Holographic Euclidean thermal correlator

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[0] Based on arXiv 2308.13518 with Song He

1. The holographic prescription
2. $U(1)$ current
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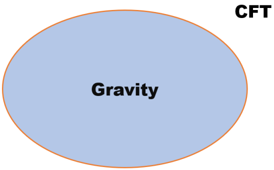
AdS/CFT and GKPW relation

- AdS/CFT holography



- Gubser-Klebanov-Polyakov-Witten relation

$$\langle e^{\int \psi_0 O} \rangle_{CFT} = Z_{grav}(\psi_0) \sim e^{-I_{grav}(\psi_0)} \tag{1}$$

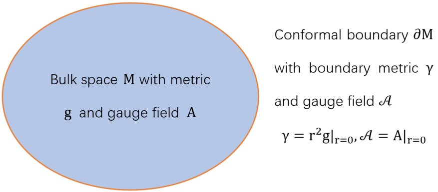
ψ_0 source for operators in the CFT, boundary condition for bulk fields in the gravity

- Holographic correlator

$$\langle O(x_1) \dots O(x_n) \rangle_{CFT} \sim \frac{\delta^n I_{grav}}{\delta\psi_0(x_1) \dots \delta\psi_0(x_n)} \tag{2}$$

The boundary value problem

- We study Euclidean thermal correlators of the stress tensor and $U(1)$ current, holographically described by Einstein's gravity and Maxwell theory.
- The boundary value problem of asymptotically AdS Einstein space



- Near boundary solution is in one-to-one correspondence with the source and the one-point correlator that satisfies the holographic Ward identity (Commun. Math. Phys. 217 (2001) 595-622, Nucl. Phys. B 631 (2002) 159).

The boundary value problem cont.

- For Einstein's gravity (stress tensor)

$$ds^2 = \frac{dr^2}{r^2} + \frac{1}{r^2}\mathbf{g}_{ij}(r,x)dx^i dx^j,$$

$$\mathbf{g}_{ij} = \mathbf{g}_{ij}^{(0)} + r^2\mathbf{g}_{ij}^{(2)} + r^4\mathbf{g}_{ij}^{(4)} + r^4\log r\mathbf{h}_{ij}^{(4)} + \dots, \tag{3}$$

$$\langle T_{ij}\rangle = \frac{4}{16\pi G}\left[\mathbf{g}_{ij}^{(4)} - \frac{1}{8}\mathbf{g}_{ij}^{(0)}(\mathbf{P}^{(0)2} - \mathbf{P}_{ij}^{(0)}\mathbf{P}^{(0)ij}) - \frac{1}{2}\mathbf{P}_{ik}^{(0)}\mathbf{P}_{j}^{(0)k} + \frac{1}{4}\mathbf{P}^{(0)}\mathbf{P}_{ij}^{(0)}\right] \tag{4}$$

- For Maxwell theory ($U(1)$ current)

$$A = \mathbf{A}_i(r,x)dx^i,$$

$$\mathbf{A}_i = \mathbf{A}_i^{(0)} + r^2\mathbf{A}_i^{(2)} + r^2\log r\mathbf{B}_i^{(2)} + \dots, \tag{5}$$

$$\langle J_i\rangle = -2\mathbf{A}_i^{(2)} \tag{6}$$

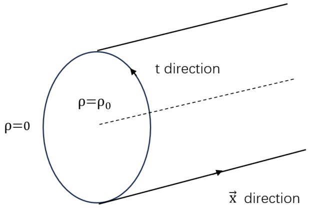
- The global boundary value problem is much more complicated, e.g. for pure gravity (hep-th 0403087).

The Euclidean AdS_5 planar black hole

- Thermal states of CFT_4 holographically described by AdS_5 planar black hole
- The black hole is a solid cylinder $\mathbb{B}^2 \times \mathbb{R}^3$ with the metric

$$ds^2 = \frac{1}{\rho^2}[(1 - \frac{\rho^4}{\rho_0^4})^{-1} d\rho^2 + (1 - \frac{\rho^4}{\rho_0^4}) dt^2 + d\vec{x}^2] \tag{7}$$

The conformal boundary is at $\rho = 0$, and the horizon is at $\rho = \rho_0$



- The period of Euclidean time t, namely the inverse temperature, is $\beta = \pi \rho_0$. Set $\rho_0 = 1$ for simplicity and recover ρ_0 in the final results.

