# Elliptic stable envelopes for certain non-symplectic varieties, and dynamical R-matrices for super spin chains from 3d $\mathcal{N} = 2$ quiver gauge theories

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Joint HEP-TH Seminar, BIMSA

September 13, 2023

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Consider a chain of $L$ spin-$1/2$ with nearest-neighbor interactions and periodic boundary condition:

$$H=-\frac{1}{2}\sum_{i=1}^{L}\left(J_x\sigma_i^x\sigma_{i+1}^x+J_y\sigma_i^y\sigma_{i+1}^y+J_z\sigma_i^z\sigma_{i+1}^z\right).$$

This is a Heisenberg spin chain model. Depending on coupling $J_x, J_y, J_z$, it is called

- XYZ model, if $J_x, J_y, J_z$ are different from each other,
- XXZ model, if $J_x = J_y \neq J_z$,
- XXX model, if $J_x = J_y = J_z$.

# Introduction: R-Matrix and YBE

For the XXX model, [H. Bethe (1931)] completely solved the eigenvalues and eigenvectors, using a method which is nowadays called *Coordinate Bethe Ansatz*. Bethe's work became the starting point of quantum integrability.

Later, [L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan (1979)] developed the *Algebraic Bethe Ansatz (ABA)* (also called Quantum Inverse Scattering Method (QISM)).

The ABA, among other things, gives a transfer matrix $T(u)$ which satisfies

$$[T(u), T(v)] = 0, \quad \forall (u,v) \in \mathbb{C}^2,$$

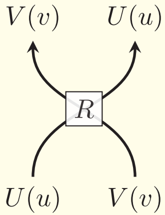
such that $T(u) = \mathrm{Id} + \sum_{i\geq 1} T_i u^{-i}$ with $T_1$ =Hamiltonian, and Bethe's eigenvector is a common eigenvector for $T(u)$ for all $u \in \mathbb{C}$.

# Introduction: R-Matrix and YBE

In the framework of ABA, the key to the integrability is a collection of **R-matrices**

$$R_{UV}(u, v) : U \otimes V \longrightarrow U \otimes V,$$

meromorphically depends on spectral parameters $(u, v)$. Pictorially presented as:

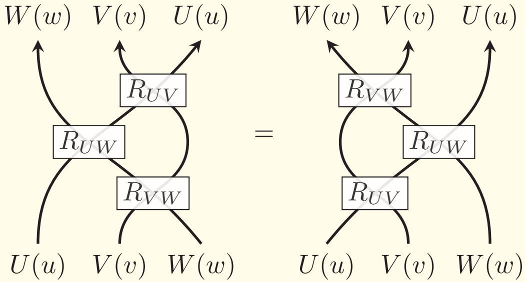



which satisfies the **Yang-Baxter Equation(YBE)**

## Introduction: R-Matrix and YBE

$$R_{UV}(u,v)R_{UW}(u,w)R_{VW}(v,w) = R_{VW}(v,w)R_{UW}(u,w)R_{UV}(u,v).$$

Pictorially presented as:



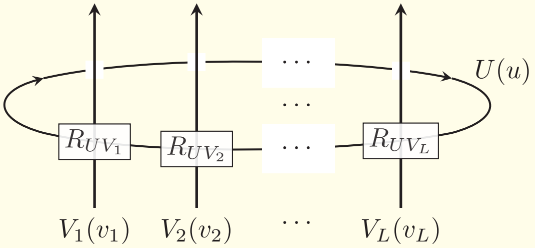

For simplicity, we assume that $R_{UV}(u,v) = R_{UV}(u-v)$.

# Introduction: R-Matrix and YBE

Given the spin chain Hilbert space $\mathcal{H} = \bigotimes_{i=1}^{L} V_i$, we add an auxiliary site $U$ to it with spectral parameter $u$. Then we define the transfer matrix:

$$T(u) := \mathrm{tr}_U [R_{UV_1}(u - v_1) R_{UV_2}(u - v_2) \cdots R_{UV_L}(u - v_L)] \in \mathrm{End}(\mathcal{H}),$$

which can be depicted in a diagram as:

$T(u) =$ 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# Introduction: R-Matrix and YBE

Then YBE $\Longrightarrow T(u)$ commutes with each other, i.e.

$$[T(u), T(v)] = 0, \quad \forall (u, v) \in \mathbb{C}^2,$$

More generally, we can define

$$\mathcal{T}^i_j(u) := \mathrm{tr}_U [E^i_j R_{UV_1}(u - v_1) R_{UV_2}(u - v_2) \cdots R_{UV_L}(u - v_L)] \in \mathrm{End}(\mathcal{H}),$$

where $E^i_j$ is the elementary matrix in $\mathrm{End}(U)$. In general, $\mathcal{T}^i_j(u)$ do not commute with each other, they satisfy the *RTT relations*:

$$R_{12}(u - v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)R_{12}(u - v)$$

where $\mathcal{T}_a(u) = E^j_i \otimes \mathcal{T}^i_j(u) \in \mathrm{End}(U_a \otimes \mathcal{H})$, $a = 1, 2$.

# Introduction: R-Matrix and YBE

## Definition

Fix $U$ together with R-matrix $R(u)$, we define the **spectrum generating algebra** $\mathcal{A}$ to be the algebra generated by the the modes $\mathcal{T}^i_{j;n}$ in $\mathcal{T}^i_j(u) = \mathrm{Id} + \sum_{n=1}^{\infty} \mathcal{T}^i_{j;n} u^{-n}$ subject to RTT relations.

## Example

In the XXX spin chain model, $U = \mathbb{C}^2$ and

$$R(u) = \mathrm{Id} + \frac{\hbar}{u} P,$$

where $\Pi(v_a \otimes v_b) = v_b \otimes v_a$. In this case

$$\mathcal{A} \cong \mathsf{Y}_\hbar(\mathfrak{gl}(2)),$$

the Yangian of $\mathfrak{gl}(2)$.

# Introduction: R-Matrix and YBE

In general, $R(u)$ depends on the spectral parameter $u$ in a periodic way

$$R(u+\Lambda) = R(u)$$

for certain discrete subgroup $\Lambda \subset \mathbb{C}$, we call $\mathbb{C}/\Lambda$ the **spectral curve**.

For Heisenberg spin chains, the spectral curves and spectral generating algebras are the following.

| Spin chain | Spectral curve | Spectrum generating algebra $\mathcal{A}$ |
|---|---|---|
| XXX | $\mathbb{C}$ | Yangian, $\mathsf{Y}_\hbar(\mathfrak{gl}(2))$ |
| XXZ | $\mathbb{C}^\times = \mathbb{C}/\mathbb{Z}$ | Quantum affine, $\mathcal{U}_\hbar(\widehat{\mathfrak{gl}}(2))$ |
| XYZ | $\mathbb{E}_\tau = \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$ | Elliptic quantum group, $E_{\tau,\hbar}(\mathfrak{gl}(2))$ |

# Introduction: R-Matrix and YBE

**Question:** How to find R-matrices?

- Algebraically, one way is to replace $\mathfrak{gl}(2)$ by more general Lie (super) algebras and play with quantum groups.
- In this talk, we focus on a different (but related) approach, which is hinted from the gauge theories.

In the work of [N. Nekrasov, S. Shatashvili (2009)], a correspondence between integrable spin chain models and SUSY gauge theories with 4 supercharges was proposed.

# Introduction: Bethe/Gauge

For example

XXX model with $L$ sites and $N$ magnons (excitation) $\longleftrightarrow$
2d $\mathcal{N}=(2,2)$ $U(N)$ with $L$ fundamental hypermultiplets

A particular essence of the Bethe/Gauge correspondence is

Hilbert space $\mathcal{H}$ $\longleftrightarrow$ Cohomology of Higgs branch $H(\mathcal{M}_H)$

e.g. $\mathcal{H}(N,L) :=$ $N$ magnon sector in XXX model with $L$ sites,

$$\mathcal{H}(N,L) \cong H(T^*\mathrm{Gr}(N,L)).$$

This suggests a geometric approach to quantum integrability.

# Introduction: Bethe/Gauge

- It was not clear in Nekrasov-Shatashvili's original paper that how to see the R-matrix or the full spectrum generating algebra from the Higgs branch geometry.
- There were works on the math side concerning quantum algebras acting on (generalized) cohomology of certain spaces:

$$\text{[Nakajima (1999)]}\,\mathcal{U}_q(L\mathfrak{g}_Q) \curvearrowright K_{\mathrm{eq}}(\mathcal{M}_Q),$$
$$\text{[Varagnolo (2000)]}\,\mathsf{Y}_\hbar(\mathfrak{g}_Q) \curvearrowright H_{\mathrm{eq}}(\mathcal{M}_Q).$$

$Q$: a quiver.
$\mathfrak{g}_Q$: Kac-Moody algebra associated to $Q$.
$\mathcal{M}_Q$: Nakajima quiver varieties (recalled later in this talk).

## Introduction: R-Matrix from Stable Envelope

The rational R-matrix (i.e. $R(u)$ is a rational function of $u$) from the Higgs branch geometry was later found by [D. Maulik and A. Okounkov (2013)], using a construction called the **stable envelope**.

