New classical integrable systems from generalized $T\bar{T}$-deformation

Takato Yoshimura (University of Oxford)

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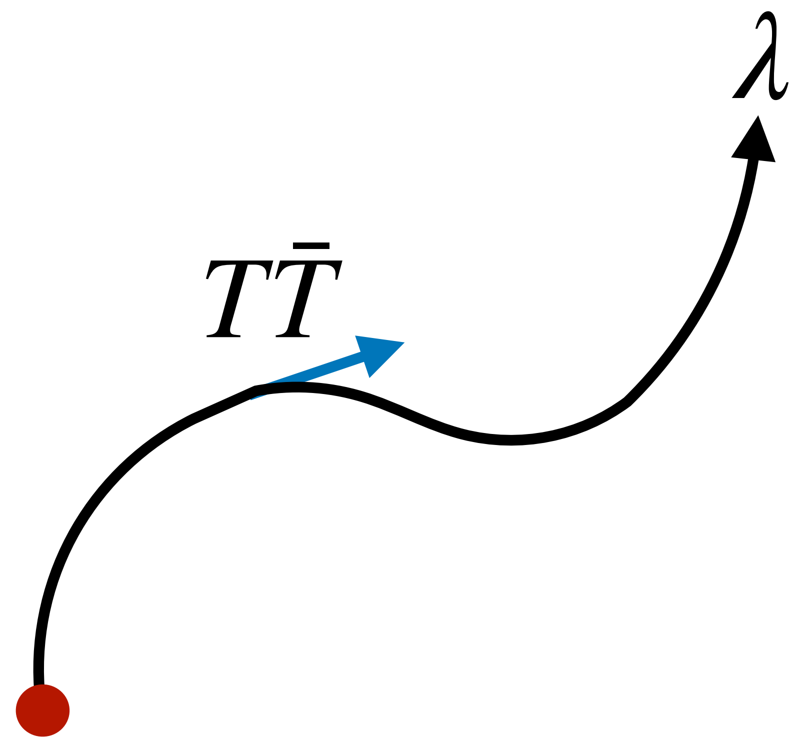
Based on arXiv:2311.06369 and 2312.14855
with **B. Doyon** and **F. Hübner** (Kings College London)



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Plan

- $T\bar{T}$-deformation and Generalised $T\bar{T}$-deformation
- Semiclassical Bethe systems: classical free particles deformed by generalised $T\bar{T}$-deformation
- Thermodynamics of the semiclassical Bethe systems
- Hydrodynamics of the semiclassical Bethe systems

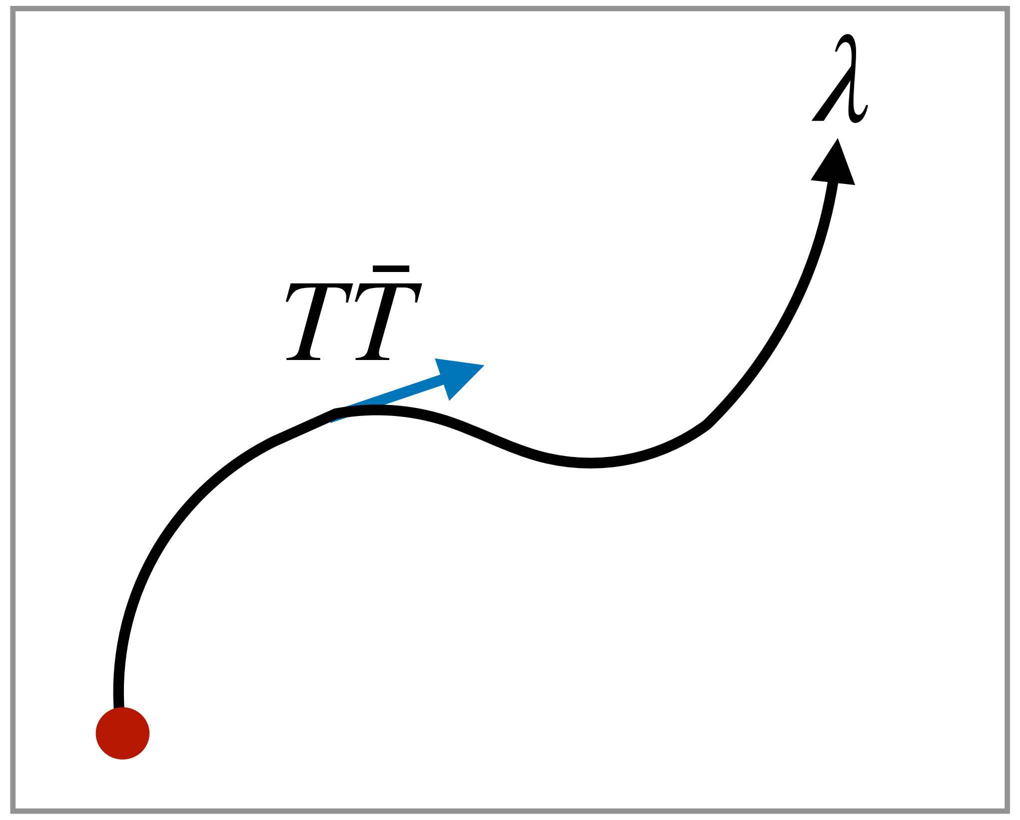
$T\bar{T}$-deformation: definition

$T\bar{T}$-deformation is a deformation of relativistic systems

$$H^{(\lambda+\delta\lambda)} = H^{(\lambda)} + \delta H^{(\lambda)}$$

[Zamolodchikov, 2004; Smirnov and Zamolodchikov, 2016; Cavaglià, Negro, Szécsényi, Tateo, 2016]

$$\delta H^{(\lambda)} = \delta\lambda \int \mathrm{d}x (q_1^{(\lambda)}(x-\epsilon) j_2^{(\lambda)}(x) - j_1^{(\lambda)}(x-\epsilon) q_2^{(\lambda)}(x)), \quad Q_1 = P, Q_2 = H$$



$T\bar{T}$-deformations can be obtained by the bilinear operator

$$X^{(\lambda)} = \int_0^L \mathrm{d}x \int_0^{x-\varepsilon} \mathrm{d}y q_1^{(\lambda)}(x) q_2^{(\lambda)}(y), \quad \frac{\delta Q_2^{(\lambda)}}{\delta\lambda} = \mathrm{i}[X^{(\lambda)}, Q_2^{(\lambda)}] + Q_1^{(\lambda)} j_2^{(\lambda)}(0) - j_1^{(\lambda)}(0) Q_2^{(\lambda)}$$