Gauge fixing and boundary conditions

- Gauge fixing: set $A_\rho = 0$ in the region $0 \leq \rho < 1$ (excluding the horizon) by a $U(1)$ gauge transformation

$$A = \mathbf{A}_i dx^i \tag{8}$$

- Boundary condition at the horizon: the solution has a regular limit as $\rho \to 1$ after a gauge transformation parametrized by Λ
- Introduce the the "cylindrical radial coordinate" $\mathfrak{s} = \frac{1}{2} \cosh^{-1} \frac{1}{\rho^2}$

$$ds^2 \sim d\mathfrak{s}^2 + \mathfrak{s}^2 d(2t)^2 + d\vec{x}^2 \tag{9}$$

and the "Cartesian coordinates"

$$\begin{aligned} X &= \mathfrak{s} \cos 2t \\ Y &= \mathfrak{s} \sin 2t \\ \vec{x} &= \vec{x} \end{aligned} \tag{10}$$

which properly covers the horizon

Gauge fixing and boundary conditions cont.

- Components in the "Cartesian coordinates" are regular

$$\lim_{\mathfrak{s}\to 0} A + d\Lambda = A^*_X(\vec{x})dX + A^*_Y(\vec{x})dY + A^*_a(\vec{x})dx^a \tag{11}$$

that is

$$\lim_{\mathfrak{s}\to 0} \partial_{\mathfrak{s}}\Lambda = A^*_X(\vec{x})\cos 2t + A^*_Y(\vec{x})\sin 2t \tag{12}$$

$$\lim_{\mathfrak{s}\to 0} \frac{\mathbf{A}_t + \partial_t\Lambda}{\mathfrak{s}} = -2A^*_X(\vec{x})\sin 2t + 2A^*_Y(\vec{x})\cos 2t \tag{13}$$

$$\lim_{\mathfrak{s}\to 0} \mathbf{A}_a + \partial_a\Lambda = A^*_a(\vec{x}) \tag{14}$$

- Some simple analysis, we find

$$\mathbf{A}_a \text{ regular as } \rho \to 1 \tag{15}$$

$$\int_0^\pi dt \mathbf{A}_t|_{\rho=1} = 0 \tag{16}$$

- Turned-on source

$$\mathbf{A}_i|_{\rho=0} = \mathcal{A}_i \tag{17}$$

Equations of motion

- The Maxwell equation

$$d \star F = 0 \tag{18}$$

- Work with Fourier modes $\tilde{\mathbf{A}}_i$ with Matsubara frequency $\omega = 2m, m \in \mathbb{Z}$ and spatial momentum $\vec{p}$ rotated to the x^1 direction for simplicity. Also use the substitution $z = \rho^2$
- Transverse channel

$$(\partial_z^2 - \frac{2z}{1-z^2}\partial_z - \frac{\omega^2 + p^2(1-z^2)}{4z(1-z^2)^2})\tilde{\mathbf{A}}_2 = 0 \tag{19}$$

- Longitudinal channel

$$\partial_z^2 \tilde{\mathbf{A}}_t - \frac{p^2}{4z(1-z^2)}\tilde{\mathbf{A}}_t + \frac{2mp}{4z(1-z^2)}\tilde{\mathbf{A}}_1 = 0 \tag{20}$$

$$\partial_z^2 \tilde{\mathbf{A}}_1 - \frac{2z}{1-z^2}\partial_z \tilde{\mathbf{A}}_1 - \frac{4m^2}{4z(1-z^2)^2}\tilde{\mathbf{A}}_1 + \frac{2mp}{4z(1-z^2)^2}\tilde{\mathbf{A}}_t = 0 \tag{21}$$

$$\frac{2m}{1-z^2}\partial_z \tilde{\mathbf{A}}_t + p\partial_z \tilde{\mathbf{A}}_1 = 0 \tag{22}$$