- Assume that $X$ is a complex symplectic variety with a torus $T$ action, and a subtorus $A \subset T$ fixing the symplectic form.
- A stable envelope is a map

  $$\mathrm{Stab} : H_T(X^A) \to H_T(X),$$

  subject to certain conditions (reviewed later in this talk).
- $\mathrm{Stab}$ depends on a choice of *chamber*

  $$\mathfrak{C} \subset \mathrm{Lie}(A)_{\mathbb{R}}$$

  and the R-matrix of a pair of chambers $\mathfrak{C}_2, \mathfrak{C}_1$ is defined as

  $$R_{\mathfrak{C}_2,\mathfrak{C}_1} := \mathrm{Stab}_{\mathfrak{C}_2}^{-1} \circ \mathrm{Stab}_{\mathfrak{C}_1} : H_T(X^A) \to H_T(X^A).$$

# Introduction: R-Matrix from Stable Envelope

- It follows from definition that

$$R_{\mathfrak{C}_n,\mathfrak{C}_{n-1}} R_{\mathfrak{C}_{n-1},\mathfrak{C}_{n-2}} \cdots R_{\mathfrak{C}_2,\mathfrak{C}_1} = R_{\mathfrak{C}_n,\mathfrak{C}_1}.$$

- If there are three chambers $\mathfrak{C}_3, \mathfrak{C}_2, \mathfrak{C}_1$, then we can write $R_{-\mathfrak{C}_1,\mathfrak{C}_1}$ in two ways:

$$R_{-\mathfrak{C}_1,\mathfrak{C}_3} R_{\mathfrak{C}_3,\mathfrak{C}_2} R_{\mathfrak{C}_2,\mathfrak{C}_1} = R_{-\mathfrak{C}_1,\mathfrak{C}_1} = R_{-\mathfrak{C}_1,-\mathfrak{C}_2} R_{-\mathfrak{C}_2,-\mathfrak{C}_3} R_{-\mathfrak{C}_3,\mathfrak{C}_1}$$

- In the above case, if assume moreover that each pair $\mathfrak{C}_i, \mathfrak{C}_j$ is separated by a wall $\mathfrak{C}_{ij}$, then Maulik-Okounkov's theory of stable envelope implies that

$$R_{-\mathfrak{C}_i,-\mathfrak{C}_j} = R_{\mathfrak{C}_j,\mathfrak{C}_i}, \text{ denoted by } R_{ji}$$

  If we set $R_{-\mathfrak{C}_1,\mathfrak{C}_3} = R_{31}$, $R_{\mathfrak{C}_3,\mathfrak{C}_2} = R_{32}$, $R_{\mathfrak{C}_2,\mathfrak{C}_1} = R_{21}$, then we have YBE:

$$R_{31}R_{32}R_{21} = R_{21}R_{32}R_{31}$$

# Introduction: R-Matrix from Stable Envelope

- The main examples are Nakajima quiver varieties $\mathcal{M}_Q$, in this case the spectrum generating algebra (Maulik-Okounkov Yangian), denoted by $\mathsf{Y}^{\mathrm{MO}}(Q)$, is expected to be isomorphic to a Cartan-doubled version of $\mathsf{Y}_\hbar(\mathfrak{g}_Q)$ (proven in finite ADE case by [M. McBreen (2013)]).
- The particular example $Q = A_1$ gives the R-matrix of XXX spin chain, and $\mathsf{Y}^{\mathrm{MO}}(A_1) \cong \mathsf{Y}_\hbar(\mathfrak{gl}(2))$.
- Cohomology can be replaced by K-theory or elliptic cohomology, the corresponding stable envelopes for hypertoric varieties and Nakajima quiver varieties were constructed by [M. Aganagic and A. Okounkov (2016)].

$$\text{K-theory} \longrightarrow \text{trigonometric R-matrix},$$
$$\text{elliptic cohomology} \longrightarrow \text{elliptic dynamical R-matrix}.$$

- Physical realization of elliptic stable envelopes were recently worked out by [M. Dedushenko and N. Nekrasov (2021)], and independently by [M. Bullimore and D. Zhang (2021)].

# Introduction: R-Matrix from Stable Envelope

In the above formulation of stable envelope, $X$ is assumed to be complex symplectic. Typically it is a Higgs branch of 3d $\mathcal{N} = 4$ gauge theory.

For gauge theory with $4$ supercharges, e.g. 3d $\mathcal{N} = 2$, the Higgs branch is not necessarily symplectic.

**Question:** Can we extend the construction of stable envelopes to the Higgs branch of some 3d $\mathcal{N} = 2$ theory which do not have $\mathcal{N} = 4$ SUSY?

# Introduction: R-Matrix from Stable Envelope

- [R. Rimányi and L. Rozanky (2021)] studied $\mathrm{Tot}(V \to \mathrm{Gr}(N, L))$ for certain vector bundles $V$, e.g. $\mathcal{O}(-1)^{\oplus 2}$ on $\mathbb{P}^1$. They show that cohomological stable envelopes exist for these varieties, and the R-matrix is

$$\mathrm{Id} + \frac{\hbar}{u}\Pi \in \mathrm{End}(\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1})$$

  $\Pi(v_a \otimes v_b) = (-1)^{|v_a|\cdot|v_b|} v_b \otimes v_a$. This is the rational R-matrix for $\mathfrak{gl}(1|1)$.
- $\mathrm{Tot}(V \to \mathrm{Gr}(N, L))$ is the Higgs branch of a 3d $\mathcal{N} = 2$ $U(N)$ theory with $L$ fundamental hypermultiplets.
- In [S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)], we show that elliptic stable envelopes exist for the Higgs branches of 3d $\mathcal{N} = 2$ quiver gauge theories. $\mathrm{Tot}(V \to \mathrm{Gr}(N, L))$ is the special case when $Q = A_1$.

# Higgs Branch of 3d $\mathcal{N}=2$ Gauge Theories: Generalities

The essential data extracted from a 3d $\mathcal{N}=2$ gauge theory is

1. a complex algebraic group $G$,
2. a complex $G$-representation $\mathbf{M}$,
3. a $G$-invariant algebraic function $\mathcal{W}:\mathbf{M}\to\mathbb{C}$,
4. and a character $\zeta: G\to\mathbb{C}^\times$.

The Higgs branch of the 3d $\mathcal{N}=2$ gauge theory associated to $(G,\mathbf{M},\mathcal{W},\zeta)$ is then the GIT quotient

$$\mathcal{M}_H(G,\mathbf{M},\mathcal{W},\zeta) := \mathrm{Crit}(\mathcal{W})^{\zeta-ss}/G.$$

**Assumption.** We assume that the semistable locus $\mathrm{Crit}(\mathcal{W})^{\zeta-ss}$ is smooth and the action of $G$ on it is free.

Under the above assumption, $\mathcal{M}_H(G,\mathbf{M},\mathcal{W},\zeta)$ is smooth.

# Higgs Branch: Generalities

A typical example is as follows.

- Take $G = G_{\mathrm{ev}} \times G_{\mathrm{odd}}$, $\mathcal{R} \in \mathrm{Rep}(G)$, and then take $\mathbf{M} := \mathcal{R} \oplus \mathcal{R}^\vee \oplus \mathfrak{g}_{\mathrm{ev}}$.
- We choose a complex moment map $\mu : \mathcal{R} \oplus \mathcal{R}^\vee \to \mathfrak{g}$ for the $G$ action, and define $\mu_{\mathrm{ev}} : \mathcal{R} \oplus \mathcal{R}^\vee \to \mathfrak{g}_{\mathrm{ev}}^\vee$ to be the composition $\mathrm{pr}_{\mathrm{ev}} \circ \mu$, where $\mathrm{pr}_{\mathrm{ev}} : \mathfrak{g}^\vee \to \mathfrak{g}_{\mathrm{ev}}^\vee$ is the projection to the even part.
- We take $\mathcal{W} = \langle X, \mu_{\mathrm{ev}} \rangle$ where $X$ is the coordinate on $\mathfrak{g}_{\mathrm{ev}}$ and $\langle \cdot, \cdot \rangle$ is the pairing between $\mathfrak{g}_{\mathrm{ev}}$ and $\mathfrak{g}_{\mathrm{ev}}^\vee$.
- Then we choose a generic character $\zeta : G \to \mathbb{C}^\times$.

In this case, the Higgs branch is then isomorphic to

$$\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta) \cong \mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss}/G.$$

Note that $\mu^{-1}(0)^{\zeta-ss}/G \hookrightarrow \mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss}/G$.

# Higgs Branch: Abelian Gauge Theories

When $G$ is abelian, choose $\mathcal{R}$ such that we get exact sequence of abelian groups:

$$1 \longrightarrow G \longrightarrow (\mathbb{C}^\times)^{\mathrm{rk}\mathcal{R}} \longrightarrow Q \longrightarrow 1.$$

Then we have a commutative diagram

$$\begin{array}{ccccc}
\mu^{-1}(0)^{\zeta-ss}/G & \hookrightarrow & \mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss}/G & \hookrightarrow & (\mathcal{R}\oplus\mathcal{R}^\vee)^{\zeta-ss}/G \\
\downarrow & & \downarrow & & \downarrow \bar{\mu} \\
\{0\} & \hookrightarrow & \mathfrak{g}_{\mathrm{ev}}^{\perp} & \hookrightarrow & \mathfrak{g}^\vee
\end{array}$$

- The squares are Cartesian.
- $\bar{\mu}$ is flat.
- $\mu^{-1}(0)^{\zeta-ss}/G$ is a hypertoric variety, and $(\mathcal{R}\oplus\mathcal{R}^\vee)^{\zeta-ss}/G$ is known as the Lawrence toric variety.