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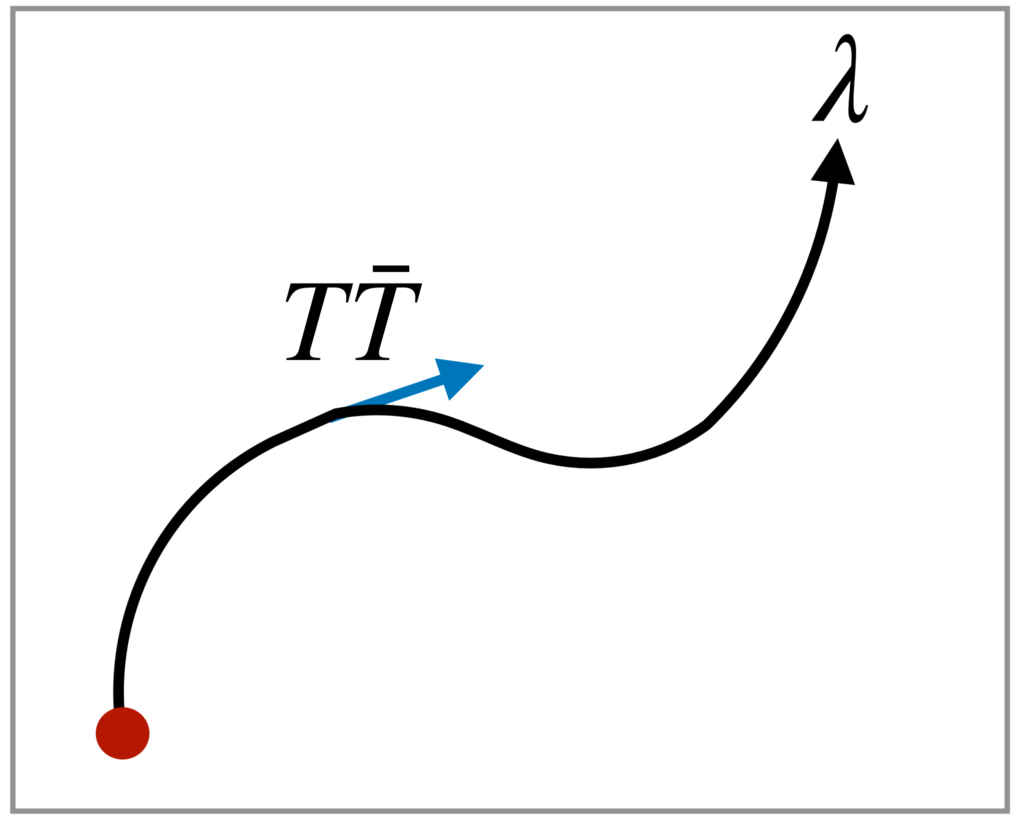

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In the nonrelativistic limit, the $T\bar{T}$-deformation reduces to the qj-deformation

$$\delta H^{(\lambda)} = \delta\lambda \int \mathrm{d}x (q_0^{(\lambda)}(x-\epsilon) j_1^{(\lambda)}(x) - j_0^{(\lambda)}(x-\epsilon) q_1^{(\lambda)}(x)), \quad Q_0 = N$$

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qj-deformation as a hard-rod deformation

[Cardy and Doyon, 2020; Jiang, 2020; Medenjak, Policastro, TY, 2020]

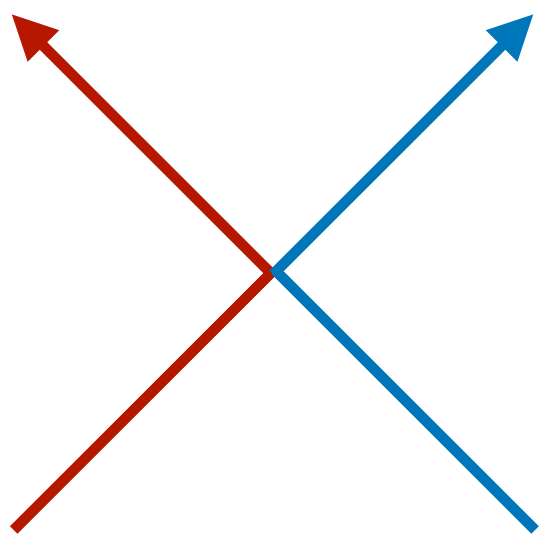
Consider a pair of classical free particles that are not allowed to cross in space (reflective potential)

$$H = \frac{p_1^2 + p_2^2}{2m} + V(x_1, x_2), \quad V(x_1, x_2) = \begin{cases} \infty & x_1 \geq x_2 \\ 0 & x_1 < x_2 \end{cases}$$

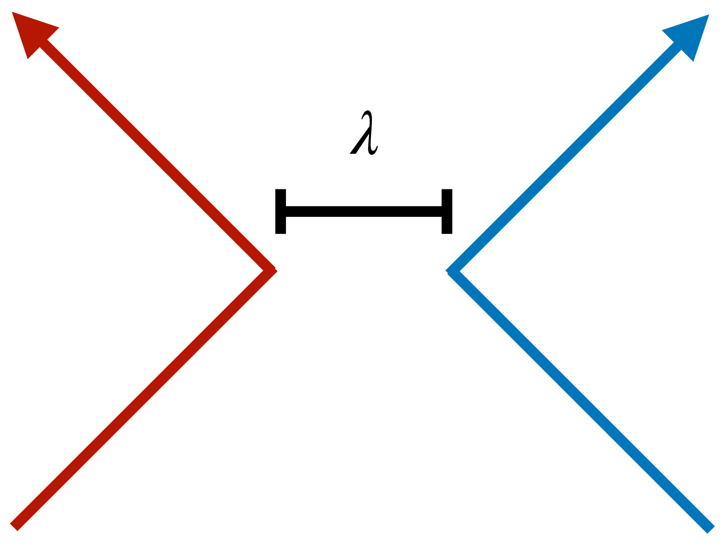
The bilinear generator is given by $X = -p_2$, which deforms the phase-space coordinates as

$$p_1^{(\lambda)} = p_1, \quad p_2^{(\lambda)} = p_2, \quad x_1^{(\lambda)} = x_1, \quad x_2^{(\lambda)} = x_2 - \lambda,$$

When $\lambda > 0$, the particles have less space to explore

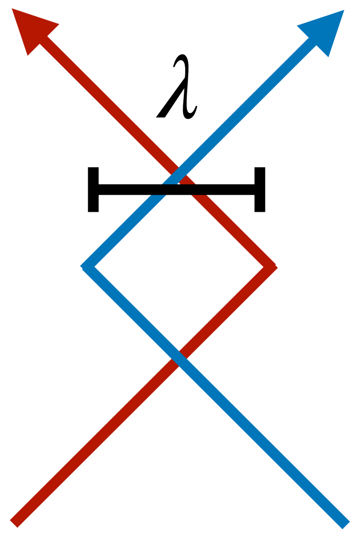


$V(x_1, x_2)$



$$V(x_1^{(\lambda)}, x_2^{(\lambda)}) = \begin{cases} \infty & \lambda \geq x_2 - x_1 \\ 0 & \lambda < x_2 - x_1 \end{cases}$$