Transverse channel

- By the substitution $\tilde{\mathbf{A}}_2(z) = (1-z^2)^{-\frac{1}{2}} w(z)$, we get a Heun equation in the normal form (see (52)) for $w(z)$

$$(\partial_z^2 + \frac{\frac{1}{4}-(\frac{1}{2})^2}{z^2} + \frac{\frac{1}{4}-(\frac{m}{2})^2}{(z-1)^2} + \frac{\frac{1}{4}-(\frac{m}{2}i)^2}{(z+1)^2} + \frac{p^2+4m^2-2}{8z(z-1)} - \frac{p^2+4m^2+2}{8z(z+1)})w(z) = 0,$$

$$t=-1, a_0 = \frac{1}{2}, a_1 = \frac{|m|}{2}, a_t = \frac{m}{2}i, a_\infty = \frac{1}{2}, u = -\frac{p^2+4m^2+2}{8} \tag{23}$$

- By the boundary condition $\mathbf{A}_2$ regular as $z \to 1$, we must have

$$\tilde{\mathbf{A}}_2(z) \sim (1-z^2)^{-\frac{1}{2}} w_+^{(1)}(z) \tag{24}$$

- By the connection relation (56) and the boundary condition $\mathbf{A}_2|_{z=0} = \mathcal{A}_2$, we find

$$\tilde{\mathbf{A}}_2(\omega = 2m, p, z) = \tilde{\mathcal{A}}_2(\omega, p)(1-z^2)^{-\frac{1}{2}} \Big[w_-^{(0)} + \frac{p^2+4m^2}{4}(-2\psi(1) - 1$$
$$+ \frac{1}{2}\sum_{\theta,\sigma=\pm} \psi(\theta\frac{m}{2} + \sigma a) - \frac{1}{2}\partial_{a_0}^2 F - \frac{2}{p^2+4m^2}(1 + 2\partial_t\partial_{a_0} F))w_+^{(0)} \Big] \tag{25}$$

Longitudinal channel

- Plugging (22) into $\partial_z\big(z(1-z^2)(20)\big)$ to eliminate $\tilde{\mathbf{A}}_1$, we obtain

$$(\partial_z^2+\frac{1-3z^2}{z(1-z^2)}\partial_z-\frac{p^2(1-z^2)+\omega^2}{4z(1-z^2)^2})\partial_z\tilde{\mathbf{A}}_t=0 \qquad (26)$$

- By the substitution $\partial_z\tilde{\mathbf{A}}_t=z^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}w(z)$, it's transformed to

$$(\partial_z^2+\frac{\frac{1}{4}-0^2}{z^2}+\frac{\frac{1}{4}-(\frac{m}{2})^2}{(z-1)^2}+\frac{\frac{1}{4}-(\frac{m}{2}i)^2}{(z+1)^2}+\frac{p^2+4m^2-6}{8z(z-1)}-\frac{p^2+4m^2+6}{8z(z+1)})w(z)=0,$$

$$t=-1, a_0=0, a_1=\frac{|m|}{2}, a_t=\frac{m}{2}i, a_\infty=1, u=-\frac{p^2+4m^2+6}{8} \qquad (27)$$

- If $m\neq 0$, by (22) we must have $\partial_z\tilde{\mathbf{A}}_t\sim z^{-\frac{1}{2}}(1-z^2)^{-\frac{1}{2}}w_+^{(1)}$ for $\tilde{\mathbf{A}}_1$ to be regular at $z=1$. Using the connection relation (56) and evaluating (20) at $z=0$, we find

$$z^{\frac{1}{2}}\sqrt{1-z^2}\partial_z\tilde{\mathbf{A}}_t=\frac{2mp\tilde{\mathcal{A}}_1-p^2\tilde{\mathcal{A}}_t}{4}\Big[-w_-^{(0)}(z)$$

$$+(2\psi(1)-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2}+\theta\frac{m}{2}+\sigma a)+\frac{1}{2}\partial_{a_0}^2F)w_+^{(0)}\Big] \qquad (28)$$

Longitudinal channel cont.