# Higgs Branch: Abelian Gauge Theories

- If the charge matrix $A : \mathbb{Z}^{\mathrm{rk}\mathcal{R}} \to \mathrm{Char}(G)$ is surjective and unimodular, i.e. every $\mathrm{rk}G \times \mathrm{rk}G$ submatrix has determinant $\in \{0, \pm 1\}$, then $\bar{\mu}$ is smooth.

**Assumption.** When we talk about Higgs branch of abelian gauge theories, we always assume the charge matrix is surjective and unimodular.

- Under the above assumption, we have isomorphisms

$$H(\mu^{-1}(0)^{\zeta-ss}/G) \cong H(\mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss}/G) \cong H((\mathcal{R} \oplus \mathcal{R}^{\vee})^{\zeta-ss}/G)$$

In fact, every fiber $\bar{\mu}^{-1}(x)$ is diffeomorphic to $\bar{\mu}^{-1}(0)$, which is $\mu^{-1}(0)^{\zeta-ss}/G$ [T. Hausel and B. Sturmfels (2002)].

# Higgs Branch: Quiver Gauge Theories

A large class of GIT quotients comes from 3d $\mathcal{N} = 2$ quiver gauge theories.

- Let $Q = (Q_0, Q_1)$ be a quiver, $Q_0$ =set of nodes, $Q_1$ =set of arrows.
- $h, t : Q_1 \to Q_0$ maps an edge to its head and tail, respectively.
- We separate $Q_0$ into two parts $Q_0 = Q_0^{\mathrm{ev}} \sqcup Q_0^{\mathrm{odd}}$, called even and odd respectively. Notations:


- Let $\mathbf{w}, \mathbf{v} \in \mathbb{N}^{Q_0}$ be $Q_0$-tuples of natural numbers, called framing dimension vector and gauge dimension vector respectively.

# Higgs Branch: Quiver Gauge Theories

- The gauge group $G = G_{\mathrm{ev}} \times G_{\mathrm{odd}}$ is such that

$$G_{\mathrm{ev}} = \prod_{i \in Q_0^{\mathrm{ev}}} \mathrm{GL}(\mathbf{v}_i), \quad G_{\mathrm{odd}} = \prod_{i \in Q_0^{\mathrm{odd}}} \mathrm{GL}(\mathbf{v}_i).$$

- The representation space $\mathcal{R}$ is the following

$$\mathcal{R} = \bigoplus_{i \in Q_0} \mathrm{Hom}(V_i, W_i) \oplus \bigoplus_{a \in Q_1} \mathrm{Hom}(V_{t(a)}, V_{h(a)}),$$

- The flavour symmetry group $F = G_W \times \mathbb{C}_\hbar^\times$, such that

$$G_W = \prod_{i \in Q_0} \mathrm{GL}(\mathbf{w}_i),$$

$\mathrm{GL}(\mathbf{w}_i)$ acts on $W_i$ by fundamental representation, and $\mathbb{C}_\hbar^\times$ acts on $\mathcal{R} \oplus \mathcal{R}^\vee$ by scaling $\mathcal{R}^\vee$ with weight $\hbar^{-1}$ and fixing $\mathcal{R}$.

# Higgs Branch: Quiver Gauge Theories

- Notations of elements in $\mathcal{R}$:

$$\alpha_i \in \mathrm{Hom}(V_i, W_i), \quad x_a \in \mathrm{Hom}(V_{t(a)}, V_{h(a)})$$

For the dual representation $\mathcal{R}^\vee$:

$$\widetilde{\alpha}_i \in \mathrm{Hom}(W_i, V_i), \quad \widetilde{x}_a \in \mathrm{Hom}(V_{h(a)}, V_{t(a)})$$

- The holomorphic symplectic form on $\mathcal{R} \oplus \mathcal{R}^\vee$ is

$$\omega = \sum_{a \in Q_1} \mathsf{d}x_a \wedge \mathsf{d}\widetilde{x}_a + \sum_{i \in Q_0} \mathsf{d}\alpha_i \wedge \mathsf{d}\widetilde{\alpha}_i,$$

- There is a moment map $\mu : \mathcal{R} \oplus \mathcal{R}^\vee \to \mathfrak{g}^\vee$ which is given by

$$\mu(x_a, \widetilde{x}_a, \alpha_i, \widetilde{\alpha}_i) = \sum_{a \in Q_1} [x_a, \widetilde{x}_a] + \sum_{i \in Q_0} \widetilde{\alpha}_i \alpha_i.$$

# Higgs Branch: Quiver Gauge Theories

- We choose a character $\zeta : G \to \mathbb{C}^\times$, $\zeta$ can be written as $\zeta(g) = \prod_{i \in Q_0} \det(g_i)^{\zeta_i}$.

**Definition**

The quiver variety is defined to be the GIT quotient

$$\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w}) = \mu_{\text{ev}}^{-1}(0)^{\zeta-ss}/G.$$

For a fixed $\mathbf{w}$, we write

$$\mathcal{M}^\zeta(\mathbf{w}) := \bigsqcup_{\mathbf{v} \in \mathbb{N}^{Q_0}} \mathcal{M}^\zeta(\mathbf{v}, \mathbf{w}).$$

For a generic $\zeta$, we have $\mu_{\text{ev}}^{-1}(0)^{\zeta-ss} = \mu_{\text{ev}}^{-1}(0)^{\zeta-s}$, and

$$\mathcal{M}_H(G, \mathbf{M}, \mathcal{W}, \zeta) \cong \mathcal{M}^\zeta(\mathbf{v}, \mathbf{w}).$$

# Higgs Branch: Quiver Gauge Theories

**Lemma**

*Assume that $\zeta$ is generic then, $\mathcal{M}^{\zeta}(\mathbf{v},\mathbf{w})$ is a smooth variety and the quotient map $\mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss} \to \mathcal{M}^{\zeta}(\mathbf{v},\mathbf{w})$ is a principal $G$-bundle.*

Two generic $\zeta$:

$$\zeta_+ := (1,\cdots,1), \quad \zeta_- := (-1,\cdots,-1),$$

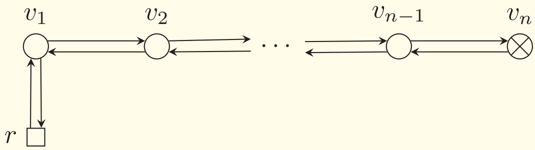
then according to the King's criterion for the stability [A. D. King (1994)], a quiver representation $(V, x_a, \widetilde{x}_a, \alpha_i, \widetilde{\alpha}_i) \in \mathcal{R} \oplus \mathcal{R}^\vee$ is $\zeta$-semistable if and only if

- $(\zeta_+)$ If $S_i \subset V_i$ are subspaces such that $S$ is preserved under the maps $(x_a, \widetilde{x}_a)$, and that $S_i \supset \mathrm{Im}(\widetilde{\alpha}_i)$ for all $i \in Q_0$, then $S = V$.
- $(\zeta_-)$ If $T_i \subset V_i$ are subspaces such that $T$ is preserved under the maps $(x_a, \widetilde{x}_a)$, and that $T_i \subset \mathrm{Ker}(\alpha_i)$ for all $i \in Q_0$, then $T = 0$.

# Higgs Branch: Quiver Gauge Theories

**Example 1.** If there is no odd node, i.e. $Q_0^{\mathrm{odd}}$ is empty, then $\mu_{\mathrm{ev}} = \mu$ and in this case $\mathcal{M}^{\zeta}(\mathbf{v}, \mathbf{w})$ is a Nakajima quiver variety.

**Example 2.** Let $Q$ be an $A_n$ quiver. Below is the doubled quiver $\overline{Q}$:



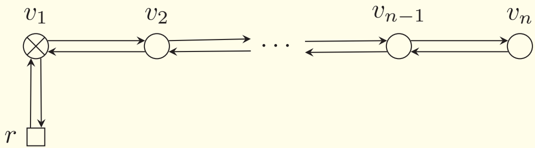

- $\mathcal{M}^{\zeta_+}(\mathbf{v}, \mathbf{w})$ is nonempty if and only if $r \geq v_1 \geq \cdots \geq v_n$.
- If nonempty then $\mathcal{M}^{\zeta_+}(\mathbf{v}, \mathbf{w}) \cong \mathrm{GL}_r \times^P \mathfrak{m}$, where $P \subset \mathrm{GL}_r$ stabilizes a fixed flag

$$F_\bullet = F_1 \subset F_2 \subset \cdots \subset F_n \subset F_{n+1} = \mathbb{C}^r, \quad \dim F_{n+1}/F_i = v_i$$

  $\mathfrak{m} \subset \mathrm{Lie}(P)$ such that $\mathfrak{m}(F_{i+1}) \subset F_i$ for $i < n$.
- $\mathcal{M}^{\zeta_+}(\mathbf{v}, \mathbf{w})$ contains $T^*\mathrm{Fl}_{\mathbf{v}}$ as a closed subvariety.