When instead $\lambda < 0$, the particles have an extra space

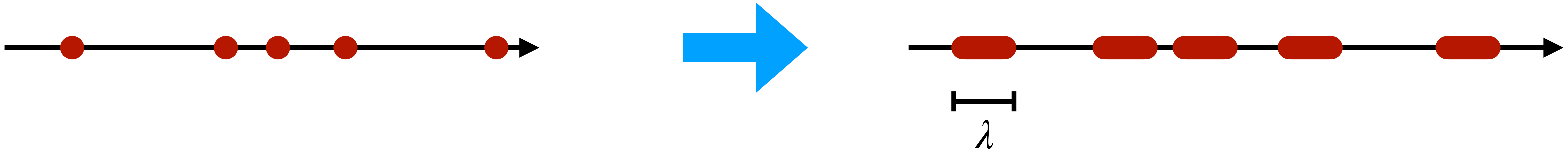


$$V(x_1^{(\lambda)}, x_2^{(\lambda)}) = \begin{cases} \infty & \lambda \leq x_1 - x_2 \\ 0 & \lambda > x_1 - x_2 \end{cases}$$

In the case of N-particles, the effective space is $L - \lambda N$

We can also interpret that these particles are jumping forward ($\lambda > 0$) or backward ($\lambda < 0$) as tracer particles (go-through picture)

The qj-deformation thus induces the nontrivial phase shift $\varphi = -\lambda$



Generalized $T\bar{T}$-deformation

The solvability of $T\bar{T}$-deformations is kept even after including other conserved charges Q_i in the system

$$\delta H^{(\lambda)} = \delta\lambda_{ij} \int \mathrm{d}x (q_i^{(\lambda)}(x-\epsilon) j_j^{(\lambda)}(x) - j_i^{(\lambda)}(x-\epsilon) q_j^{(\lambda)}(x))$$

[Smirnov and Zamolodchikov, 2016; Hernández-Chifflet, Negro, Sfondrini, 2020; Pozsgay, Jiang, Takács, 2020]

In integrable systems, one can also include all the quasi-local charges labeled by quasi-momenta θ

$$Q_\theta = \int \mathrm{d}x\, q_\theta(x), \quad q_\theta(x) = \sum_i \delta(\theta-\theta_i)\delta(x-x_i), \quad \partial_t q_\theta(x) + \partial_x j_\theta(x) = 0$$

[Ilievski, Quinn, Caux, 2017]

Generalized $T\bar{T}$-deformation is then given by

[Doyon, During, Yoshimura, 2022]

$$\delta H^{(\lambda)} = \delta\lambda \int \mathrm{d}\theta \mathrm{d}\alpha \mathrm{d}x \mathrm{d}x'\, w(x-x', \theta-\alpha)\left(q_\theta^{(\lambda)}(x) j_\alpha^{(\lambda)}(x') - j_\theta^{(\lambda)}(x) q_\alpha^{(\lambda)}(x')\right)$$

$$X_{\theta\phi}^{(\lambda)} = \int_0^L \mathrm{d}x \int_0^{x-\varepsilon} \mathrm{d}y q_\theta^{(\lambda)}(x) q_\phi^{(\lambda)}(y), \quad \frac{\delta Q_\gamma^{(\lambda)}}{\delta\lambda_{\theta\phi}} = \mathrm{i}[X_{\theta\phi}^{(\lambda)}, Q_n^{(\lambda)}] + Q_\theta^{(\lambda)} j_{\gamma\phi}^{(\lambda)}(0) - j_{\gamma\theta}^{(\lambda)}(0) Q_\phi^{(\lambda)}$$

where the generalized current $j_{\theta\phi}(x)$ satisfies $\mathrm{i}[Q_\theta, q_\phi(x)] + \partial_x j_{\theta\phi}(x) = 0$

Scatterings in generalized $T\bar{T}$-deformation

As in the standard one, generalized $T\bar{T}$-deformations induce a phase shift

$$S^{(\lambda)}(\theta) = e^{\mathrm{i}\phi(\theta)} S^{(0)}(\theta), \quad \phi(\theta) = 2\lambda \int \mathrm{d}x\, w_\theta(x,\theta)$$

[Doyon, Durning, Yoshimura, 2022]

- Conservation laws remain intact in quantum systems
- Thermodynamics is given by TBA (flow equations admit a unique solution that gives TBA)

$$\langle q_\theta \rangle = \frac{\delta f}{\delta \beta^\theta} \quad \langle j_{\theta\phi} \rangle = \frac{\delta g_\theta}{\delta \beta^\phi}$$

$$\frac{\delta f}{\delta \lambda_{\theta\phi}} = g_\theta \frac{\delta f}{\delta \beta^\phi} - g_\phi \frac{\delta f}{\delta \beta^\theta}, \quad \frac{\delta g_\gamma}{\delta \lambda_{\theta\phi}} = g_\theta \frac{\delta g_\gamma}{\delta \beta^\phi} - g_\phi \frac{\delta g_\gamma}{\delta \beta^\theta}$$

An explicit construction of the Hamiltonian is rather challenging in quantum systems

In the classical case, we have a full control over the microscopic details of the deformed theory!