- We integrate to obtain $\tilde{\mathbf{A}}_t$ and plug it into (22) to get $\tilde{\mathbf{A}}_1$

$$\tilde{\mathbf{A}}_t = \tilde{\mathcal{A}}_t + \frac{2mp\tilde{\mathcal{A}}_1 - p^2\tilde{\mathcal{A}}_t}{4}\Big[- (z\log z + \dots) + (2\psi(1) + 1 - \frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2} + \theta\frac{m}{2} + \sigma a) + \frac{1}{2}\partial_{a_0}^2 F)(z + \dots)\Big], \tag{29}$$

$$\tilde{\mathbf{A}}_1 = \tilde{\mathcal{A}}_1(1 + \dots) + \frac{2m(p\tilde{\mathcal{A}}_t - 2m\tilde{\mathcal{A}}_1)}{4} \times \Big(2\psi(1) + 1 - \frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2} + \theta\frac{m}{2} + \sigma a) + \frac{1}{2}\partial_{a_0}^2 F\Big)(z + \dots) \tag{30}$$

- With the boundary condition $\tilde{\mathbf{A}}_t(m=0)|_{z=1} = 0$ from (16), one can show the solution for $m = 0$ can be carried over from the case $m \neq 0$, with m set to zero in the expression

Exact correlators for $U(1)$ current

- Recover the dependence on ρ_0, read off $\mathbf{A}_i^{(2)}$ from the bulk gauge field $\mathbf{A}_i$ as the coefficient of z^1, use the formula for one-point correlator (6) and rotate the spatial momentum to a general direction
- We find

$$
\begin{aligned}
&\langle \tilde{J}_t(\omega,p)\tilde{J}_t(-\omega,-p)\rangle = \frac{p^2}{2}\mathcal{C}_2(\omega,p)\\
&\langle \tilde{J}_t(\omega,p)\tilde{J}_b(-\omega,-p)\rangle = -\frac{\omega}{2}\mathcal{C}_2(\omega,p)p_b\\
&\langle \tilde{J}_a(\omega,p)\tilde{J}_b(-\omega,-p)\rangle = \frac{p^2+\omega^2}{2}\mathcal{C}_1(\omega,p)(\delta_{ab}-\frac{p_a p_b}{p^2}) + \frac{\omega^2}{2}\mathcal{C}_2(\omega,p)\frac{p_a p_b}{p^2},\\
&\mathcal{C}_1(\omega=\frac{2m}{\rho_0},p) = (2\psi(1)+1-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\theta\frac{m}{2}+\sigma a)\\
&+\frac{1}{2}\partial_{a_0}^2 F + \frac{2}{\rho_0^2 p^2+4m^2}(1+\partial_t\partial_{a_0}F))\Big|_{t=-1,a_0=\frac{1}{2},a_1=\frac{|m|}{2},a_t=\frac{m}{2}i,a_\infty=\frac{1}{2},u=-\frac{\rho_0^2p^2+4m^2+2}{8}}\\
&\mathcal{C}_2(\omega=\frac{2m}{\rho_0},p) = (2\psi(1)+1-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\frac{1}{2}+\theta\frac{m}{2}+\sigma a)\\
&+\frac{1}{2}\partial_{a_0}^2 F)\Big|_{t=-1,a_0=0,a_1=\frac{|m|}{2},a_t=\frac{m}{2}i,a_\infty=1,u=-\frac{\rho_0^2p^2+4m^2+6}{8}}
\end{aligned}
\tag{31}
$$

