG is the original orbifold theory $T_{\bar{N}}$.
→ The GG expansion is invertible.

This is because σ_x is an involution.

The GG expansion is invertible

$$U_k = \exp\left(\frac{2\pi i}{k}(R_x - R_y)\right) \qquad U_k^* = \exp\left(\frac{2\pi i}{k}(-R_x - J_1)\right)$$



The boundary theories : $T_{\bar{N}}$ on $R \times S^3$.

The theories on GG : $T_{\bar{m}}^*$ on $R \times (S^3/Z_k)$.

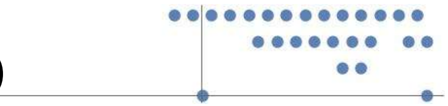
The boundary theories : $T_{\bar{m}}^*$ on $R \times (S^3/Z_k)$.

The theories on GG : $T_{\bar{N}}$ on $R \times S^3$.

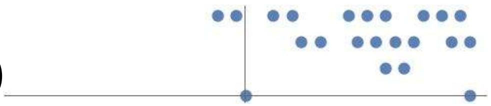
$$\frac{I_{T_{\bar{N}}}}{I_{T_\infty}} = \sum_{\bar{m}=0}^{\infty} x^{k\bar{m}\bar{N}} \sigma_x I_{T_{\bar{m}}^*} \qquad \frac{I_{T_{\bar{m}}^*}}{I_{T_\infty^*}} = \sum_{\bar{N}=0}^{\infty} x^{k\bar{m}\bar{N}} \sigma_x I_{T_{\bar{N}}}$$

Numerical test $(k = 2 \text{ case})$

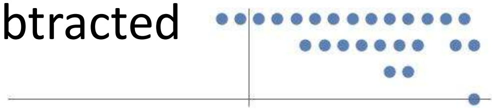
$\frac{I_{T_{\bar{N}}}}{I_{T_\infty}}$ $(\bar{N} = 2)$



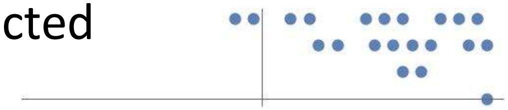
$\frac{I_{T^*_{\bar{m}}}}{I_{T^*_\infty}}$ $(\bar{m} = 2)$



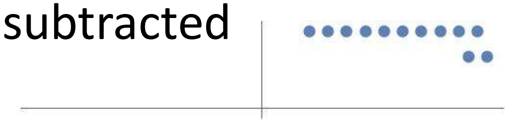
$\bar{m} = 0$ subtracted



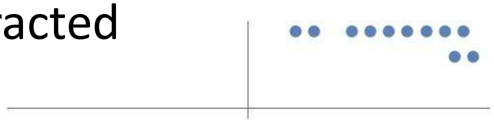
$\bar{N} = 0$ subtracted



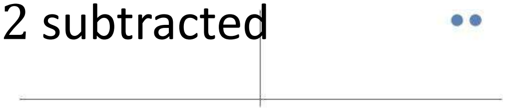
$\bar{m} = 0,1$ subtracted



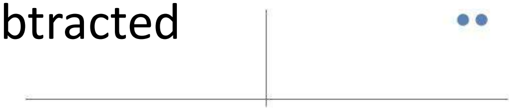
$\bar{N} = 0,1$ subtracted



$\bar{m} = 0,1,2$ subtracted



$\bar{N} = 0,1,2$ subtracted



The expansions works well!

U and SU

- In IR the diagonal U(1) factors of the $U(N)$ gauge groups become global symmetries (baryonic symmetries.)

- In the analysis above we treated the $U(1)$ factors as gauge symmetries. This is equivalent to pick up only contribution from sector with vanishing baryonic charges.

- This corresponds to "the equal-rank condition" on the other side of the relation.