Semiclassical Bethe systems

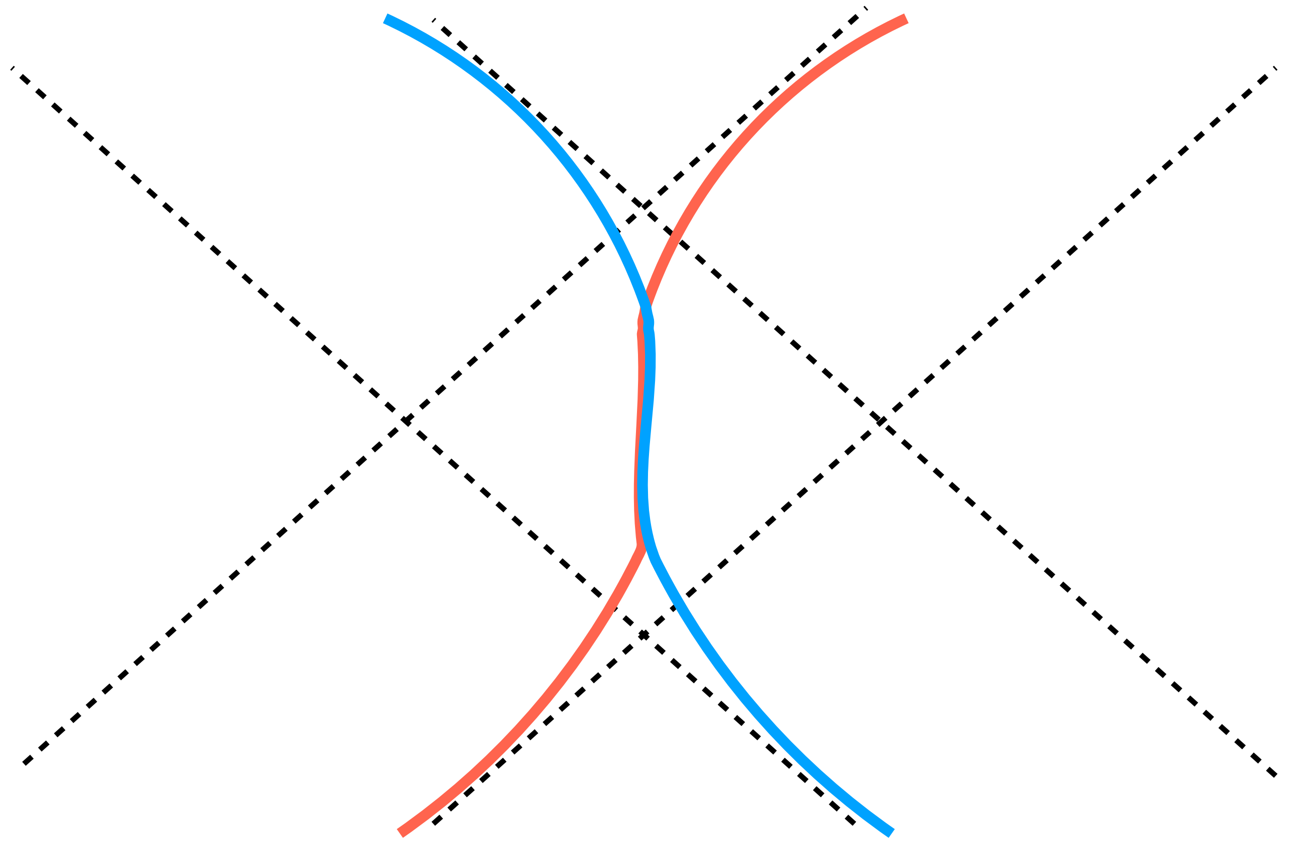
[Doyon, Hübner TY, 2023]

For a given function ψ, consider the phase space trajectories given by the equations

$$y_i = x_i + \sum_{j\neq i} \partial_\theta \psi(x_i - x_j, \theta_i - \theta_j)$$

$$p_i = \theta_i + \sum_{j\neq i} \partial_x \psi(x_i - x_j, \theta_i - \theta_j)\,.$$

$$\begin{cases} \psi(-x, -\theta) = \psi(x,\theta) \\ |x|\psi_x(x,\theta) \to 0, |x| \to \infty \end{cases}$$



and define the associated Hamiltonian

$$H^{[\psi]} = \sum_i \frac{1}{2}\theta_i^2(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i=1}^N \frac{p_i^2}{2} + V^{[\psi]}(\boldsymbol{x}, \boldsymbol{p})$$

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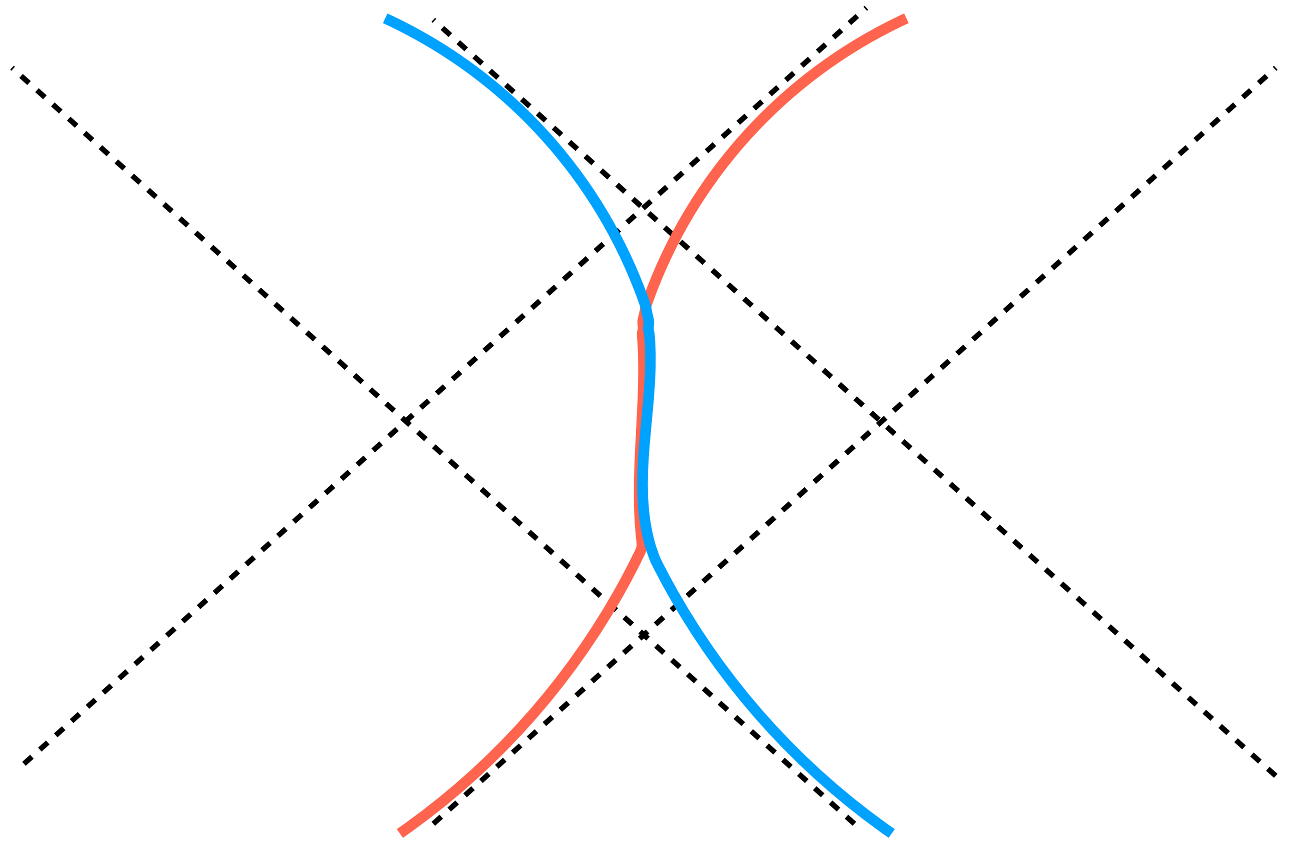