Gauge fixing and boundary conditions

- Gauge fixing: make the solid cylinder coordinates $\rho, t, \vec{x}$ the Fefferman-Graham coordinates of the perturbed bulk metric in the region $0 \leq \rho < 1$ by a diffeomorphism

$$\delta ds^2 = \delta \mathbf{g}_{ij} dx^i dx^j \tag{32}$$

- Boundary condition at the horizon: the solution has a regular limit as $\rho \to 1$ after a diffeomorphism parametrized by V
- Some simple analysis, we find

$$\delta \mathbf{g}_{ab} \text{ regular as } \rho \to 1 \tag{33}$$

$$\int_0^\pi dt \delta \mathbf{g}_{ta}|_{\rho=1} = 0 \tag{34}$$

Equations of motion

- The linearized Einstein equation

$$\frac{1}{2}(\nabla^{\lambda}\nabla_{\mu}\delta g_{\lambda\nu}+\nabla^{\lambda}\nabla_{\nu}\delta g_{\lambda\mu}-\nabla^{\lambda}\nabla_{\lambda}\delta g_{\mu\nu}-\nabla_{\mu}\nabla_{\nu}\delta g^{\lambda}_{\lambda})+4\delta g_{\mu\nu}=0 \quad (35)$$

- We work in Fourier modes and rotate the spatial momentum to the x^1 direction. And for simplicity, we use the variable $\mathbf{h}_{ij}=\rho^2\delta\mathbf{g}_{ij}$
- Scalar channel

$$\partial_z^2\tilde{\mathbf{h}}_{23}-\frac{1+z^2}{z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{23}-\frac{p^2(1-z^2)+\omega^2}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{23}=0 \quad (36)$$

- Shear channel

$$\partial_z^2\tilde{\mathbf{h}}_{t2}-\frac{1}{z}\partial_z\tilde{\mathbf{h}}_{t2}-\frac{p^2}{4z(1-z^2)}\tilde{\mathbf{h}}_{t2}+\frac{2mp}{4z(1-z^2)}\tilde{\mathbf{h}}_{12}=0 \quad (37)$$

$$\partial_z^2\tilde{\mathbf{h}}_{12}-\frac{1+z^2}{z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{12}-\frac{4m^2}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{12}+\frac{2mp}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{t2}=0 \quad (38)$$

$$\frac{2m}{1-z^2}\partial_z\tilde{\mathbf{h}}_{t2}+p\partial_z\tilde{\mathbf{h}}_{12}=0 \quad (39)$$

Equations of motion cont.

- Sound channel

$$\partial_z^2\tilde{\mathbf{h}}_{tt} - \frac{3-5z^2}{2z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{1+z^2}{2z}\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{-4z+12z^3-p^2(1-z^2)}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{tt} - \frac{4m^2}{4z(1-z^2)}(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{2mp}{2z(1-z^2)}\tilde{\mathbf{h}}_{t1} = 0 \quad (40)$$

$$\partial_z^2\tilde{\mathbf{h}}_{11} - \frac{3+z^2}{2z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{11} - \frac{1}{2z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{1}{2z}\partial_z(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{4m^2}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{11} - \frac{p^2+4z}{4z(1-z^2)^2}\tilde{\mathbf{h}}_{tt} - \frac{p^2}{4z(1-z^2)}(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{2mp}{2z(1-z^2)^2}\tilde{\mathbf{h}}_{t1} = 0 \quad (41)$$

$$\partial_z^2(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{2}{z(1-z^2)}\partial_z(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{1}{z(1-z^2)}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{1}{z}\partial_z\tilde{\mathbf{h}}_{11} - \frac{4m^2+p^2(1-z^2)}{4z(1-z^2)^2}(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - \frac{2}{(1-z^2)^2}\tilde{\mathbf{h}}_{tt} = 0 \quad (42)$$

$$\partial_z^2\tilde{\mathbf{h}}_{t1} - \frac{1}{z}\partial_z\tilde{\mathbf{h}}_{t1} - \frac{2mp}{4z(1-z^2)}(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) = 0 \quad (43)$$

$$\partial_z^2(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{1}{1-z^2}\partial_z^2\tilde{\mathbf{h}}_{tt} - \frac{z}{1-z^2}\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{z}{(1-z^2)^2}\partial_z\tilde{\mathbf{h}}_{tt} + \frac{2}{(1-z^2)^3}\tilde{\mathbf{h}}_{tt} = 0 \quad (44)$$