- If you take the sector with non-vanishing baryonic charges, you have to include the contribution from GGs with different ranks.

$$I_{T_{\overline{N}}} \to I_{T_{\overline{N}}}^{[B_1,\dots,B_k]} \qquad\qquad I_{T^*_{\overline{m}}} \to I_{T^*_{m_1,\dots,m_k}}$$

Important remark

- For the simple-sum expansion to work, the decoupling of two cycles is necessary. This is NOT always the case.

- For example, in the previous example, we can use Y=0 instead of X=0 (because of $SU(2)_R$), while we cannot use Z=0. The projection P_k eliminates all negative degree terms from $\sigma_x f_{vec}$ and $\sigma_y f_{vec}$ and X=0 and Y=0 do not decouple. (no matter what degree assignment we use.)

Generalizations

Example 2: Orientifolds

Orientifolds

Let us consider N=4 SYM with orthogonal and symplectic gauge groups realized by the orientifolds (with $\mathrm{O3}^-$-plane).

$$U_{O3} = e^{\pi i S}, \qquad S = R_x + R_y + R_z + A$$

$A : SO(2)_R$ charge of type IIB SUGRA
$e^{\pi i A}$ is the worldsheet parity.

The boundary theory is $\mathcal{N}$=4 SYM with $O(2N)$.

Z_2 refinement

To study orientifold, it is convenient to define Z_2 refined index.

$$I(q,p,x,y,z,\eta) = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F q^{J_1} p^{J_2} x^{R_x} y^{R_y} z^{R_z} \eta^S\right], \qquad (qp = xyz)$$

$$S = R_x + R_y + R_z + A$$

Orientifold projection

$$P_{\mathrm{O3}}[\dots] = \frac{[\dots]_{\eta=+1} + [\dots]_{\eta=-1}}{2}$$

O3 projection

Taking account of the action on the Chan-Paton factor, we define Z_2-refined character

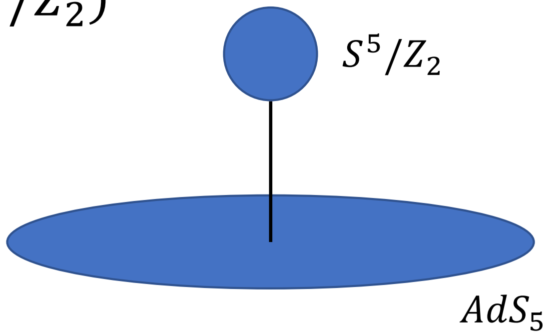
$$\chi^{\mathrm{ref}} = \chi_{adj}^{O(2N)} + \eta\chi_{adj}^{Sp(N)}$$

The index of N=4 O(2N) SYM is given by

$$I_{O(2N)} = \int_{O(2N)} d\mu \ \mathrm{Pexp}\left(P_{\mathrm{O3}}\left[f_{\mathrm{vec}}\chi^{\mathrm{ref}}\right]\right) = \int_{O(2N)} d\mu \ \mathrm{Pexp}\left(f_{\mathrm{vec}}\chi_{\mathrm{adj}}^{O(2N)}\right)$$

Dual geometry [Witten, hep-th/9805112]

The dual geometry is $AdS_5 \times (S^5/Z_2)$



$$I_{O(\infty)} = \mathrm{Pexp}\left(\mathrm{P_{O3}}\left[i_{\mathrm{KK}}^{\mathrm{ref}}\right]\right)$$

$$= \mathrm{Pexp}\left[\frac{1}{2}\left(\frac{x}{1-x}+\frac{y}{1-y}+\frac{z}{1-z}-\frac{q}{1-q}-\frac{p}{1-p}\right) + \frac{1}{4}\left(\frac{(1-x)(1-y)(1-z)(1+q)(1+p)}{(1+x)(1+y)(1+z)(1-q)(1-p)}-1\right)\right]$$

Variable change

As the $AdS_5 \times S^5$ case, the theory on GG can be obtained by using σ_x, σ_y, and σ_z.

