Non-Invertible Duality Symmetries via Relative QFTs and “Polarization Pairs”

Based on 2306.11783
with Craig Lawrie and Xingyang Yu

Hao Y. Zhang / 张昊

University of Pennsylvania → Kavli IPMU, University of Tokyo

Sep. 27, 2023 @ BIMSA / Joint HEP-TH Seminars

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Global Symmetries

Symmetries are the most fundamental properties of a QFT.

- E.g., every conservation law has an underlying symmetry (Noether)
- E.g., discrete symmetries $\rightarrow$ selection rules, anomaly constraints

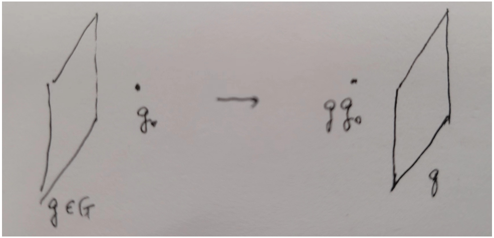
Robust across different descriptions, does not require a Lagrangian

Usually: a group acting on operators/excitations of a QFT.

Generalized Global Symmetries

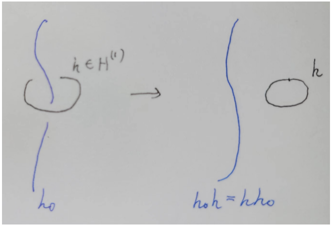
[GKSW '14] redefines global symmetries by topological operators.

When a topological operator of a G symmetry crosses the charged operator, it does the $g \in G$ transformation.



Higher Form Symmetries

Higher ($n > 0$) form symmetry via codim-($n+1$) top ops. Abelian. charging n-dimensional operators/excitations.



- E.g.: 4D U(1) theory has $U(1)_e^{(1)} \times U(1)^{(1)}$
- 4D $SU(N)$ theory ($\mathbb{Z}_N^{(e)}$) vs $SU(N)/\mathbb{Z}_N$ theory ($\mathbb{Z}_N^{(m)}$)

Remark: both $SU(N)$ and $PSU(N)$ have identical gauge dynamics (of $\mathfrak{su}(n)$), but differ in extended operators!

Later, we will explain that $SU(N)$ and $PSU(N)$ absolute theories both come from the $\mathfrak{su}(N)$ relative theory

Gauging a (Discrete) Higher-Form Symmetry

Gauging a continuous symmetry: coupling the gauge field to a conserved current.

Gauging a discrete $n \geq 0$-form symmetry $G^{(n)}$: summing over all G-values of the $(n+1)$-form bkf field B.

$$Z[B] \xrightarrow{\text{Gauge } G^{(n)}} \sum_{B \in H^{n+1}(M,G)} e^{i \int_M b \wedge B} Z[B] \tag{1}$$

Applies to both ordinary and higher-form symmetries. [Vafa '89][GKSW '14])

In order to gauge a symmetry, it has to be *non-anomalous* in the first place.

Physically: sum over all insertions of $G^{(n)}$ symmetry ops. in M.

Higher gauging: sum over $G^{(n)}$ symmetry ops. on a *submanifold* $N \subset M$. [Roumpedakis, Seifnashri, Shao '22]

Non-Invertible Symmetries

Relax the condition that symmetries obey group laws, s.t. a symmetry operator might not have an inverse.

Then, the general math structure is a fusion category $\mathcal{C}$.

- e.g., Tambara-Yamagami category: $\{1, \eta, \mathcal{N}\}$ s.t.

$$\eta^2 = 1, \quad \eta\mathcal{N} = \mathcal{N}\eta = \mathcal{N}, \quad \mathcal{N}^2 = 1 + \eta \tag{2}$$

  (so $\mathcal{N}$ has no inverse).

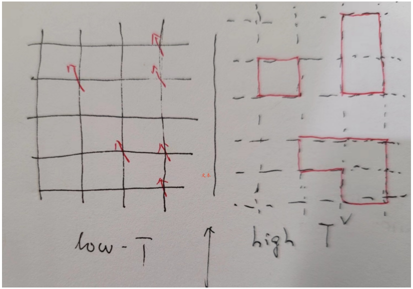
Physical realization? "duality defects"!