# Higgs Branch: Quiver Gauge Theories

**Example 3.** Let $Q$ be an $A_n$ quiver. Below is the doubled quiver $\overline{Q}$.




- $\mathcal{M}^{\zeta_+}(\mathbf{v}, \mathbf{w})$ is nonempty if and only if $v_n \leq r$ and $v_{i+1} \leq v_i \leq v_{i+1} + r$.
- If nonempty, $\mathcal{M}^{\zeta_+}(\mathbf{v}, \mathbf{w})$ is the total space of a vector bundle on

$$\mathrm{Gr}^{\omega_{d_n}}_{\mathrm{GL}_r} \widetilde{\times} \mathrm{Gr}^{\omega_{d_{n-1}}}_{\mathrm{GL}_r} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}^{\omega_{d_1}}_{\mathrm{GL}_r},$$

where $d_i = v_i - v_{i+1}$ for $1 \leq i \leq n-1$ and $d_n = v_n$, and $\omega_i$ is the $i$-th fundamental coweight of $\mathrm{GL}_r$.

**Remark.** In the last example, if we replace the odd node with a even one, then $\mathcal{M}^{\zeta_+}(\mathbf{v},\mathbf{w})$ is nonempty if and only if $r \geq v_1 \geq v_2 \geq \cdots \geq v_n$. In particular, if $n \geq 2$ then there exists $\mathbf{v}$ such that $\mu^{-1}(0)^{\zeta_+-ss}$ is empty but $\mu_{\mathrm{ev}}^{-1}(0)^{\zeta_+-ss}$ is nonempty, for instance $\mathbf{v} = (nr, (n-1)r, \cdots, r)$.

**Lemma**

*Let $\mathsf{A} \subset T_W$ be a subtorus, such that $W$ decomposes as eigenspaces*

$$W = \bigoplus_{\lambda \in \mathrm{Char}(\mathsf{A})} W^\lambda,$$

*and we write $\mathbf{w} = \sum_\lambda \mathbf{w}^\lambda$ for the dimension vector, then*

$$\mathcal{M}^\zeta(\mathbf{w})^{\mathsf{A}} \cong \prod_{\lambda \in \mathrm{Cochar}(\mathsf{A})} \mathcal{M}(\mathbf{w}^\lambda).$$

# Review of Equivariant Elliptic Cohomology

Let $q$ be a nonzero complex number such that $|q| < 1$, then take the elliptic curve $\mathbb{E} = \mathbb{C}^\times / q^{\mathbb{Z}}$.

For a reductive algebraic group $G$, the zeroth degree $G$-equivariant elliptic cohomology is a functor

$$\{G\text{-varieties}\} \to \{\text{schemes finite over } \mathcal{E}_G\}$$
$$X \mapsto \mathrm{Ell}_G(X),$$

$\mathcal{E}_G$ is the moduli scheme of semistable principal $G$-bundles of trivial topological type on the dual elliptic curve $\mathbb{E}^\vee$.

We will not encounter nonzero degree elliptic cohomology in this talk.

# Equivariant Elliptic Cohomology Base

- For a torus $T$,

$$\mathcal{E}_T = \mathbb{E} \otimes_{\mathbb{Z}} \mathrm{Cochar}(T).$$

- If $T$ is maximal torus of $G$ then [R. Friedman, J. W. Morgan and E. Witten (1997)]

$$\mathcal{E}_G \cong \mathcal{E}_T / W.$$

- When $G$ is simple and simply-connected, [Looijenga (1976)] showed that $\mathcal{E}_G$ is isomorphic to the weighted projective space $\mathbb{P}(1, g_1, \cdots, g_r)$, where $g_i$ are coefficients in the decomposition

$$\theta^\vee = \sum_i g_i \alpha_i^\vee,$$

of the dual of highest root into simple coroots.

# Chern Class

If $H$ is another reductive group, and $P \to X$ is a $G$-equivariant principal $H$-bundle, then $P$ induces the *Chern class* map:

$$c : \mathrm{Ell}_G(X) \to \mathrm{Ell}_H(\mathrm{pt}) = \mathcal{E}_H.$$

## Definition

For a vector bundle $V$ of rank $r$, the *Thom line bundle* associated to $V$ is defined as

$$\Theta(V) := c^*\mathcal{O}(D_\Theta), \quad D_\Theta = \{0\} + S^{r-1}\mathbb{E} \subset S^r\mathbb{E} = \mathcal{E}_{\mathrm{GL}_r}.$$

- $\Theta(V)$ inherits a canonical section $\vartheta(V)$ from the effective divisor $D_\Theta$.

# Theta Bundles

- $\Theta(V) = \Theta(V_1) \otimes \Theta(V_1)$ for a short exact sequence

$$0 \to V_1 \to V \to V_2 \to 0,$$

  so $\Theta : V \to \Theta(V)$ descends to a group homomorphism $K_G(X) \to \mathrm{Pic}(\mathrm{Ell}_G(X))$.
- The canonical section simply multiplies: $\vartheta(V) = \vartheta(V_1)\vartheta(V_2)$.
- We also have

$$\Theta(V^\vee) \cong \Theta(V),$$

  such that the canonical section picks up a sign $\vartheta(V^\vee) = (-1)^{\mathrm{rk}\, V}\vartheta(V)$.
- $\vartheta(x) = (x^{\frac{1}{2}} - x^{-\frac{1}{2}}) \prod_{n>0}(1 - q^n x)(1 - q^n x^{-1})$.

# Gysin Map

For a proper $G$-equivariant map $f : X \to Y$, assume that $f$ factors as a regular embedding $i : X \hookrightarrow Z$ and a smooth projection $p : Z \to Y$, and that both $i$ and $p$ are $G$-equivariant, then there exists a distinguished element (Gysin map):

$$f_{\circledast} \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{Ell}_G(Y)}}(f_* \Theta(T_f), \mathcal{O}_{\mathrm{Ell}_G(Y)}),$$

where $f_* : \mathrm{Ell}_G(X) \to \mathrm{Ell}_G(Y)$ is the induced map between elliptic cohomologies, and $T_f$ is the relative tangent bundle.

# Gysin Map

If $T_f$ equals to $f^*V$ for some $V \in K_G(Y)$, then we denote by $[X]$ the section $\Gamma(\mathrm{Ell}_G(Y), \Theta(-V))$ induced by

$$\mathcal{O}_{\mathrm{Ell}_G(Y)} \longrightarrow f_*\mathcal{O}_{\mathrm{Ell}_G(X)} \xrightarrow{f_\circledast} \Theta(-V).$$

For example, let $N \to X$ be a $G$-equivariant vector bundle and let $i : X \hookrightarrow N$ be the zero section, then $f_* : \mathrm{Ell}_G(X) \to \mathrm{Ell}_G(N)$ is an isomorphism, and $i_\circledast \in \Gamma(\mathrm{Ell}_G(X), \Theta(N))$ is the section $\vartheta(N)$.

# Supports

For a section $\alpha$ of a coherent sheaf $\mathcal{F}$ on $\mathrm{Ell}_G(X)$, and a $G$-invariant open subset $j : U \hookrightarrow X$, we say that $\alpha$ is supported on $X \setminus U$ if $j^*(\alpha) = 0$ in $\mathrm{Ell}_G(U)$.

We define $\mathrm{supp}(\alpha)$ to be the intersection of $G$-invariant closed subset that $\alpha$ is supported on.

The Gysin map can be defined for compactly supported sections. Namely there exists a distinguished element

$$f_{\circledast} \in \mathrm{Hom}_{\mathcal{O}_{\mathrm{Ell}_G(Y)}}(f_* \Theta(T_f)_c, \mathcal{O}_{\mathrm{Ell}_G(Y)}),$$

where $\Theta(T_f)_c \subset \Theta(T_f)$ is the subsheaf of sections $\alpha$ such that $f|_{\mathrm{supp}(\alpha)}$ is proper.

# Correspondences

Consider the diagram

$$\begin{array}{ccccc} & & X_2 \times X_1 & & \\ & \swarrow^{p_2} & & \searrow^{p_1} & \\ X_2 & & & & X_1 \\ & \searrow_{q_2} & & \swarrow_{q_1} & \\ & & \mathrm{pt} & & \end{array} .$$

Assume that $X_1$ is smooth, then for a pair of line bundles $\mathcal{L}_i \in \mathrm{Pic}(\mathrm{Ell}_G(X_i))$, and for any section

$$\alpha \in \Gamma(\mathrm{Ell}_G(X_2 \times X_1), \mathcal{L}_2 \boxtimes (\mathcal{L}_1^\vee \otimes \Theta(T_{X_1}))),$$

such that $\mathrm{supp}(\alpha)$ is proper over $X_2$, $\alpha$ induces a map

$$q_{1*}\mathcal{L}_1 \xrightarrow{p_{2\circledast}(\alpha p_1^*(\cdot))} q_{2*}\mathcal{L}_2,$$

# Degree of Line Bundle

For a line bundle $\mathcal{L}$ on an abelian variety $\mathcal{A}$, we define

$$\deg \mathcal{L} := [\mathcal{L}] \in \text{Néron-Severi group} = \mathrm{Pic}(\mathcal{A})/\mathrm{Pic}^0(\mathcal{A}).$$

- It is known that $\mathrm{NS}(\mathcal{A}) \cong \{f \in \mathrm{Hom}(\mathcal{A}, \mathcal{A}^\vee) : f = f^\vee\}$, and the isomorphism is given by [D. Mumford (1974)]

$$\mathcal{L} \mapsto (\phi_{\mathcal{L}} : x \mapsto x^*\mathcal{L} \otimes \mathcal{L}^{-1}).$$