Z_2 refinement of the variable change σ_x is given as follows. [Arai, YI, arXiv:1904.09776]

σ_x

$$(J_1, J_2) \leftrightarrow (R_y, R_z) \qquad (q, p) \leftrightarrow (\eta y, \eta z)$$

$$R_x \leftrightarrow -R_x \qquad x \leftrightarrow x^{-1}$$

$$A \leftrightarrow -A \qquad \eta \leftrightarrow \eta^{-1} = \eta$$

Functions F_{m_x,m_y,m_z}

$$F_{m_x,m_y,m_z} = \int d\mu \, \mathrm{Pexp}(i_x[m_x] + i_y[m_y] + i_z[m_z] + \cdots)$$

$$i_x[m_x] = P_{\mathrm{O3}}[\sigma_x(f_{vec}\chi)] \text{ etc.}$$

If we take the degree assignment $\deg(q,p,x,y,z) = (1,1,0,1,1)$

F_{m_x,m_y,m_z} with $m_y + m_z \geq 1$ decouple.

➡ GG expansion reduces to the simple sum

$$\frac{I_{O(2N)}}{I_{O(\infty)}} = \sum_{m=0}^{\infty} x^{2mN} \sigma_x I_{O(2m)^*},$$

O(2m)* is the theory on GG wrapped on X=0.

Theory on GG

The theories on GG are obtained by σ_x from U_{O3}.

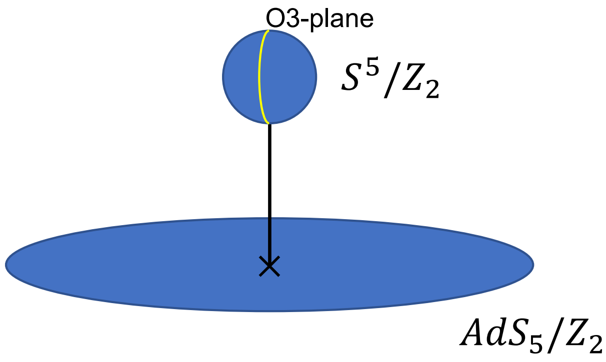
$$U_{O3} = e^{\pi i(R_x+R_y+R_z+A)} \xrightarrow{\sigma_x} U_{O3}^* = \sigma_x U_{O3} \sigma_x = e^{\pi i(-R_x+J_1+J_2-A)}$$

This gives another orientifold of $AdS_5 \times S^5$.

This is locally $\mathcal{N}$=4 SYM with $G = U(2m)$ in $S^3 \times R$, but G is broken by non-trivial holonomy to $O(2m)$.

We denote this theory by $O(2m)^*$.

Dual geometry of O(m)*



$$U_{O3}^{*} = e^{\pi i(-R_x+J_1+J_2-A)}$$

- The dual geometry is $(AdS_5 \times S^5)/Z_2$ with O3-plane wrapped around the internal space at the AdS center.
- The large N limit of the index can be obtained by the simple projection of KK modes.

$$\begin{aligned} I_{O(\infty)^*} &= \mathrm{Pexp}\left(P_{O3}^{*}\left[i_{\mathrm{KK}}^{\mathrm{ref}}\right]\right) \\ &= \mathrm{Pexp}\left[\begin{array}{l} \dfrac{1}{2}\left(\dfrac{x}{1-x}+\dfrac{y}{1-y}+\dfrac{z}{1-z}-\dfrac{q}{1-q}-\dfrac{p}{1-p}\right) \\ +\dfrac{1}{4}\left(\dfrac{(1-x)(1+y)(1+z)(1-q)(1-p)}{(1+x)(1-y)(1-z)(1+q)(1+p)}-1\right) \end{array}\right] \end{aligned}$$

GG expansion of O(2m)*

We can consider giant graviton expansion of $\mathrm{O}(2m)^*$.