Lattice: Kramer-Wannier Duality

Discrete story: 2D Ising model.

$$E = -\frac{1}{2} J \sum_{n.n.} \sigma_i \sigma_k - mH \sum_i \sigma_i \tag{3}$$

Krammer-Wannier duality: Ordered states $\leftrightarrow$ disordered states



Exact match of Z_{low-T} with Z_{hi-T}. [Kramers, Wannier '41]

Lattice: Non-invertible duality defect in 2D Ising Models

$K = \frac{J}{2k_B T}, \quad K^* = \frac{J}{2k_B T^*}, \quad$ s.t. $\sinh(2K)\sinh(2K^*) = 1$

Criticality @ $k_B T_c = \frac{2J}{\ln(1+\sqrt{2})}$, where $T = T^\vee = T_c$.

Two ingredients to build non-invertible topological line:

- I_{KW}: duality interface within T_c system! (Ordered σ to disorder μ state.)
- Gauge $\mathbb{Z}_2$ spin flip symmetry ($\Sigma_{\mathbb{Z}_2}$)
  - Doing once gives a dual $\mathbb{Z}_2^\vee$ symmetry
  - Doing twice projects onto the "parity-even" sector.

Combined interface $\mathcal{N} = \Sigma_{\mathbb{Z}_2} \circ I_{KW}$ is non-invertible!

$$\mathcal{N} \times \mathcal{N} = 1 + \eta_{\mathbb{Z}_2,\text{spin flip}} \tag{4}$$

Continuum: 2D Ising CFT and Non-Invertible Symmetries

Continuum / IR limit of the lattice model: a Majorana fermion

$$\mathcal{L} = \frac{1}{2}\left(\psi \frac{\partial}{\partial z}\psi + \overline{\psi}\frac{\partial}{\partial \bar{z}}\overline{\psi}\right) + m\psi\overline{\psi} \tag{5}$$

Try: combine "gauging $\mathbb{Z}_2$" line with "KW duality" Line.

$$Ising(T) \xrightarrow{\text{gauge } \mathbb{Z}_2} Ising^{\vee}(T) \xrightarrow{\text{KW duality}} Ising(T^{\vee}) \tag{6}$$

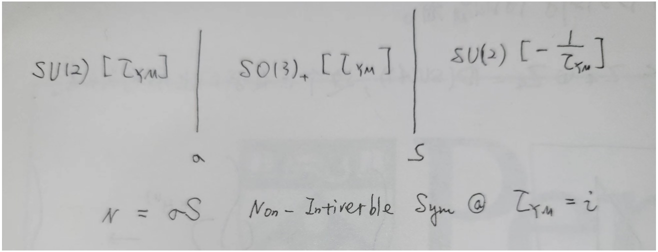
Needs critical temperature $T = T^{\vee} = T_c$!: Then $m = 0$, Ising CFT becomes KW self-dual. EOM gives $\psi = \psi(z)$ and $\overline{\psi} = \overline{\psi}(\bar{z})$.

$\mathcal{N} = \Sigma_{\mathbb{Z}_2} \circ I_{KW}$: non-inv. duality symmetry w/ action on states:

Non-Invertible Symmetries in 4D: Review

In 2021, non-invertible symmetries have been identified in 4D. [Choi, Cordova, Hsin, Lam, Shao '21], [Kaidi, Ohmori, Zheng '21]

Today: "half-space gauging" approach. [Choi, Cordova, Hsin, Lam, Shao '21]



$\mathcal{N} = \Sigma S$: non-invertible duality defect, $S \in SL(2, \mathbb{Z})$. Illustration / convention following [Kaidi, Zafrir, Zheng '22].

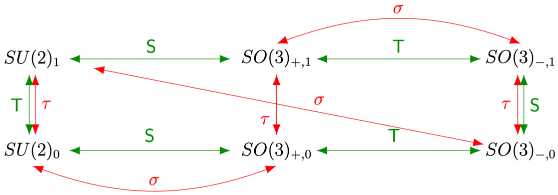
Order operators (W) maps to disorder operators (H).