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The Hamiltonian of the generalized $T\bar{T}$-deformed $H^{(\lambda)}$ is given by

$$H^{(\lambda)} = H^{[\psi^\lambda]}, \quad \psi^\lambda(x,\theta) = \lambda \int_{-\infty}^{\infty} \mathrm{d}x' \,\mathrm{sgn}(x-x') w(x',\theta)$$

These equations can be thought of as a canonical coordinate transformation $(\boldsymbol{y}, \boldsymbol{\theta}) \mapsto (\boldsymbol{x}, \boldsymbol{p})$ induced by the generating function

$$\Phi^{[\psi]}(\boldsymbol{x}, \boldsymbol{\theta}) = \sum_i x_i \theta_i + \frac{1}{2} \sum_{i \neq j} \psi(x_i - x_j, \theta_i - \theta_j)$$

$$y_i = \partial_{\theta_i} \Phi(\boldsymbol{x}, \boldsymbol{p}), \quad p_i = \partial_{x_i} \Phi(\boldsymbol{x}, \boldsymbol{p})$$

- The Hamiltonian $H^{[\psi]}$ generates trajectories in the phase space $t \mapsto (\boldsymbol{x}(t), \boldsymbol{p}(t))$
- In terms of the $(\boldsymbol{y}, \boldsymbol{\theta})$ coordinates, the time-evolution is trivial: $\boldsymbol{y}(t) = \boldsymbol{y} + t\boldsymbol{\theta}, \boldsymbol{\theta}(t) = \boldsymbol{\theta}$
- The generating function is similar to the phase appearing in the semiclassical Bethe wave function in the Lieb-Liniger model $\Psi = e^{\mathrm{i}\Phi}$

[Doyon and Hübner, 2023]

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The semiclassical Bethe systems are Liouville integrable and finite-range interacting with the scattering shift

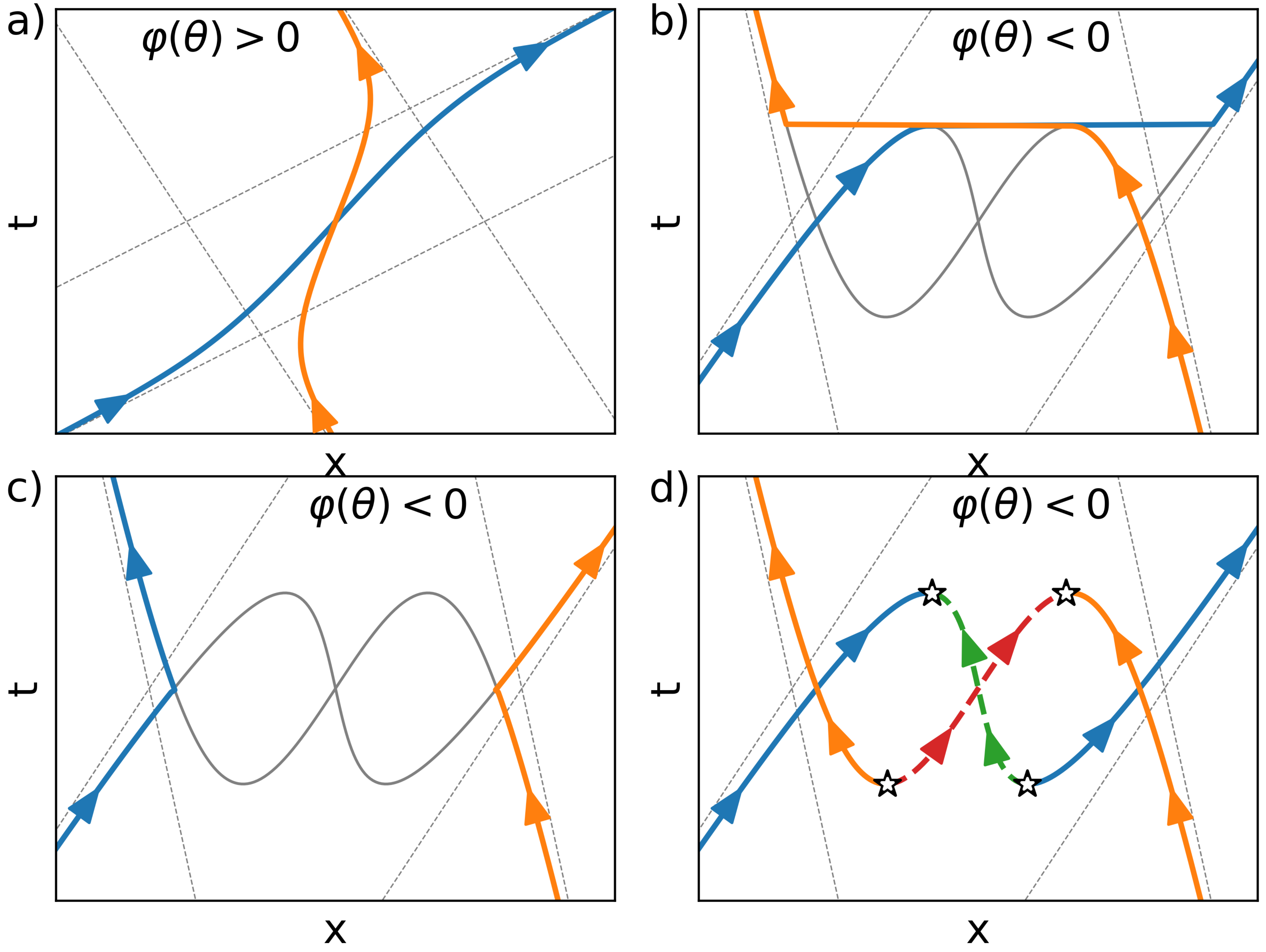
$$\varphi(\theta) = \lim_{x\to\infty}\left(\partial_\theta\psi(x,\theta) - \partial_\theta\psi(-x,\theta)\right)$$

The conserved charges $Q_i(\boldsymbol{x},\boldsymbol{p})$ can be constructed as

$$Q_n(\boldsymbol{x},\boldsymbol{p}) = \sum_i \theta_i^n(\boldsymbol{x},\boldsymbol{p}), \quad H = \frac{1}{2}Q_2(\boldsymbol{x},\boldsymbol{p})$$

Trajectories in the semiclassical Bethe systems

When $\varphi(\theta) > 0$, there is a unique trajectory. This is not necessarily the case if $\varphi(\theta) < 0$



"Go-through" picture

"Hard-core" picture

Particle-antiparticle creation

Time-symmetric

Corresponding **Hamiltonian**?