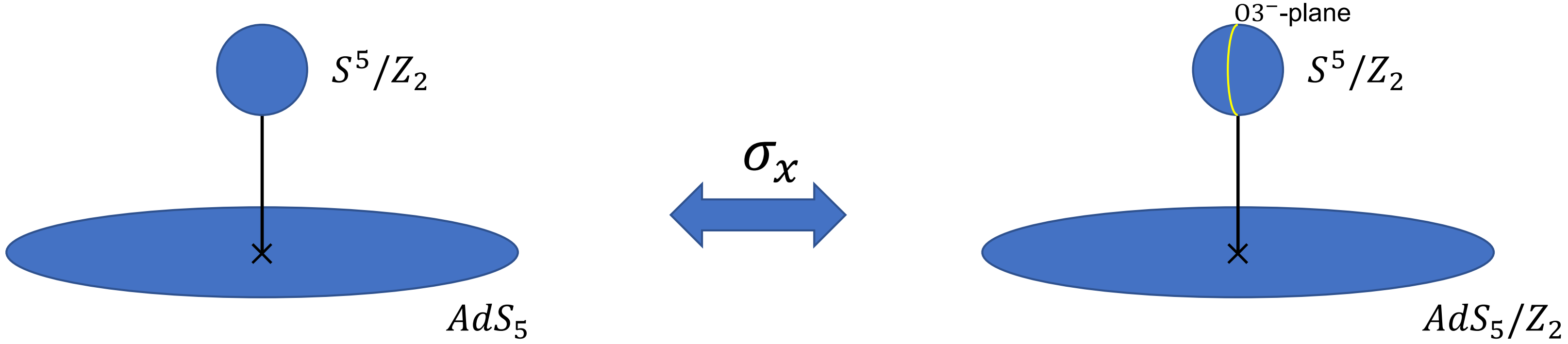
We can show the decoupling of two cycles for the degree assignment $\deg(q,p,x,y,z)=(1,1,0,1,1) \rightarrow$ GG expansion with simple sum.

$$\frac{I_{O(2m)^*}}{I_{O(\infty)^*}} = \sum_{N=0}^{\infty} x^{2mN} \sigma_x I_{O(2N)}$$

The theory on GG is the original orbifold theory $\mathrm{O}(2N)$.
→ The GG expansion is invertible.

GG expansion is invertible

$U_{O3} = e^{\pi i (R_x + R_y + R_z + A)}$ $\qquad$ $U^* = e^{\pi i (-R_x + J_1 + J_2 - A)}$



The boundary theories : $O(2N)$ on $R \times S^3$. $\qquad$ The boundary theories : $O(2m)^*$ on $R \times (S^3/Z_2)$.

The theories on GG : $O(2m)^*$ on $R \times (S^3/Z_2)$. $\qquad$ The theories on GG : $O(2N)$ on $R \times S^3$.

(O is replaced by Sp for $O3^+$)

$$\frac{I_{O(2N)}}{I_{O(\infty)}} = \sum_{m=0}^{\infty} x^{2mN} \sigma_x I_{O(2m)^*}, \qquad \frac{I_{O(2m)^*}}{I_{O(\infty)^*}} = \sum_{N=0}^{\infty} x^{2mN} \sigma_x I_{O(2N)}$$

GG expansions for $O3^{\pm}$

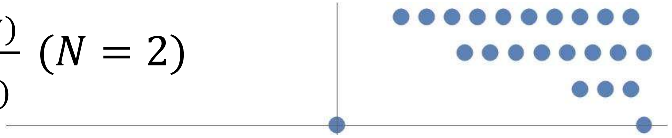
We found the following GGEs hold.

$$\frac{I_{G(2N)}}{I_{G(\infty)}} = \sum_{m=0}^{\infty} x^{2mN} \sigma_x I_{G(2m)^*}, \qquad \frac{I_{G(2m)^*}}{I_{G(\infty)^*}} = \sum_{N=0}^{\infty} x^{2mN} \sigma_x I_{G(2N)}$$