Extra data: SPT phases

Symmetry Protected Topological (SPT) phases shows up when looking for more ways to identify non-invertible symmetries.

4D: $Z[B] \to Z[B] \exp(\frac{2\pi i}{2N} \int_{M_4} B \cup B)$, B: bkg field of 1-form symmetry.

Affects the outcome of gauging 1-form symmetry!



(Figure form [Kaidi, Zafrir, Zheng '22])

[Lawrie, Yu, Zhang '23]:

Reinterpret and generalize all this via relative and absolute QFT!

Give a concrete prescription, "polarization pairs", which

- Refines our understanding of "absolute QFTs" in some cases;
- Incorporates a large family of half-space gauging constructions of non-invertible symmetries.

Weird type of Quantum Theories in 6D as Relative QFTs

Mysterious 6D theories [Witten '95] [Seiberg '96] that has a partition *vector*, labeled only by a *Lie algebra*.

This has to do with Heisenberg non-commutativity of flux operators, labeled by the defect group $\mathbb{D}$ (more on the next slide)

Are they QFT?

But maybe one should manage to think of them as QFTs.

Defect Groups

Relative and absolute QFTs all comes from a via *defect group*.

We focus on intermediate dimensional defects (Dirac S.D.)

- Λ-charged dynamical particles in 4D or dynamic strings in 6D
- Λ^*-charged heavy defects: *W*/*H* in 4D, surface defects in 6D

Defect group $\mathbb{D}$ via 't Hooft screening argument ['t Hooft '78]:

$$\mathbb{D} = \Lambda^*/\Lambda. \tag{7}$$

$\mathbb{D} \neq 0 =>$ Heisenberg alg. non-commutative $=>$ relative QFT

Relative and Absolute QFTs

[Freed, Teleman '12]: primarily about 6D (or 4k+2 D), where a QFT admit a partition vector space (rather than a partition function).

A $(2k)D$ relative QFT lives on the boundary of, (and thus is "relative" to), a $(2k+1)D$ Topological QFT (TQFT).

The defect group $\mathbb{D}$ controls braiding relations of k-dim'l ops. in $(2k+1)D$

Absolute QFTs

Only some $\mathbb{D}$ gives one (or more) *absolute* QFTs.

Need to specify a Lagrangian subgroup $L \subset \mathbb{D}$, such that

- $|L|^2 = |\mathbb{D}|$, and
- Dirac pairing on $\mathbb{D}$ vanish on identically L.

Specifying an L involves specifying a direction in the partition vector space.

$L \subset \mathbb{D}$ is called a Polarization, resulting in a $(k-1)$-form global symmetry valued in $\mathbb{D}/L$.

Physically, L means Topological B.C. of operators in the 7D TQFT.

Global Structures of 4D SYM, Revisited

Relative and absolute QFTs does not appear often in 4d?

All 4D relative QFTs can be made absolute since
$\mathbb{D} = \mathbb{D}_{ele.} \oplus \mathbb{D}_{mag.}$, and at least $L \cong \mathbb{D}_{mag.}$ is always possible.

E.g., $SU(2)$ and $SO(3)$ SYM are both absolute QFTs coming from the SAME $\mathfrak{su}(2)$ relative theory.

Only in $(4k+2)$D are there some relative QFT which cannot be made absolute.

Polarization Pairs

When $\mathbb{D} = L \oplus \overline{L}$ splits, $\mathbb{D}/L$ is non-anomalous / gaugeable.

Polarization pair Ordered pair $(L, \overline{L})$ of Lagrangian subgroups $L, \overline{L} \subset \mathbb{D}$ s.t. $\mathbb{D} = L \oplus \overline{L}$.

So that, the global symmetry $\mathbb{D}/L$, whose uplift to $\mathbb{D}$ is specified by $\overline{L}$. (This key statement is unfortunately formal.)

L: "polarization" as usually known, $\overline{L}$: a choice of SPT phase.

- Gauging $\mathbb{D}/L$: exchanging $(L, \overline{L}) \to (\overline{L}, L)$
- Stacking SPT phase: changing $\overline{L} \to \overline{L}'$
- Duality(e.g. $SL(2, \mathbb{Z})$: action on $(L, \overline{L})$ via action on $\mathbb{D}$.