- Let $T$ be a torus, then $\mathrm{Hom}(\mathcal{E}_T, \mathcal{E}_T^\vee)$ is isomorphic to $\mathrm{Char}(T)^{\otimes 2} \otimes_{\mathbb{Z}} \mathrm{End}(\mathbb{E})$.
- We choose $q$ generic so that $\mathrm{End}(\mathbb{E}) = \mathbb{Z}$, so $\mathrm{NS}(\mathcal{E}_T) \cong S^2\mathrm{Char}(T)$.
- Explicitly, any $\mu \in \mathrm{Char}(T)$ gives a map $\phi_\mu : \mathcal{E}_T \to \mathbb{E}$, then $\deg \phi_\mu^* \mathcal{O}(D_\Theta) = \mu \otimes \mu \in S^2\mathrm{Char}(T)$. For $V = \sum_\mu V_\mu \cdot \mu \in K_T(\mathrm{pt})$,

$$\deg \Theta(V) = \sum_\mu (\dim V_\mu) \mu \otimes \mu.$$

# Elliptic Stable Envelope: Partial Polarization

In [M. Aganagic, A. Okounkov (2016)], an ingredient for defining elliptic stable envelope is a polarization:

$$T_X = T_X^{1/2} + \hbar^{-1}(T_X^{1/2})^\vee \in K_T(X).$$

This structure is absent for Higgs branch of a general 3d $\mathcal{N} = 2$ gauge theory. E.g. $\exists$ polarization $\Longrightarrow \dim T_X$ is even, which is not true for $\mathrm{Tot}(\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1)$.

We introduce a generalization, called **partial polarization**, which will cover the 3d $\mathcal{N} = 2$ abelian or quiver gauge theories.

**Setting.** We denote by X a smooth quasi-projective complex variety with a torus T action, we fix a nontrivial group homomorphism $\mathsf{T} \to \mathbb{C}^\times_\hbar$ and a subtorus $\mathsf{A} \subset \ker(\mathsf{T} \to \mathbb{C}^\times_\hbar)$.

# Partial Polarization

**Definition ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])**

A partial polarization on X is the following data:

- a decomposition of the tangent bundle

$$T_{\mathsf{X}} = \mathrm{Pol}_{\mathsf{X}}^{+} + \hbar^{-1}(\mathrm{Pol}_{\mathsf{X}}^{+})^{\vee} + \mathrm{Pol}_{\mathsf{X}}^{-} + \hbar(\mathrm{Pol}_{\mathsf{X}}^{-})^{\vee} + \mathcal{E} \in K_{\mathsf{T}}(\mathsf{X}),$$

such that

(1) $\mathcal{E} = \mathcal{E}^{\vee}$ in $K_{\mathsf{T}}(\mathsf{X})$,

(2) $\Theta(\mathcal{E})$ admits a square root.

We define the opposite partial polarization to be

$$\mathrm{Pol}_{\mathsf{X}}^{\mathrm{op}} = \mathrm{Pol}_{\mathsf{X}}^{\mathrm{op},+} + \mathrm{Pol}_{\mathsf{X}}^{\mathrm{op},-} = \hbar^{-1}(\mathrm{Pol}_{\mathsf{X}}^{+})^{\vee} + \hbar(\mathrm{Pol}_{\mathsf{X}}^{-})^{\vee}.$$

# Partial Polarization

In practice, the existence of $\sqrt{\Theta(\mathcal{E})}$ is subtle. We give one criterion as follows:

**Lemma** ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])

*Let $G$ be a reductive group whose derived subgroup $[G, G]$ is simply connected and every simple constituent is of type $\mathrm{A}, \mathrm{C}, \mathrm{D}, \mathrm{E}_6$, or $\mathrm{G}_2$. Let $\mathcal{P}$ be a $\mathsf{T}$-equivariant principal $G$-bundle on $\mathsf{X}$, then $\Theta(\mathrm{adj}(\mathcal{P}))$ has a square root in $\mathrm{Pic}(\mathrm{Ell}_{\mathsf{T}}(\mathsf{X}))$.*

**Remark.** When $G = \mathrm{GL}_n$, the existence of $\sqrt{\Theta(\mathrm{adj}(\mathcal{P}))}$ follows from the "trick of diagonal" in the enumerative geometry.

# Partial Polarization

**Lemma** ([N. Nekrasov, A. Okounkov (2014)])

*Let $\mathcal{L}$ be a line bundle on $Y \times Y$ such that $(12)^*\mathcal{L} \cong \mathcal{L}$, where $(12)$ permutes two copies of $Y$, then $\Delta^*\mathcal{L}$ admits a square root.*

Back to the case when $G = \mathrm{GL}_n$, $\mathrm{adj}(\mathcal{P}) \cong \mathrm{End}(\mathcal{V})$ for a T-equivariant vector bundle $\mathcal{V}$. Notice that $\Theta(\mathcal{V} \otimes \mathcal{V}^\vee)$ is isomorphic to pullback $\Delta^*\Theta(\mathcal{V}_1 \boxtimes \mathcal{V}_2^\vee)$ along the diagonal morphism

$$\Delta : \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}) \hookrightarrow \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}) \times \mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$$

Moreover,

$$(12)^*\Theta(\mathcal{V}_1 \boxtimes \mathcal{V}_2^\vee) \cong \Theta(\mathcal{V}_1^\vee \boxtimes \mathcal{V}_2) \cong \Theta(\mathcal{V}_1 \boxtimes \mathcal{V}_2^\vee)$$

Thus $\Theta(\mathcal{V} \otimes \mathcal{V}^\vee)^{1/2}$ exists.

# Partial Polarization

**Example.** Suppose $G = G_{\mathrm{ev}} \times G_{\mathrm{odd}}$, $\mathcal{R} \in \mathrm{Rep}(G)$ and $\mu_{\mathrm{ev}} : \mathcal{R} \oplus \mathcal{R}^\vee \to \mathfrak{g}_{\mathrm{ev}}$ is the moment map for $G_{\mathrm{ev}}$. Assume moreover that $G_{\mathrm{odd}}$ is the reductive group whose derived subgroup $[G_{\mathrm{odd}}, G_{\mathrm{odd}}]$ is simply connected and every simple constituent is of type $\mathrm{A}, \mathrm{C}, \mathrm{D}, \mathrm{E}_6$, or $\mathrm{G}_2$. Then $\mathsf{X} := \mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss}/G$ is a partially-polarized variety with

$$\mathrm{Pol}_{\mathsf{X}} = \mathrm{Pol}_{\mathsf{X}}^{+} = \mathscr{R} - \mathrm{adj}(\mathcal{P}_{\mathrm{ev}}),$$

$$T_{\mathsf{X}} = \mathrm{Pol}_{\mathsf{X}} + \hbar^{-1}\mathrm{Pol}_{\mathsf{X}}^{\vee} - \mathrm{adj}(\mathcal{P}_{\mathrm{odd}}).$$

- $\mathcal{P}_{\mathrm{ev}} \times \mathcal{P}_{\mathrm{odd}}$ is the principal $G_{\mathrm{ev}} \times G_{\mathrm{odd}}$ bundle $\mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss} \to \mathsf{X}$.
- $\mathscr{R}$ is the bundle associated to the representation $\mathcal{R}$.

# Partial Polarization

Let $Q$ be a quiver with decomposition of nodes $Q_0 = Q_0^{\mathrm{ev}} \sqcup Q_0^{\mathrm{odd}}$. In this case $\mathsf{T} = \mathsf{A} \times \mathbb{C}_\hbar^\times$, where $\mathsf{A} = T_W$ is the maximal torus of the flavour group.

We further decompose $Q_0 = Q_0^+ \sqcup Q_0^-$, and set $c_i = \begin{cases} 1, & i \in Q_0^+, \\ -1, & i \in Q_0^-. \end{cases}$

We define the action of $\mathbb{C}_\hbar^\times$ on the doubled quiver $\overline{Q}$ by

| | $x_a$ | $\widetilde{x}_a$ | $\alpha_i$ | $\widetilde{\alpha}_i$ |
|---|---|---|---|---|
| $\mathbb{C}_\hbar^\times$-weight | $0$ | $-c_{h(a)}$ | $0$ | $-c_i$ |

**Assumption.** We assume that if $t(a) \in Q_0^{\mathrm{ev}}$, then $c_{h(a)} = c_{t(a)}$.

# Partial Polarization

In this setting,

$$\mathrm{Pol}^{+}_{\mathcal{M}} := \sum_{i \in Q_0^{+}} \mathcal{W}_i \mathcal{V}_i^{\vee} + \sum_{h(a) \in Q_0^{+}} \mathcal{V}_{h(a)} \mathcal{V}_{t(a)}^{\vee} - \sum_{j \in Q_0^{\mathrm{ev},+}} \mathcal{V}_j \mathcal{V}_j^{\vee},$$

$$\mathrm{Pol}^{-}_{\mathcal{M}} := \sum_{i \in Q_0^{-}} \mathcal{W}_i \mathcal{V}_i^{\vee} + \sum_{h(a) \in Q_0^{-}} \mathcal{V}_{h(a)} \mathcal{V}_{t(a)}^{\vee} - \sum_{j \in Q_0^{\mathrm{ev},-}} \mathcal{V}_j \mathcal{V}_j^{\vee},$$

is a partial polarization on $\mathcal{M}^{\zeta}(\mathbf{v}, \mathbf{w})$, and we have

$$T_{\mathcal{M}^{\zeta}(\mathbf{v},\mathbf{w})} = \mathrm{Pol}^{+}_{\mathcal{M}} + \hbar^{-1}(\mathrm{Pol}^{+}_{\mathcal{M}})^{\vee} + \mathrm{Pol}^{-}_{\mathcal{M}} + \hbar(\mathrm{Pol}^{-}_{\mathcal{M}})^{\vee} - \sum_{i \in Q_0^{\mathrm{odd}}} \mathcal{V}_i \mathcal{V}_i^{\vee},$$