At large scales all of these trajectories give rise to the same dynamics

Thermodynamics of the semiclassical Bethe systems

We are interested in computing the free energy density of the system in a GGE $e^{-\sum_a \beta_a Q_a(\boldsymbol{x},\boldsymbol{p})}$

$$f_L = -\frac{1}{L}\log Z_L, \quad Z_L = \sum_{N=0}^{\infty}\int \frac{\mathrm{d}^N x \mathrm{d}^N p}{(2\pi)^N N!} e^{-\sum_{a=0}^{b} \beta_a Q_a(\boldsymbol{x},\boldsymbol{p})}, \quad b \in \mathbb{Z}_{\geq 0}$$

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We first rewrite the finite-volume partition function Z_L as

$$Z_L = \sum_{N=0}^{\infty}\frac{1}{(2\pi)^N N!}\int \prod_{j=1}^{N} \mathrm{d}x_j \mathrm{d}\theta_j \,|\,\Gamma(\boldsymbol{x},\boldsymbol{\theta})\,|\, e^{-\sum_{i=1}^{N}\beta(\theta_i)}, \quad \beta(\theta) = \sum_a \beta_a \theta^a$$

$\Gamma_{ij} = \delta_{ij} + L_{ij}$ is the Gaudin matrix where L_{ij} is a Laplace matrix

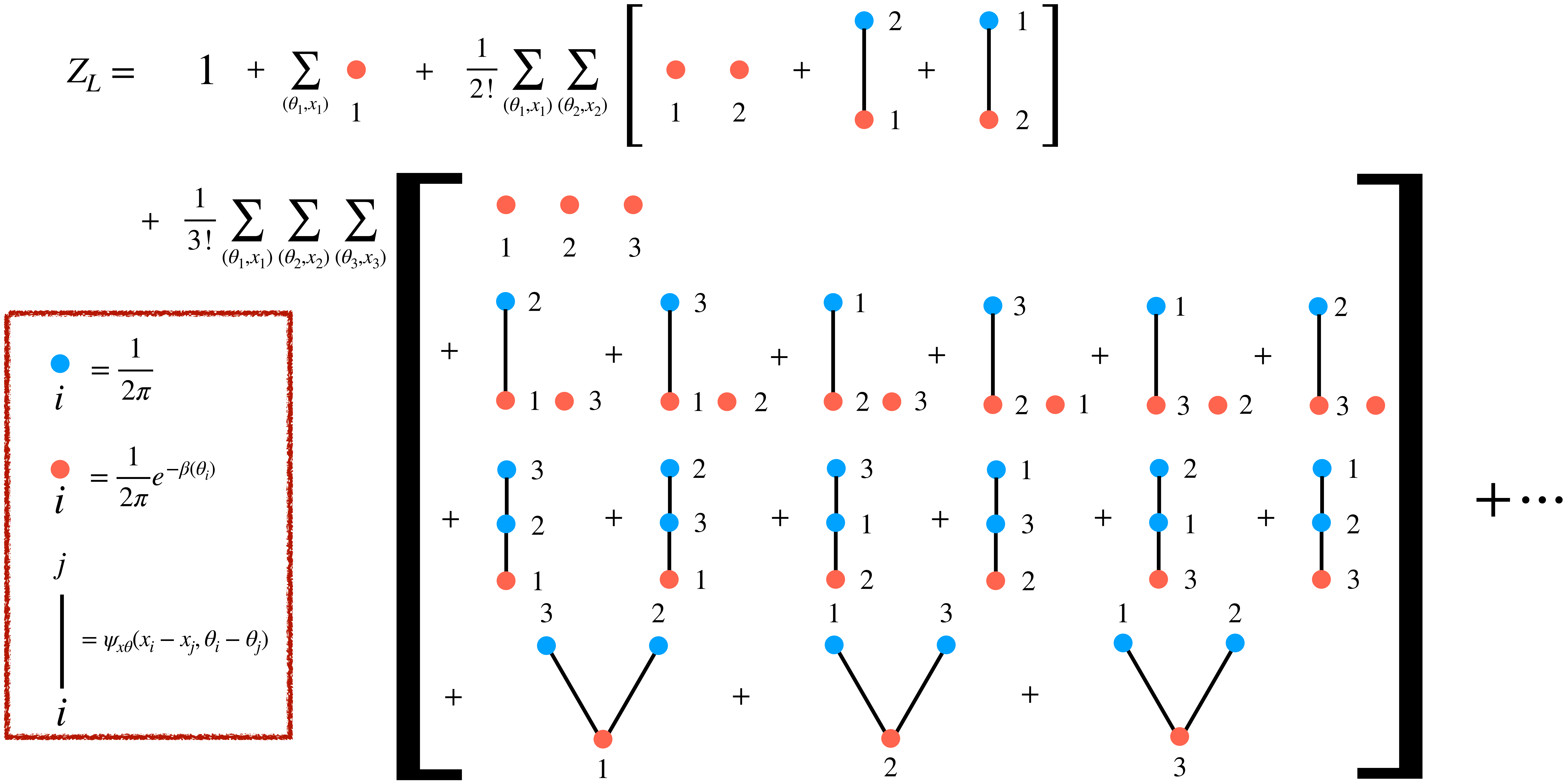
$$L_{ij} = \delta_{ij}\sum_{k\neq i}\psi_{x\theta}(x_{ik},\theta_{ik}) - (1-\delta_{ij})\psi_{x\theta}(x_{ij},\theta_{ij})$$

Matrix tree theorem allows the Gaudin determinant $|\,\Gamma\,|$ to be written as a sum of spanning forests made of N vertices

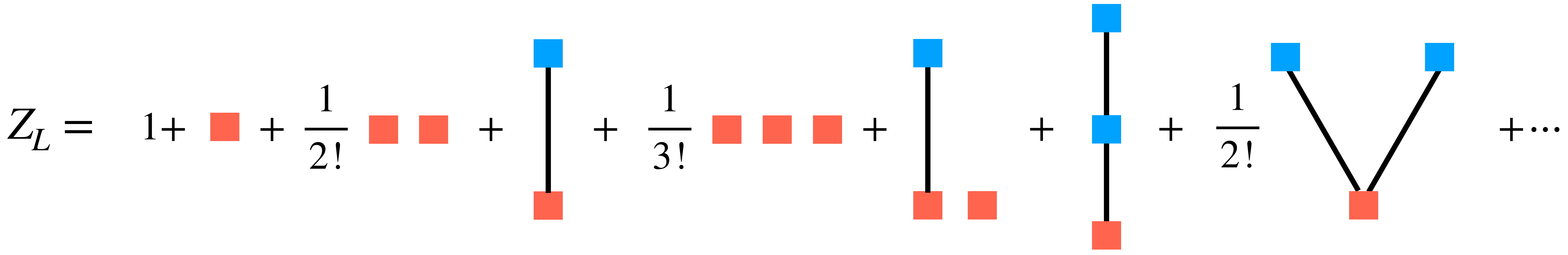
[Chiken and Kleitman, 1978]

$$|\,\Gamma\,| = \sum_{\alpha\subset\{1,\cdots,N\}}\ \sum_{F\in\mathcal{F}_\alpha}\ \prod_{\langle jk\rangle\in F}\psi_{x\theta}(x_{jk},\theta_{jk})$$