Simple algebraic approach to duality defects via **Polarization Pair**!

Duality Defects in 2D Ising Model, Revisited

E.g., $\mathbb{D} = \mathbb{Z}_2 \times \mathbb{Z}_2$, 6 choices of $(L, \overline{L})$

$(1,0)$: $\mathbb{Z}_2$ of the original Ising; $(0,1)$: $\mathbb{Z}_2$ of the dual Ising.
E.g. in the original Ising, the dual $\mathbb{Z}_2$ is gauged.

$\mathcal{T}^{c=\frac{1}{2}}_{(0,1)(1,0)}$: Ising, $\mathcal{T}^{c=\frac{1}{2}}_{(1,0)(0,1)}$: Ising$^\vee$.

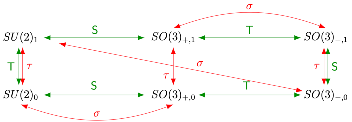
$$\mathcal{T}^{c=\frac{1}{2}}_{(0,1)(1,0)} \xrightarrow{\text{gauge } \mathbb{Z}_2} \mathcal{T}^{c=\frac{1}{2}}_{(1,0)(0,1)} \xrightarrow{\text{KW duality}} \mathcal{T}^{c=\frac{1}{2}}_{(0,1)(1,0)} \tag{8}$$

the two interfaces combines into a non-invertible symmetry in 2D.

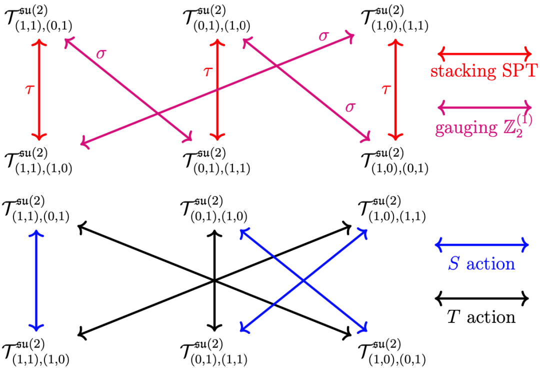
Q: $\mathcal{T}^{c=\frac{1}{2}}_{(0,1),(1,1)}$: $\mathcal{T}^{c=\frac{1}{2}}_{(0,1)(1,0)} +$ SPT phase?

A: Fermionic SPT phase [Kapustin, Thorngren, Turzillo, Wang '14]

4D $\mathcal{N} = 4$ $su(2)$ SYM Case



[Kaidi, Zafrir, Zheng '22]



[Lawrie, Yu, Zhang '23]

Illustration: $su(3)$

We exactly reproduces the $su(3)$ case in [Kaidi, Zafrir, Zheng '22]

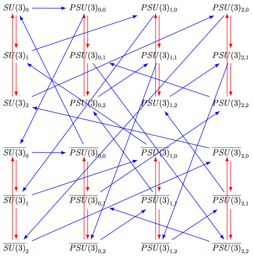


Figure 6: Web of $SL(2, \mathbb{Z}_3)$ transformations for theories with gauge algebra $\mathfrak{su}(3)$. We have denoted the action of σ in blue and the action of τ in red, where σ and τ represent gauging the $\mathbb{Z}_3^{(1)}$ symmetry and coupling to an invertible phase, respectively.

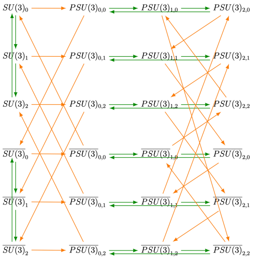


Figure 7: Web of $SL(2, \mathbb{Z})$ transformations for theories with gauge algebra $\mathfrak{su}(3)$. We have denoted the action of S in orange and the action of T in green, where S and T represent the usual modular transformations.

together with all other cases of $\mathcal{N} = 4$ SYM (next slide ...)

Illustration: $su(3)$, Cont.