# Elliptic Stable Envelope: Chambers and Attracting Sets

Let $\mathrm{Cochar}(\mathsf{A})$ be the cocharacter lattice of A, and we denote

$$\mathfrak{a}_{\mathbb{R}} := \mathrm{Cochar}(\mathsf{A}) \otimes_{\mathbb{Z}} \mathbb{R} \subset \mathrm{Lie}(\mathsf{A}).$$

We define

1. **Roots** of the pair $(\mathsf{X}, \mathsf{A})$ = set of weights $\{\alpha\}$ appearing in the normal bundle to $\mathsf{X}^{\mathsf{A}}$.
2. A **chamber** = a connected component of the complement of hyperplanes cut out by roots, i.e.

$$\mathfrak{a}_{\mathbb{R}} \setminus \bigcup_{\alpha \in \mathrm{roots}} \alpha^{\perp} = \bigsqcup_i \mathfrak{C}_i.$$

3. Let $\mathfrak{C}$ be a chamber, then we say that a root $\alpha$ is **attracting** (resp. **repelling**) if $\alpha$ is positive (resp. negative) on $\mathfrak{C}$.

# Chambers and Attracting Sets

We define attracting subvariety

$$\mathsf{Attr}_{\mathfrak{C}}(\mathcal{F}) := \{x \in \mathsf{X} \mid \lim_{t\to 0} \sigma(t) \cdot x \in \mathcal{F}\},$$

for some $\sigma \in \mathfrak{C} \cap \mathrm{Cochar}(\mathsf{A})$.

- $\mathsf{Attr}_{\mathfrak{C}}(\mathcal{F})$ does not depend on the choice of $\sigma$.
- $\mathsf{Attr}_{\mathfrak{C}}(\mathcal{F})$ is the exponential of attracting part of the normal bundle $N^+_{\mathsf{X}/\mathcal{F}}$.

Define the union $\mathsf{Attr}_{\mathfrak{C}} := \coprod_{\mathcal{F}} \mathsf{Attr}_{\mathfrak{C}}(\mathcal{F})$, it admits an immersion

$$\mathsf{Attr}_{\mathfrak{C}} \hookrightarrow \mathsf{X} \times \mathsf{X}^{\mathsf{A}}, \quad x \mapsto (x, \lim_{t\to 0} \sigma(t) \cdot x).$$

- $\mathsf{Attr}_{\mathfrak{C}}$ is not closed in $\mathsf{X} \times \mathsf{X}^{\mathsf{A}}$.

# Chambers and Attracting Sets

We define $\mathsf{Attr}^f_{\mathfrak{C}}$ to be the set of pairs $(x, y)$ that belongs to a chain of closures of attracting A-orbits.

- $\mathsf{Attr}^f_{\mathfrak{C}}$ is closed in $\mathsf{X} \times \mathsf{X}^\mathsf{A}$.

We define a partial order $\preceq$ on the set of connected components of $\mathsf{X}^\mathsf{A}$ by

$$\mathcal{F}_j \cap \overline{\mathsf{Attr}_{\mathfrak{C}}(\mathcal{F}_i)} \neq \emptyset \Longrightarrow \mathcal{F}_j \preceq \mathcal{F}_i.$$

We define closed subvarieties $\mathsf{Attr}^<_{\mathfrak{C}} \subset \mathsf{Attr}^\leq_{\mathfrak{C}} \subset \mathsf{X} \times \mathsf{X}^\mathsf{A}$ by

$$\mathsf{Attr}^<_{\mathfrak{C}} := \bigcup_{\mathcal{F}_j \prec \mathcal{F}_i} \mathsf{Attr}_{\mathfrak{C}}(\mathcal{F}_j) \times \mathcal{F}_i,$$

$$\mathsf{Attr}^\leq_{\mathfrak{C}} := \bigcup_{\mathcal{F}_j \preceq \mathcal{F}_i} \mathsf{Attr}_{\mathfrak{C}}(\mathcal{F}_j) \times \mathcal{F}_i.$$

- Note that $\mathsf{Attr}^f_{\mathfrak{C}} \cap (\mathsf{Attr}^\leq_{\mathfrak{C}} \setminus \mathsf{Attr}^<_{\mathfrak{C}}) = \mathsf{Attr}_{\mathfrak{C}}$.

# Chambers and Attracting Sets: Quiver Variety

Consider $\mathsf{X} = \mathcal{M}^{\zeta_-}(\mathbf{v}, \mathbf{w})$ for a quiver $Q$ with dimension vectors $(\mathbf{v}, \mathbf{w})$ and the stability condition $\zeta_-$. Let $\mathsf{A} = \mathbb{C}_a^\times \curvearrowright W$ such that $W = W^{(1)} \oplus aW^{(2)}$ (dimension decomposes as $\mathbf{w} = \mathbf{w}^{(1)} + \mathbf{w}^{(2)}$), then

$$\mathsf{X}^{\mathsf{A}} = \bigsqcup_{\mathbf{v}^{(1)}+\mathbf{v}^{(2)}=\mathbf{v}} \mathcal{M}^{\zeta}(\mathbf{v}^{(1)}, \mathbf{w}^{(1)}) \times \mathcal{M}^{\zeta}(\mathbf{v}^{(2)}, \mathbf{w}^{(2)}).$$

Then for all $\mathbf{u} \in \mathbb{N}^{Q_0}$,

$$\mathcal{M}^{\zeta}(\mathbf{v}^{(1)}, \mathbf{w}^{(1)}) \times \mathcal{M}^{\zeta}(\mathbf{v}^{(2)}, \mathbf{w}^{(2)}) \preceq \mathcal{M}^{\zeta}(\mathbf{v}^{(1)} + \mathbf{u}, \mathbf{w}^{(1)}) \times \mathcal{M}^{\zeta}(\mathbf{v}^{(2)} - \mathbf{u}, \mathbf{w}^{(2)}),$$

# Chambers and Attracting Sets

The restriction of partial polarization $\mathrm{Pol}_{\mathsf{X}}$ to $\mathsf{X}^{\mathsf{A}}$ decomposes according to the chamber $\mathfrak{C}$ as

$$\mathrm{Pol}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}}} = \mathrm{Pol}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}},>0} + \mathrm{Pol}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}},\mathrm{fixed}} + \mathrm{Pol}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}},<0}.$$

**Definition**

We define the index bundle

$$\mathsf{ind} = \mathrm{Pol}^{+}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}},>0} - \mathrm{Pol}^{-}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}},>0} \in K_{\mathsf{T}}(\mathsf{X}^{\mathsf{A}}),$$

**Lemma ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])**

$\mathrm{Pol}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}},\mathrm{fixed}}$ *is a partial polarization on* $\mathsf{X}^{\mathsf{A}}$*.*

# Elliptic Stable Envelope: Attractive Line Bundle

**Definition ([A. Okounkov (2020)])**

A line bundle $\mathcal{L}$ on $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$ is called attractive for a given chamber $\mathfrak{C}$ if

$$\deg_{\mathsf{A}} \mathcal{L} = \deg_{\mathsf{A}} \Theta(N^{-}_{\mathsf{X}/\mathsf{X}^{\mathsf{A}}}),$$

where $\deg_{\mathsf{A}} \mathcal{L}$ is the degree of the restriction of $\mathcal{L}$ to the fiber along the projection $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X}^{\mathsf{A}}) \to \mathrm{Ell}_{\mathsf{T}/\mathsf{A}}(\mathsf{X}^{\mathsf{A}})$.

- Each fiber of $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X}^{\mathsf{A}}) \to \mathrm{Ell}_{\mathsf{T}/\mathsf{A}}(\mathsf{X}^{\mathsf{A}})$ is isomorphic to $\mathcal{E}_{\mathsf{A}}$.
- The $\deg_{\mathsf{A}}$ takes value in $H^0(\mathsf{X}^{\mathsf{A}}, S^2\mathrm{Char}(\mathsf{A}))$.

# Attractive Line Bundle

Consider a partial polarization $\mathrm{Pol}_{\mathsf{X}} = \mathrm{Pol}^{+}_{\mathsf{X}} + \mathrm{Pol}^{-}_{\mathsf{X}}$ with

$$T_{\mathsf{X}} = \mathrm{Pol}^{+}_{\mathsf{X}} + \hbar^{-1}(\mathrm{Pol}^{+}_{\mathsf{X}})^{\vee} + \mathrm{Pol}^{-}_{\mathsf{X}} + \hbar(\mathrm{Pol}^{-}_{\mathsf{X}})^{\vee} + \mathcal{E}$$

From now on we fix a square root for $\Theta(\mathcal{E})$.