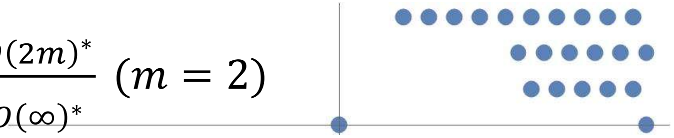
$(G = O \text{ or } USp)$

Numerical test (G=O)

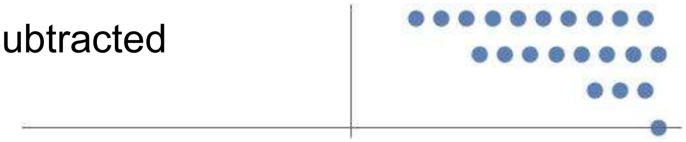
$\frac{I_{O(2N)}}{I_{O(\infty)}}$ $(N = 2)$



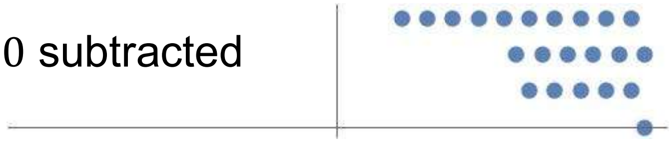
$\frac{I_{O(2m)^*}}{I_{O(\infty)^*}}$ $(m = 2)$



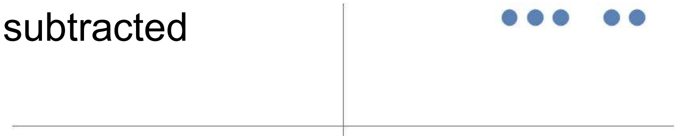
$2m = 0$ subtracted



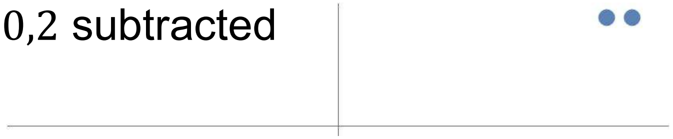
$2N = 0$ subtracted



$2m = 0{,}2$ subtracted

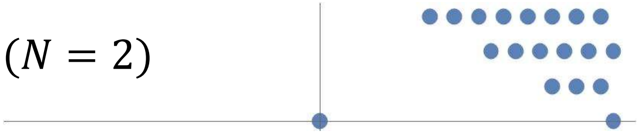


$2N = 0{,}2$ subtracted

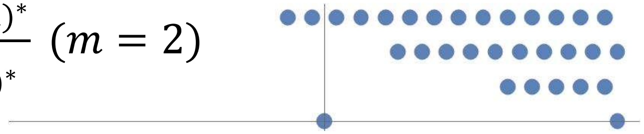


Numerical test (G=USp)

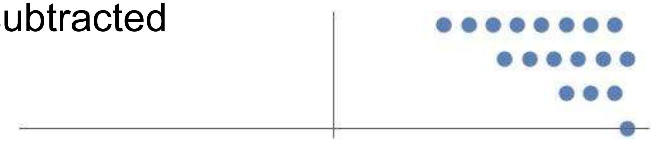
$\frac{I_{USp(2N)}}{I_{USp(\infty)}}\ (N=2)$



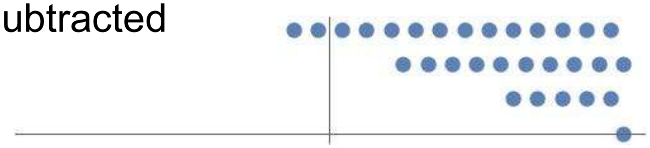
$\frac{I_{USp(2m)^*}}{I_{USp(\infty)^*}}\ (m=2)$