[Lawrie, Yu, Zhang '23]

$SU(3)_0 : (0,2),(1,0)$	$PSU(3)_{0,0} : (2,0),(0,2)$	$PSU(3)_{1,0} : (2,1),(0,2)$	$PSU(3)_{2,0} : (2,2),(0,2)$
$SU(3)_1 : (0,2),(1,2)$	$PSU(3)_{0,1} : (2,0),(2,2)$	$PSU(3)_{1,1} : (2,1),(2,0)$	$PSU(3)_{2,1} : (2,2),(2,1)$
$SU(3)_2 : (0,2),(1,1)$	$PSU(3)_{0,2} : (2,0),(1,2)$	$PSU(3)_{1,2} : (2,1),(1,1)$	$PSU(3)_{2,2} : (2,2),(1,0)$
$\overline{SU(3)}_0 : (0,1),(2,0)$	$\overline{PSU(3)}_{0,0} : (1,0),(0,1)$	$\overline{PSU(3)}_{1,0} : (1,2),(0,1)$	$\overline{PSU(3)}_{2,0} : (1,1),(0,1)$
$\overline{SU(3)}_1 : (0,1),(2,1)$	$\overline{PSU(3)}_{0,1} : (1,0),(1,1)$	$\overline{PSU(3)}_{1,1} : (1,2),(2,0)$	$\overline{PSU(3)}_{2,1} : (1,1),(1,2)$
$\overline{SU(3)}_2 : (0,1),(2,2)$	$\overline{PSU(3)}_{0,2} : (1,0),(2,1)$	$\overline{PSU(3)}_{1,2} : (1,2),(2,2)$	$\overline{PSU(3)}_{2,2} : (1,1),(2,0)$

Then all the arrows can be visually checked via our "three rules".

General Duality Actions on $\mathbb{D}$

Dualities we have up till now:

- Kramers-Wannier duality in 2D Ising CFT
- $SL(2,\mathbb{Z})$ duality in 4D $\mathcal{N}=4$ SYM
  - $Sp(N,\mathbb{Z})$ in 4D $\mathcal{N}=2$ class $\mathcal{S}$. [Bashmakov, Del Zotto, Hasan, Kaidi '22]
- 2D $T^{6D}[M_4]$ via mapping class group of M_4 [Chen, Cui, Haghighat, Wang '23], [Bashmakov, Del Zotto, Hasan, '23]

Common feature: some automorphism of the QFT acting on $\mathbb{D}$.

Look for other such dualities acting on $\mathbb{D}$, potentially in higher dim and not via 6D compactification!

6D (2,0) SCFTs

6D (2,0): strongly-coupled, non-Lagrangian, ADE classified.

- Realized as worldvolume theory of M5 branes,
- or type IIB on $\mathbb{R}^{1,5} \times \mathbb{C}^2/\Gamma_{ADE}$.

On the "tensor branch", there are $\mathrm{rk}(\mathfrak{g}_{ADE})$ tensor multiplets, couple to effective strings.

Their Dirac pairing matrix (add def!) $\cong$ Cartan matrix of $\mathfrak{g}_{ADE}$

$$\mathbb{D} = \Lambda^*/\Lambda = \Gamma_{ADE}/[\Gamma_{ADE}, \Gamma_{ADE}] = \mathrm{Ab}[\Gamma_{ADE}]! \tag{9}$$

(We used McKay correspondence: $\mathfrak{g}_{ADE} \leftrightarrow \Gamma_{ADE} \subset SU(2)$.)

Automorphisms of 6D SCFTs Acting on $\mathbb{D}$

"Green-Schwarz Automorphisms" [Apruzzi, Heckman, Rudelius '17]!
For 6D $(2,0)$, outer auto. of ADE Dynkin diagrams.

Acts (on Λ^*, and thus) on the tensor multiplets. Thus on the string charge lattice Λ and $\mathbb{D} = \Lambda^*/\Lambda$. Exchange different absolute theories.

So we propose to rename them as "Green-Schwarz *Dualities*"

(analog of S-duality $S : SU(2)[\tau_{YM}] \to SO(3)_+[-\frac{1}{\tau_{YM}}]$,
which also exchange different absolute theories of 4D $\mathfrak{su}(2)$)!