The partition function can be expressed graphically as



Upon the integration over the phase space, we get

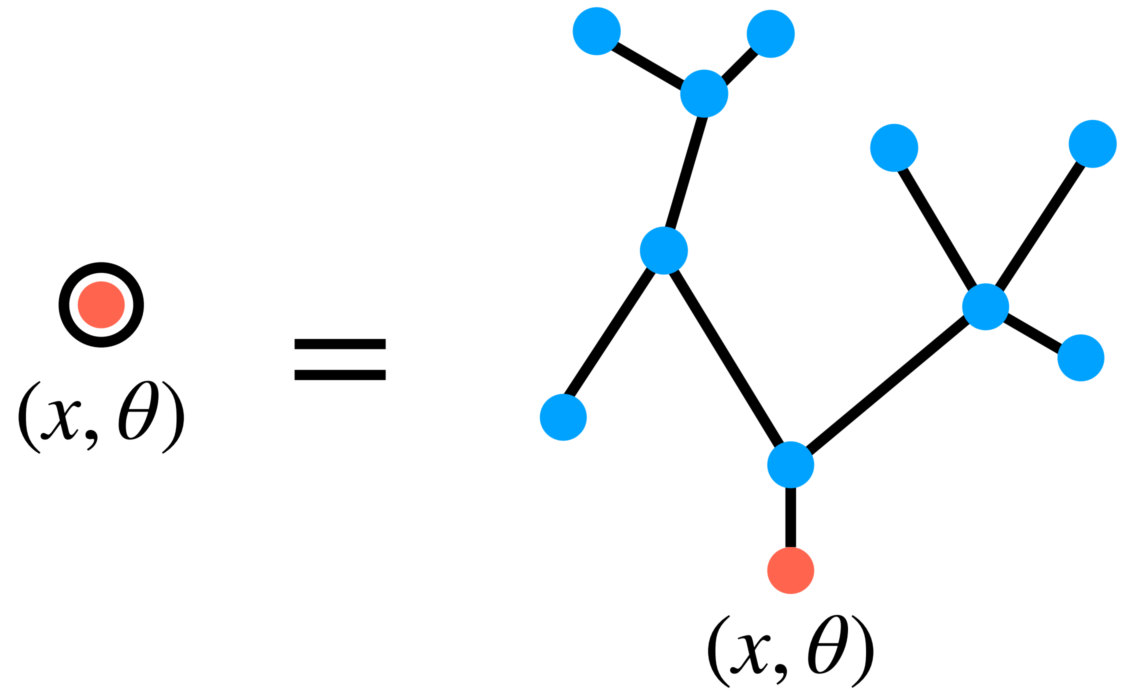


$$= \sum_{M=0}^{\infty} \sum_{\substack{n_1, n_2, \cdots \\ \sum_m n_m = M}} \frac{1}{n_1!} V_1^{n_1} \frac{1}{n_2!} V_2^{n_2} \cdots \frac{1}{n_M!} V_M^{n_M} = \exp\left(\sum_{j=1}^{\infty} V_j \right)$$

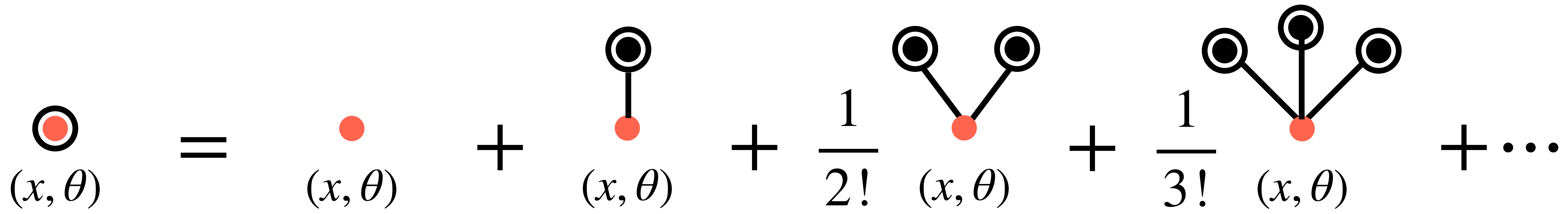
$$= \exp\left[\int_{[-L/2, L/2]\times\mathbb{R}} \frac{\mathrm{d}x\mathrm{d}\theta}{2\pi} Y_L(x, \theta) \right]$$

where $Y_L(x, \theta)$ is the generating function of the spanning tree rooted at (x, θ)

$Y_L(x, \theta) =$



The generating function $Y_L(x,\theta)$ satisfies the Schwinger-Dyson equation



This amounts to the integral equation for the pseudo-energy defined by $Y_L(x,\theta) = e^{-\varepsilon_L(x,\theta)}$

$$\varepsilon_L(x,\theta) = \beta(\theta) - \frac{1}{2\pi}\int_{[-L/2,L/2]\times\mathbb{R}} \mathrm{d}x'\mathrm{d}\theta'\, \psi_{x\theta}(x-x',\theta-\theta')e^{-\varepsilon_L(x',\theta')}$$

The finite-volume free energy density is given by $f_L = -(2\pi L)^{-1}\int_{[-L/2,L/2]\times\mathbb{R}} \mathrm{d}x\mathrm{d}\theta\, e^{-\varepsilon_L(x,\theta)}$

In the infinite volume limit $L \to \infty$, the integral equation reduces to the TBA equation for $\varepsilon(x,\theta) = \lim_{L\to\infty} \varepsilon_L(x,\theta)$

$$\varepsilon(\theta) = \beta(\theta) - \frac{1}{2\pi}\int \mathrm{d}\theta'\,\varphi(\theta-\theta')e^{-\varepsilon(\theta')}$$

- Works also in the presence of a trap

Hydrodynamics of the semiclassical Bethe systems

Generalized hydrodynamics (GHD) emerges at large space-time in the semiclassical Bethe systems