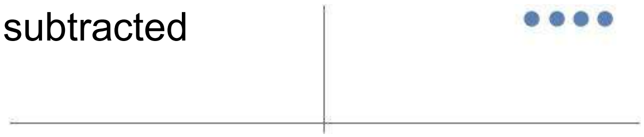
$2m=0$ subtracted



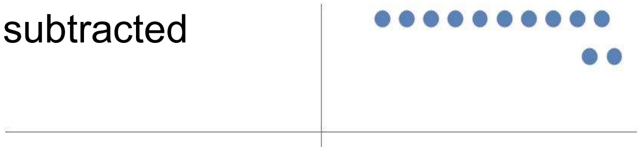
$2N=0$ subtracted



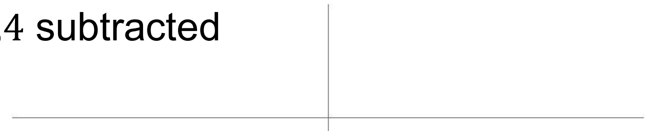
$2m=0,2$ subtracted



$2N=0,2$ subtracted



$2m=0,2,4$ subtracted



The expansion works well.

GG expansion of SO(2N)

SO(2N) O(2N)

We can regard $O(2N)$ as the Z_2 gauging of $SO(2N)$.

$$O(2N) = SO(2N) \rtimes Z_2$$

From the holographic viewpoint this comes from $H_3(S^5/Z_2, Z) = Z_2$

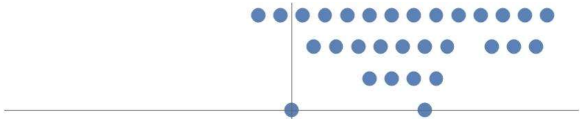
Ungauging of the Z_2 gives

$$\frac{I_{SO(2N)}}{I_{SO(\infty)}} = \sum_{m=0}^{\infty} x^{mN} \sigma_x I_{O(m)^*}$$

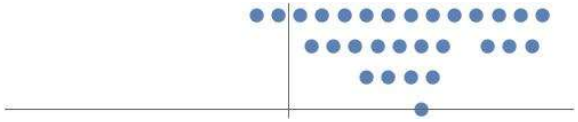
0, 1, 2, 3, ...

Numerical test (SO(2N))

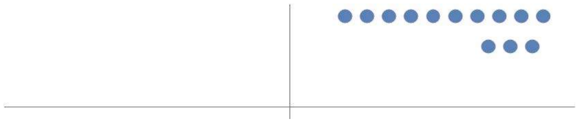
$$\frac{I_{SO(2N)}}{I_{SO(\infty)}} \ (N=3)$$



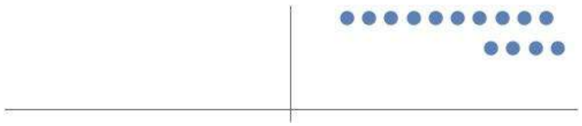
$m = 0$ subtracted



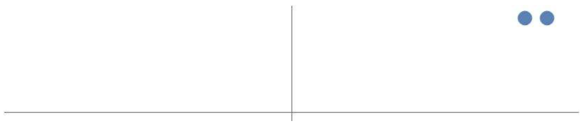
$m = 0,1$ subtracted



$m = 0,1,2$ subtracted



$m = 0,1,2,3$ subtracted



Discussion

Analytic proof of the invertibility?

For examples we have checked the simple-sum GGE always has the inverse.
Is this always true?