**Definition**

Define a line bundle on $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$:

$$\mathcal{S}_{\mathsf{X}} := \Theta(\mathrm{Pol}_{\mathsf{X}}) \otimes \Theta(\mathcal{E})^{\otimes \frac{1}{2}}$$

**Proposition** ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])

*$\mathcal{S}_{\mathsf{X}}$ is an attractive line bundle for every chamber $\mathfrak{C} \subset \mathfrak{a}_{\mathbb{R}}$.*

# Attractive Line Bundle

For a line bundle $\mathcal{L}$ on $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$, define twisted dual $\mathcal{L}^{\nabla} := \mathcal{L}^{\vee} \otimes \Theta(T_{\mathsf{X}})$. Then

$$\mathcal{S}_{\mathsf{X}}^{\nabla} \cong \Theta(\mathrm{Pol}_{\mathsf{X}}^{\mathrm{op}}) \otimes \Theta(\mathcal{E})^{\otimes \frac{1}{2}}$$

therefore $\mathcal{S}_{\mathsf{X}}^{\nabla}$ is the attractive line bundle associated to the opposite partial polarization.

We define

$$\mathcal{S}_{\mathsf{X},\mathsf{A}} := i^* \mathcal{S}_{\mathsf{X}} \otimes \Theta(-N^{-}_{\mathsf{X}/\mathsf{X}^{\mathsf{A}}}), \quad i : \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}^{\mathsf{A}}) \to \mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$$

On the other hand, we also have the line bundle $\mathcal{S}_{\mathsf{X}^{\mathsf{A}}}$ defined using the restriction of the partial polarization $\mathrm{Pol}_{\mathsf{X}^{\mathsf{A}}}$. Then we have:

$$\mathcal{S}_{\mathsf{X},\mathsf{A}} \otimes \mathcal{U} \cong \mathcal{S}_{\mathsf{X}^{\mathsf{A}}} \otimes \Theta(\hbar)^{-\mathrm{rk}\,\mathsf{ind}} \otimes \tau(-\hbar \det \mathsf{ind})^* \mathcal{U}.$$

$\mathcal{U}$ and $\tau(\cdots)$ will be explained soon.

# Elliptic Stable Envelope: Kähler Parameters

**Assumption.** We assume that $\mathrm{Pic}(\mathsf{X})$ is finitely generated as an abelian group, and we fix a set of generators $\{\mathcal{L}_i^\circ\}_{i=1}^r$, which induces

$$\mathsf{K} = (\mathbb{C}^\times)^r \twoheadrightarrow \mathrm{Pic}(\mathsf{X}) \otimes_{\mathbb{Z}} \mathbb{C}^\times.$$

$\mathsf{K}$ is called the Kähler torus.

- For quiver variety $\mathcal{M}^\zeta(\mathbf{v}, \mathbf{w})$, $\mathsf{K}$ can be chosen to be $(\mathbb{C}^\times)^{Q_0}$.

We choose an equivariant lift $\mathcal{L}_i \in \mathrm{Pic}_{\mathsf{T}}(\mathsf{X})$ for each $\mathcal{L}_i^\circ$, then get Chern classes $c_i : \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}) \to \mathbb{E}$. We define

$$\mathcal{U}(\mathcal{L}_i, z_i) := (c_i \times 1)^* \mathcal{U}_{\text{Poincaré}}, \quad c_i \times 1 : \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}) \times \mathcal{E}_{z_i} \to \mathbb{E} \times \mathcal{E}_{z_i},$$

where we identify $\mathbb{E} \cong \mathcal{E}_{z_i}^\vee$, and $\mathcal{U}_{\text{Poincaré}}$ is the universal line bundle on $\mathbb{E} \times \mathbb{E}^\vee$.

# Kähler Parameters

**Definition ([M. Aganagic, A. Okounkov (2016)])**

We define the **extended equivariant elliptic cohomology**

$$\mathsf{E}_{\mathsf{T}}(\mathsf{X}) := \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}) \times \mathcal{E}_{\mathsf{K}}$$

which is a scheme over

$$\mathcal{B}_{\mathsf{T},\mathsf{X}} := \mathcal{E}_{\mathsf{T}} \times \mathcal{E}_{\mathsf{K}}.$$

And we define a line bundle on $\mathsf{E}_{\mathsf{T}}(\mathsf{X})$:

$$\mathcal{U} := \bigotimes_{i=1}^{r} \mathcal{U}(\mathcal{L}_i, z_i).$$

# Kähler Parameters

For a homomorphism between abelian varieties $g : \mathcal{E}_\mathsf{T} \to \mathcal{E}_\mathsf{K}$, we associate an automorphism $\tau(g) : \mathcal{B}_{\mathsf{T},\mathsf{X}} \cong \mathcal{B}_{\mathsf{T},\mathsf{X}}$ by

$$(t, z) \mapsto (t, z + g(t)).$$

We use the same notation $\tau(g) : \mathsf{E}_\mathsf{T}(\mathsf{X}) \cong \mathsf{E}_\mathsf{T}(\mathsf{X})$.

- $\tau(g)$ does not affect $\mathrm{Ell}_\mathsf{T}(\mathsf{X})$, it only shift the Kähler parameters.
- The line bundle $\tau(g)^*\mathcal{U} \otimes \mathcal{U}^{-1}$ is trivial along $\mathcal{E}_\mathsf{K}$ direction, i.e. it is the pullback of a line bundle from $\mathrm{Ell}_\mathsf{T}(\mathsf{X})$.
- The line bundle $\tau(g)^*\mathcal{U} \otimes \mathcal{U}^{-1}$ depends linearly on $g$, i.e.

$$\frac{\tau(g_1)^*\mathcal{U}}{\mathcal{U}} \otimes \frac{\tau(g_2)^*\mathcal{U}}{\mathcal{U}} \cong \frac{\tau(g_1 \cdot g_2)^*\mathcal{U}}{\mathcal{U}}.$$

# Kähler Parameters

An example is as follows: for a pair

$$\mu \in \mathrm{Char}(\mathsf{T}) = \mathrm{Hom}(\mathcal{E}_{\mathsf{T}}, \mathbb{E}), \quad \lambda \in \mathrm{Cochar}(\mathsf{K}) = \mathrm{Hom}(\mathbb{E}, \mathcal{E}_{\mathsf{K}}),$$

$\lambda\mu \in \mathrm{Hom}(\mathcal{E}_{\mathsf{T}}, \mathcal{E}_{\mathsf{K}})$, so we have $\tau(\lambda\mu)$.

It is known [M. Aganagic, A. Okounkov (2016)] that there is a meromorphic section

$$\frac{\vartheta(\lambda \cdot \mu)}{\vartheta(\lambda)\vartheta(\mu)},$$

of the line bundle $\tau(\lambda\mu)^*\mathcal{U} \otimes \mathcal{U}^{-1}$ on $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$, here $\lambda \cdot \mu$ is the coordinate multiplication.

# Elliptic Stable Envelope: Main Results

For a line bundle $\mathcal{L}$ on $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$, denote

$$\mathcal{L}_{\mathsf{A}} := i^*\mathcal{L} \otimes \Theta(-N^{-}_{\mathsf{X}/\mathsf{X}^{\mathsf{A}}}), \quad i : \mathrm{Ell}_{\mathsf{T}}(\mathsf{X}^{\mathsf{A}}) \to \mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$$

**Theorem ([A. Okounkov (2020)])**

*If $\mathcal{L}$ is an attractive line bundle for the chamber $\mathfrak{C}$, then there exists a* **unique** *meromorphic section*

$$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}} \in \Gamma(\mathsf{E}_{\mathsf{T}}(\mathsf{X} \times \mathsf{X}^{\mathsf{A}}) \backslash \boldsymbol{\Delta}, \mathcal{L} \otimes \mathcal{U} \boxtimes (\mathcal{L}_{\mathsf{A}} \otimes \mathcal{U})^{\nabla}),$$

*such that*

1. *it is supported on* $\mathrm{Attr}^{f}_{\mathfrak{C}}$
2. *its restriction to the complement of* $\mathrm{Attr}^{<}_{\mathfrak{C}}$ *is given by* $[\mathrm{Attr}_{\mathfrak{C}}]$.

Here $\boldsymbol{\Delta} \subset \mathcal{B}_{\mathsf{T},\mathsf{K}}$ is the locus where $\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}}$ has poles.

# Main Results

**Theorem** ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])

*If* $\mathsf{X}$ *is partially-polarized, then for every chamber* $\mathfrak{C}$*, there exists a unique meromorphic section*

$$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_\mathsf{X}} \in \Gamma(\mathsf{E_T}(\mathsf{X}\times\mathsf{X}^\mathsf{A})\backslash\mathbf{\Delta}, \mathcal{S}_\mathsf{X}\otimes\mathcal{U}\boxtimes(\mathcal{S}_{\mathsf{X,A}}\otimes\mathcal{U})^\triangledown),$$

*such that*

1. *it is supported on* $\mathsf{Attr}^f_\mathfrak{C}$
2. *its restriction to the complement of* $\mathsf{Attr}^<_\mathfrak{C}$ *is given by* $(-1)^{\mathrm{rk}\ \mathsf{ind}}[\mathsf{Attr}_\mathfrak{C}]$.