$$2m\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{2mz}{1-z^2}\partial_z(\tilde{\mathbf{h}}_{11}+\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) - p\partial_z\tilde{\mathbf{h}}_{t1} - \frac{2pz}{1-z^2}\tilde{\mathbf{h}}_{t1} = 0 \quad (45)$$

$$p\partial_z(\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}) + \frac{p}{1-z^2}\partial_z\tilde{\mathbf{h}}_{tt} - \frac{2m}{1-z^2}\partial_z\tilde{\mathbf{h}}_{t1} + \frac{pz}{(1-z^2)^2}\tilde{\mathbf{h}}_{tt} = 0 \quad (46)$$

Exact correlators for stress tensor

- In the scalar and shear channel we find

$$
\begin{aligned}
\langle \tilde{T}_{23}(\omega = \frac{2m}{\rho_0}, p)\tilde{T}_{23}(-\omega,-p)\rangle &= \frac{1}{4\pi G}\frac{(p^2+\omega^2)^2}{32}\mathcal{C}_3(\omega=\frac{2m}{\rho_0},p)\\
\langle \tilde{T}_{t2}(\omega = \frac{2m}{\rho_0}, p)\tilde{T}_{t2}(-\omega,-p)\rangle &= \frac{1}{4\pi G}\frac{p^2+\omega^2}{32}p^2\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p)\\
\langle \tilde{T}_{t2}(\omega = \frac{2m}{\rho_0}, p)\tilde{T}_{12}(-\omega,-p)\rangle &= -\frac{1}{4\pi G}\frac{p^2+\omega^2}{32}\omega p\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p)\\
\langle \tilde{T}_{12}(\omega = \frac{2m}{\rho_0}, p)\tilde{T}_{12}(-\omega,-p)\rangle &= \frac{1}{4\pi G}\frac{p^2+\omega^2}{32}\omega^2\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p),\\
\mathcal{C}_3(\omega=\frac{2m}{\rho_0},p) &= \Big[2\psi(1)+\frac{5}{2}-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(-\frac{1}{2}+\theta\frac{m}{2}+\sigma a)\\
&+\frac{1}{2}\partial_{a_0}^2F-\frac{16}{(\rho_0^2p^2+4m^2)^2}\big(4a^2-2a^2m^2+\frac{1}{4}m^4+4(\partial_tF)^2+(-8a^2+2m^2)\partial_tF\\
&-4\partial_tF\partial_t\partial_{a_0}F+(-2+4a^2-m^2)\partial_t\partial_{a_0}F\big)\Big]\Big|_{t=-1,a_0=1,a_1=\frac{|m|}{2},a_t=\frac{m}{2}i,a_\infty=0,u=-\frac{\rho_0^2p^2+4m^2-2}{8}}\\
\mathcal{C}_4(\omega=\frac{2m}{\rho_0},p) &= \big(2\psi(1)+1-\frac{1}{2}\sum_{\theta,\sigma=\pm}\psi(\theta\frac{m}{2}+\sigma a)\\
&+\frac{1}{2}\partial_{a_0}^2F+\frac{2}{\rho_0^2p^2+4m^2}(1+2\partial_t\partial_{a_0}F)\big)\Big|_{t=-1,a_0=\frac{1}{2},a_1=\frac{|m|}{2},a_t=\frac{m}{2}i,a_\infty=\frac{3}{2},u=-\frac{\rho_0^2p^2+4m^2+10}{8}}
\end{aligned}
\tag{47}
$$

The unsolved sound channel

- We can reduce the sound channel to first-order equations of variables $\tilde{\mathbf{h}}_{tt}, \tilde{\mathbf{h}}_{11}, \frac{\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}}{2}, \tilde{\mathbf{h}}_{t1}, \partial_z \tilde{\mathbf{h}}_{t1}$, and by the substitution