Can we show the latter from the former?

$$f_N(x) = f_\infty(x) \sum_m x^{mN}\, g_m(x^{-1}), \qquad g_m(x) = g_\infty(x) \sum_N x^{mN}\, f_m(x^{-1})$$

Naïve substitution

Naïve substitution gives

$$g_\infty(x)\sum_N x^{mN}\underbrace{\left(f_\infty(x^{-1})\sum_{m'} x^{-m'N}\, g_{m'}(x)\right)}_{f_N(x^{-1})}$$

$$= g_\infty(x) f_\infty(x^{-1}) \sum_{m'} g_{m'}(x)\left(\sum_N x^{N(m-m')}\right) \overset{?}{=} g_m(x)$$

This would hold if the following relations held, but they are not well-defined as they are.

$$g_\infty(x) f_\infty(x^{-1}) = 1, \qquad \sum_N x^{N(m-m')} = \delta_{m,m'}$$

$$(|x|<1)\ \ (|x|>1) \qquad\qquad (|x|=1)$$

Careful treatment of the analytic continuation will be necessary.

M-branes

As we saw, the simple-sum GG expansion is much more convenient than the multiple-sum GG expansion. To what extent is it applicable?

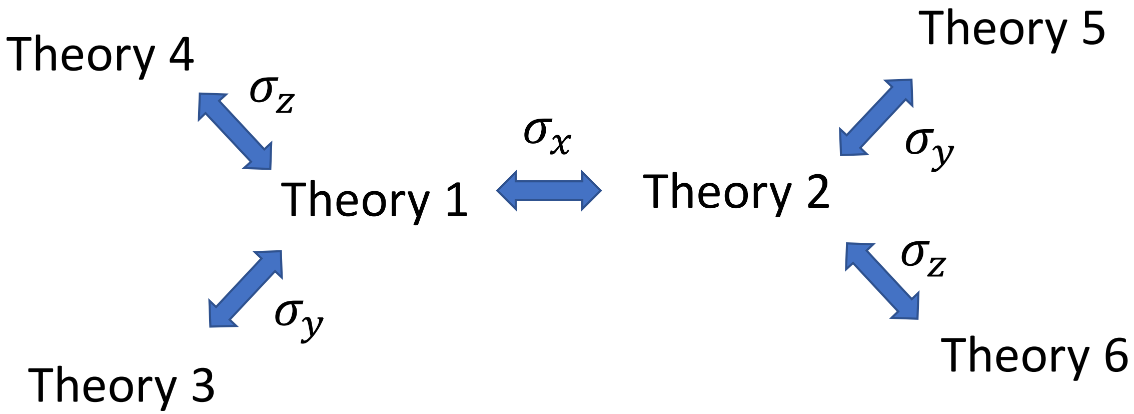
It works for other maximally supersymmetric theories like $AdS_4 \times S^7$ and $AdS_7 \times S^4$. [YI, arXiv:2205.14615]

$$\frac{I_{M5(N)}}{I_{M5(\infty)}} = \sum_{m=0}^{\infty} x^{mN} \sigma I_{M2(m)} \qquad \frac{I_{M2(N)}}{I_{M2(\infty)}} = \sum_{m=0}^{\infty} x^{mN} \sigma I_{M5(m)}$$

It may be interesting to consider orbifold of M2 and M5 theories, because GG expansion of an orbifold of M2 (M5) theory gives an orbifold of M5(M2) theories. This may be convenient to calculate SCI of N=(1,0) theory.

Web of GG-expansions

Simple-sum GG expansion relates a theory to another theory. We can use not only σ_x but also σ_y and σ_z to define the expansions. By repeating these expansions, we can connect many different theories. → Web of GG expansions



Other open problems

The multiple-sum GG expansion has technical difficulty associated with the pole selection in the localization formula. Can we derive it with localization?

To what extent does the simple-sum GG expansion work? It works in many cases of orbifolds and orientifolds. However, it does not work for more general toric SE5.

Similar, but different giant graviton expansions are proposed by Lee (based on the analysis on the gauge theory side) and Murthy (based on the character expansion formula). It is important to understand the relation among them.

For the BPS partition function of N=4 SYM, it is possible to reproduce the finite N result by using both sphere giants and AdS giants. Can we consider GG expansion associated with AdS giants?

… and more