Non-Invertible Symmetries in 6D (2,0): D_4 Example

Concrete example: $D_4(2,0)$ theory, $\mathbb{D} = \mathbb{Z}_2 \times \mathbb{Z}_2$.

$G_{GS}(D_4) = S_3$: automorphism of the D_4 Dynkin diagram.

This S_3 automorphism permutes the

$$(1,0), (0,1), (1,1) \tag{10}$$

elements of $Z(Spin(8)) = \mathbb{Z}_2 \times \mathbb{Z}_2$ in all possible ways.

Thus permuting different Absolute (2,0) theories:
"$SO(8), Sc(8), Ss(8)$" (2-form charge lattices, not gauge dynamics)

Non-Invertible Symmetries in 6D (2,0): D_4 Example

$$\mathcal{T}^{D_4}_{(1,1),(0,1)} \xrightarrow{\sigma \quad (gauging)} \mathcal{T}^{D_4}_{(0,1),(1,1)} \xrightarrow{a \quad (GS)} \mathcal{T}^{D_4}_{(1,1),(0,1)} \tag{11}$$

Thanks to many hints in [Gukov, Hsin, Pei '20]

Let $M_5 \subset \mathbb{R}^{1,5}$ be a codim-1 interface

$$\mathcal{N}(M_5) = \Sigma(M_5) \circ a_{(L_{SO}, L_{Sc})}(M_5) \tag{12}$$

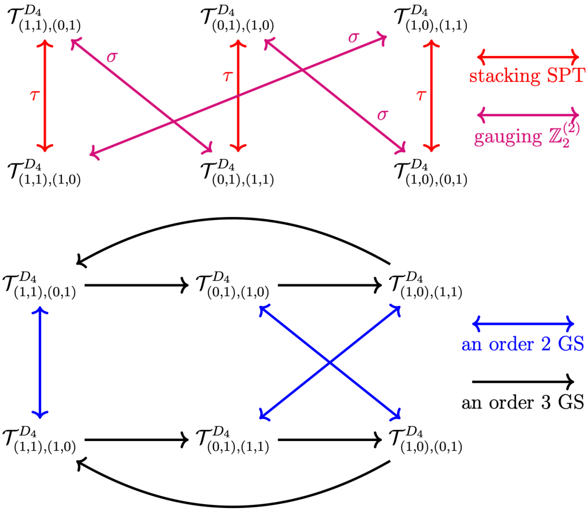
Fusion rules:

$$\mathcal{N}(M_5) \times \overline{\mathcal{N}}(M_5) = \sum_{\mathcal{S}_3 \in H_3(M_5, \mathbb{Z}_2)} \mathcal{U}(S_3) \tag{13}$$

$$\mathcal{N}(M_5) \times \mathcal{U}(S_3) = \mathcal{N}(M_5) \tag{14}$$

Non-invertible duality defects in 6D!

Non-Invertible Symmetries in 6D (2,0): D_4 Example, Cont.

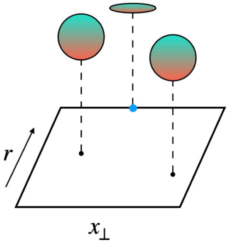


Comments: String Perspective

View 6D (2,0) as IIB on $\mathbb{C}^2/\Gamma$.

Then the Green-Schwarz duality can be Geometrized as automorphisms of $\mathbb{C}^2/\Gamma$.

Concretely, one can consider a "S^3/Γ fiber degeneration" at $(r = \infty, x_\perp = 0)$.



Top-down perspective by starting from 10D, instead of 6D.

Conclusion and Future Directions

We introduced "polarization pairs" refining polarizations in even D.

- which reproduces non-invertible symmetries in 2D and 4D
- and leads to novel non-invertible symmetries in 6D.

Upcoming work **[Lawrie, Yu, Zhang, 2311.XXXXX]**:
comparing non-invertible duality symmetries across dimensions

Anomalies of non-invertible duality symmetries via relative and absolute QFTs?


terima kasih
σας ευχαριστώ
આભાર
salamat
ačiū
dank u
Thank you
Danke
謝謝
شكرا
спасибо
ありがとう
감사합니다
merci
aitäh
ขอบคุณ
Tak
शुक्रिया
obrigado