$$\begin{pmatrix} \tilde{\mathbf{h}}_{tt} \\ \tilde{\mathbf{h}}_{11} \\ \frac{\tilde{\mathbf{h}}_{22}+\tilde{\mathbf{h}}_{33}}{2} \\ \tilde{\mathbf{h}}_{t1} \\ \partial_z \tilde{\mathbf{h}}_{t1} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{1}{3}(1-z^2)^2 & \frac{2}{3}z(1-z^2) & 0 & 0 \\ -z^2 & 1-z^2 & \frac{2}{3}z & 0 & 0 \\ \frac{1}{2}z^2 & 0 & -\frac{1}{3}z & 0 & 0 \\ 0 & 0 & 0 & 1-z^2 & 0 \\ 0 & 0 & 0 & 0 & z \end{pmatrix} H \tag{48}$$

  we can transform the equations into a Fuchsian system of normal form

$$\partial_z H = (\frac{M_0}{z} + \frac{M_1}{z-1} + \frac{M_{-1}}{z+1})H \tag{49}$$

- We don't know connection relation of local solutions of this Fuchsian system.

Fuchsian ODE and local monodromy basis

- An ODE is called Fuchsian if the coefficients are rational functions and all singularities are regular.
- Eigenvectors of the local monodromy

$$w_k^{(a)} = (z-a)^{\rho_k} \sum_{i=0}^{\infty} c_i (z-a)^i \tag{50}$$

The characteristic exponent ρ_k captures the eigenvalue and it's computed as the root of the indicial equation.

- All eigenvalues are distinct (exponents don't differ by an integer), eigenvectors span the space of local solutions.
- Repeated eigenvalues (some exponents differ by an integer), we may need generalized eigenvector (with logarithm) to span the space of local solutions

$$w_k^{(a)} = (z-a)^{\rho_k} \sum_{i=0}^{\infty} c_i (z-a)^i + \log(z-a) w_{k'}^{(a)} \tag{51}$$

Second order Fuchsian ODE

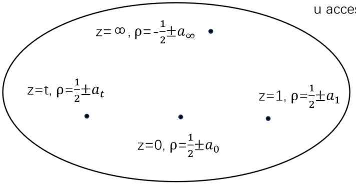
- Label the two characteristic exponents as ρ^+, ρ^-, with $\mathrm{Re}\rho_+ \geq \mathrm{Re}\rho_-$. There is always a series solution (with no logarithm) $w_+^{(a)}$ with the exponent ρ^+.
- If two exponents differ by an integer, the other solution $w_-^{(a)}$ may contain a logarithm. There is also no canonical choice of $w_-^{(a)}$ to form a basis since we can add any constant multiple of $w_+^{(a)}$ to $w_-^{(a)}$. We choose the convention that the coefficient of the power $(z-a)^{\rho_+}$ is zero in $w_-^{(a)}$.

The Heun equation

- The Heun equation is the second-order Fuchsian ODE with four regular singularities. Fuchs relation $\sum_{a,\sigma} \rho_\sigma^{(a)} = 1$
- Möbius transformation to map the singularities to $0, 1, \infty, t$. "Gauge transformation" to shift the exponents at the singularities. We have the normal form of Heun equation

$$
\Big(\partial_z^2 + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} - \frac{\frac{1}{2} - a_1^2 - a_t^2 - a_0^2 + a_\infty^2 + u}{z(z-1)} + \frac{u}{z(z-t)}\Big) w(z) = 0 \qquad (52)
$$

u accessory parameter



The connection relation of Heun equation

- Heun equation $\leftrightarrow$ semiclassical Liouville CFT $\leftrightarrow$ SUSY gauge theory
- The connection relation of the local solutions in the generic case (exponents do not differ by an integer) (Commun. Math. Phys. 397 (2023) 635-727)

$$w_\theta^{(1)}(z) = \sum_{\theta'=\pm} \mathcal{M}_{\theta\theta'}(a_1, a_0; a) e^{(\frac{\theta}{2}\partial_{a_1} - \frac{\theta'}{2}\partial_{a_0})F(\,{}^{a_t}_{a_\infty}\, a\, {}^{a_1}_{a_0};\frac{1}{t})} w_{\theta'}^{(0)}(z),$$