To see it, we take macroscopic space and time $x = L\bar{x}, t = L\bar{t}$ with the rescaled coordinates $\bar{x}_i(t) = x_i(t)/L, \bar{y}_i(t) = y_i(t)/L$. The "empirical density"

$$\rho_{\mathrm{e}}(\theta, \bar{x}, \bar{t}) = L^{-1} \sum_i \delta(\bar{x} - \bar{x}_i(\bar{t}))\delta(\theta - \theta_i) \xrightarrow{L\to\infty} \rho_{\mathrm{p}}(\theta, x, t)$$

satisfies

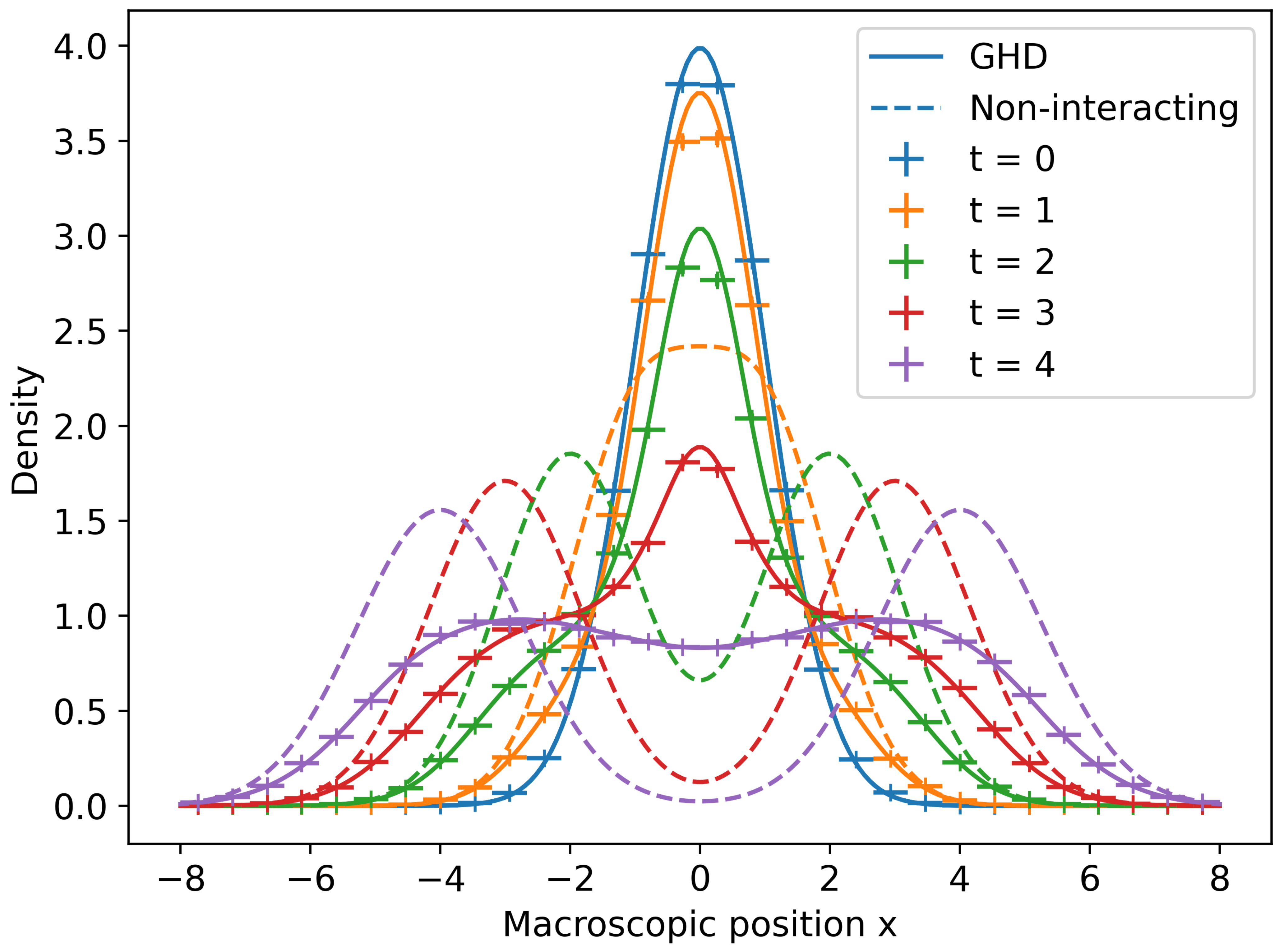
$$\partial_{\bar{t}}\rho_{\mathrm{e}}(\theta, \bar{x}, \bar{t}) + \partial_{\bar{x}}\Big(L^{-1} \sum_i \dot{\bar{x}}_i \delta(\bar{x} - \bar{x}_i)\delta(\theta - \theta_i)\Big) = 0$$

Using the equation for trajectories it follows that $\dot{\bar{x}} \to v^{\mathrm{eff}}_{[\rho_{\mathrm{p}}(\cdot,\bar{x},\bar{t})]}$ as $L \to \infty$. The GHD equation therefore emerges in the limit of $L \to \infty$:

$$\partial_{\bar{t}}\rho_{\mathrm{p}}(\theta, \bar{x}, \bar{t}) + \partial_{\bar{x}}\Big(v^{\mathrm{eff}}_{[\rho_{\mathrm{p}}]}(\theta, \bar{x}, \bar{t})\rho_{\mathrm{p}}(\theta, \bar{x}, \bar{t})\Big) = 0.$$

[Castro-Alvaredo, Doyon, TY, 2016;
Bertini, Collura, De Nardis, Fagotti, 2016]

We also numerically verified the agreement with GHD



$$L = 300, \quad N \approx 3000$$

$$\rho_{\rm p}(\theta, x, 0) = \frac{25}{2\pi} e^{-x^2/2} \left(e^{-25(\theta-1)^2/2} + e^{-25(\theta+1)^2/2} \right)$$

$$\psi(x, \theta) = \frac{x}{2\sqrt{x^2+\alpha^2}} \phi(\theta), \quad \phi(\theta) = 2 \arctan \frac{\theta}{c}$$

- Hydrodynamics of the semiclassical Bethe systems is expected reproduce all the higher derivative contributions in the classical integrable system with the same phase shift
- The trajectories of the semiclassical Bethe systems can capture the locations of solitons in the soliton gas

Conclusion

- A novel class of classical integrable systems can be obtained from generalized $T\bar{T}$-deformations
- The explicit Hamiltonian and trajectories can be obtained
- The trajectories with the negative phase shift admits an interpretation in terms of particle-antiparticle pair creations
- Thermo/hydrodynamics are described by TBA and GHD as in standard integrable systems

Outlook

- External potential
- Rigorous proof of the emergence of GHD