$$\mathcal{M}_{\theta\theta'}(a_1, a_0; a) = \frac{\Gamma(-2\theta' a_0)\Gamma(1+2\theta a_1)}{\Gamma(\frac{1}{2}+\theta a_1 - \theta' a_0 + a)\Gamma(\frac{1}{2}+\theta a_1 - \theta' a_0 - a)} \quad (53)$$

and a is to be implicitly determined from the relation

$$u = -\frac{1}{4} - a^2 + a_t^2 + a_0^2 + t\partial_t F \quad (54)$$

- F is the Nekrasov-Shatashvili function, defined as power series in $\frac{1}{t}$, with combinatorially defined rational functions in other parameters as the coefficients, e.g. see (SciPost Phys. 14, 116 (2023)).

The connection relation for degenerate local monodromy

- In our application, masslessness of bulk fields leads to degenerate local monodromy of the Heun equation at $z = 0$, that is, two exponents differ by an integer $a_0 = \frac{N}{2}, N \in \mathbb{N}$.
- A specific local solution around $z = 0$ (as long as its definition doesn't depend on the local monodromy) depends continuously on all the parameters including a_0. The connection relation in the degenerate case can be obtained by taking limit $a_0 \to \frac{N}{2}$ of $w_+^{(1)}$ (while fixing t, a_1, a_t, a_∞, a).
- Series coefficients of $w_-^{(0)}$ and $\mathcal{M}_{++}(a_1, a_0; a)$ take $a_0 = \frac{N}{2}$ as a simple pole, the limit $\frac{0}{0}$ becomes a differentiation with respect to a_0.

The connection relation for degenerate local monodromy cont.

- For $a_0 = 0$

$$w_+^{(1)} = \frac{\Gamma(1+2a_1)}{\Gamma(\frac{1}{2}+a_1+a)\Gamma(\frac{1}{2}+a_1-a)} e^{\frac{1}{2}\partial_{a_1}F} \Big[-w_-^{(0)} + \big(2\psi(1) - \psi(\frac{1}{2}+a_1+a) - \psi(\frac{1}{2}+a_1-a) + \frac{1}{2}\partial_{a_0}^2 F\big) w_+^{(0)} \Big] \quad (55)$$

- For $a_0 = \frac{1}{2}$

$$w_+^{(1)} = \frac{\Gamma(1+2a_1)}{\Gamma(a_1+a)\Gamma(a_1-a)} e^{(\frac{1}{2}\partial_{a_1} - \frac{1}{2}\partial_{a_0})F} \Big[\frac{t}{-\frac{t}{2} + t(a_0^2+a_1^2+a_t^2-a_\infty^2) + (1-t)u} w_-^{(0)} + \big(-2\psi(1) - 1 + \frac{1}{2}\psi(1+a_1+a) + \frac{1}{2}\psi(1+a_1-a) + \frac{1}{2}\psi(a_1+a) + \frac{1}{2}\psi(a_1-a) - \frac{1}{2}\partial_{a_0}^2 F - \frac{t + t(1-t)\partial_t\partial_{a_0}F}{2(-\frac{t}{2} + t(a_0^2+a_1^2+a_t^2-a_\infty^2) + (1-t)u)} \big) w_+^{(0)} \Big] \quad (56)$$

- For higher N, we can compute with Mathematica

Discussion

- An illustrative example of holographic Euclidean correlators
- The technical approach as compared to the gauge-invariants method in Phys. Rev. D 72, 086009 (2005), for example in the longitudinal channel

$$E_L = p\tilde{\mathbf{A}}_t - \omega\tilde{\mathbf{A}}_1,$$
$$\partial_z^2 E_L - \frac{2\omega^2 z}{(1-z^2)(\omega^2+p^2(1-z^2))}\partial_z E_L - \frac{\omega^2+p^2(1-z^2)}{4z(1-z^2)^2}E_L = 0 \quad (57)$$

- Retarded thermal Green's function $\rightarrow$ linear response to perturbation $\rightarrow$ (higher-order) holographic transport coefficients
- Holographic OPE (need to expand in the inverse temperature β) J. High Energ. Phys. 2022, 234 (2022)

Thank you!