# Main Results

$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}}$ gives rise to a map between sheaves

$$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}} : \mathcal{S}_{\mathsf{X},\mathsf{A}} \otimes \mathcal{U} \to \mathcal{S}_{\mathsf{X}} \otimes \mathcal{U}$$

on $\mathcal{B}_{\mathsf{T},\mathsf{X}} \backslash \boldsymbol{\Delta}$, such that

(1) The support of $\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}}$ is triangular with respect to $\prec$, i.e.

$$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}}|_{\mathcal{F}_j \times \mathcal{F}_i} = 0$$

for a pair of connected components of $\mathsf{X}^{\mathsf{A}}$ such that $\mathcal{F}_j \not\preceq \mathcal{F}_i$.

(2) The diagonal

$$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}}|_{\mathcal{F}_i \times \mathcal{F}_i} = (-1)^{\mathrm{rk}\,\mathsf{ind}} \vartheta(N^-_{\mathsf{X}/\mathcal{F}_i})$$

# Example: Abelian Case

Consider a torus $G = G_{\text{ev}} \times G_{\text{odd}} \curvearrowright \mathcal{R} \oplus \mathcal{R}^\vee$. There is a Cartesian diagram

$$\begin{array}{ccc} \mu^{-1}(0)^{\zeta-ss}/G & \hookrightarrow & \mu_{\text{ev}}^{-1}(0)^{\zeta-ss}/G \\ \downarrow & & \downarrow \\ \{0\} & \hookrightarrow & \mathfrak{g}_{\text{ev}}^{\perp} \end{array}$$

- $\mathsf{T} = \mathsf{A} \times \mathbb{C}_\hbar^\times = ((\mathbb{C}^\times)^{\text{rk}\mathcal{R}}/G) \times \mathbb{C}_\hbar^\times$, where $\mathbb{C}_\hbar^\times \curvearrowright \mathcal{R} \oplus \mathcal{R}^\vee$ by $\mathcal{R} \oplus \hbar^{-1}\mathcal{R}^\vee$.
- $\mathsf{X} := \mu_{\text{ev}}^{-1}(0)^{\zeta-ss}/G$ is a $\mathsf{T}$-equivariant smooth deformation of $\mathsf{X}_0 := \mu^{-1}(0)^{\zeta-ss}/G$ over the base $\mathfrak{g}_{\text{ev}}^{\perp}$
- The inclusion $\mathsf{X}_0 \hookrightarrow \mathsf{X}$ induces $\text{Ell}_{\mathsf{T}}(\mathsf{X}_0) \cong \text{Ell}_{\mathsf{T}}(\mathsf{X})$. And $\mathcal{S}_{\mathsf{X}} \cong \mathcal{S}_{\mathsf{X}_0}$.
- Components of $\mathsf{X}_0^{\mathsf{A}} \overset{1:1}{\longleftrightarrow}$ components of $\mathsf{X}^{\mathsf{A}}$.
- $\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}} = \mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}_0}}$.

# Example: $A_1$ Quiver

Consider an $A_1$ quiver $Q$ with one odd node. Take $\mathbf{v} = N, \mathbf{w} = L$, and $\zeta = \zeta_-$, then

- $\mathcal{M}^\zeta(N, L)$ is nonempty if and only if $N \le L$
- If nonempty then

$$\mathcal{M}^\zeta(N, L) \cong \mathrm{Tot}(\mathcal{V}^{\oplus L} \to \mathrm{Gr}(N, L)),$$

  where $\mathcal{V}$ is the tautological bundle of rank $N$.
- In this case $\mathsf{A} = (\mathbb{C}^\times)^L$ and $\mathsf{T} = \mathsf{A} \times \mathbb{C}^\times_\hbar$. $\mathsf{A}$ acts on both the base and the fiber of $\mathrm{Tot}(\mathcal{V}^{\oplus L} \to \mathrm{Gr}(N, L))$, and $\mathbb{C}^\times_\hbar$ scales the fiber with weight $\hbar^{-1}$.
- Connected components of $\mathsf{A}$-fixed points are labelled by order-preserving embedding $p : \{1, \cdots, N\} \hookrightarrow \{1, \cdots, L\}$.
- Denote by $\{m_1, \cdots, m_L\}$ the coordinates on $\mathrm{Lie}(\mathsf{A})_\mathbb{R}$, then we choose the chamber

$$\mathfrak{C} = \{m_1 < \cdots < m_L\}$$

.

# Example: $A_1$ Quiver

Equivariant parameters:

1. gauge group $\mathrm{GL}_N$: $\boldsymbol{s} = \{s_a\}_{a=1}^N$
2. flavour torus A: $\boldsymbol{x} = \{x_i\}_{i=1}^L$.

Kähler torus $\mathsf{K} = \mathbb{C}^\times$, with Kähler parameter $z$.

The natural inclusion $\mu_{\mathrm{ev}}^{-1}(0)^{\zeta-ss}/G \hookrightarrow [\mathcal{R} \oplus \mathcal{R}^\vee/G]$ induces a map

$$\mathrm{Ell}_{\mathsf{T}}(\mathsf{X}) \to \mathrm{Ell}_{\mathsf{T}}([\mathcal{R} \oplus \mathcal{R}^\vee/G]) \cong \mathcal{E}_{\mathsf{T}} \times \mathcal{E}_G.$$

- $\mathcal{S}_{\mathsf{X}}$ is actually defined on $\mathcal{E}_{\mathsf{T}} \times \mathcal{E}_G$.
- Every component $\mathcal{F}_p$ of $\mathsf{X}^{\mathsf{A}}$ is a vector space, thus

$$\mathrm{Ell}_{\mathsf{T}}(\mathsf{X} \times \mathcal{F}_p) \cong \mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$$

# Example: $A_1$ Quiver

Then the elliptic stable envelope is

$$\mathbf{Stab}_{\mathfrak{C},\mathcal{S}_{\mathsf{X}}}|_{\mathsf{X}\times\mathcal{F}_p} = \mathsf{Sym}_{S_N}\left[\left(\prod_{a=1}^{N}\mathbf{f}_{p(a)}(s_a,\boldsymbol{x},\hbar,z)\right)\cdot\left(\prod_{a>b}\frac{1}{\vartheta(s_a s_b^{-1})}\right)\right],$$

- $\mathrm{Sym}_{S_N}$ = summation over permutations $\{s_a\}\mapsto\{s_{\sigma(a)}\}_{\sigma\in S_N}$
- $\mathbf{f}_m(s,\boldsymbol{x},\hbar,z)$ is the following function

$$\mathbf{f}_m(s,\boldsymbol{x},\hbar,z) := \frac{\vartheta(s x_m \hbar^{m-L} z)}{\vartheta(\hbar^{m-L} z)}\prod_{i<m}\vartheta(s x_i)\prod_{j>m}\vartheta(s x_j \hbar),$$

# R-Matrix

Fix a partial polarization and write $\mathbf{Stab}_{\mathfrak{C}} = \mathbf{Stab}_{\mathfrak{C},\mathcal{S}_\mathsf{X}}$.

**Definition**

Let $\mathfrak{C}_1, \mathfrak{C}_2$ be two chambers in $\mathrm{Lie}(\mathsf{A})$, then we define the R-matrix

$$\mathbf{R}_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1} := \mathbf{Stab}_{\mathfrak{C}_2}^{-1} \circ \mathbf{Stab}_{\mathfrak{C}_1}.$$

This is a map from $\mathcal{S}_{\mathsf{X}^\mathsf{A}} \otimes \tau(-\hbar \det \mathsf{ind}_1)^* \mathcal{U}$ to $\mathcal{S}_{\mathsf{X}^\mathsf{A}} \otimes \tau(-\hbar \det \mathsf{ind}_2)^* \mathcal{U}$, where $\mathsf{ind}_1$ and $\mathsf{ind}_2$ are index bundles for the chambers $\mathfrak{C}_1$ and $\mathfrak{C}_2$ respectively.

It follows from definition that

$$\mathbf{R}_{\mathfrak{C}_3 \leftarrow \mathfrak{C}_2} \mathbf{R}_{\mathfrak{C}_2 \leftarrow \mathfrak{C}_1} = \mathbf{R}_{\mathfrak{C}_3 \leftarrow \mathfrak{C}_1}.$$

# Wall R-Matrix

Let $\mathfrak{C}' \subset \mathfrak{C}$ be a face, and let $\mathsf{A}' \subset \mathsf{A}$ be the subtorus associated to the span of $\mathfrak{C}'$ in $\mathrm{Lie}(\mathsf{A})$. Then

- $\mathfrak{C}/\mathfrak{C}'$ is a chamber of $\mathsf{A}/\mathsf{A}'$.
- There is a partial polarization $\mathrm{Pol}_{\mathsf{X}}|_{\mathsf{X}^{\mathsf{A}'},\mathrm{fixed}}$ on $\mathsf{X}^{\mathsf{A}'}$, so we get attractive lien bundle $\mathcal{S}_{\mathsf{X}^{\mathsf{A}'}}$.
- So we have stable envelope on $\mathsf{X}^{\mathsf{A}'}$:

$$\mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}',\mathsf{X}^{\mathsf{A}'}} : \mathcal{S}_{\mathsf{X}^{\mathsf{A}'},\mathsf{A}} \otimes \mathcal{U} \to \mathcal{S}_{\mathsf{X}^{\mathsf{A}'}} \otimes \mathcal{U}$$

- (Triangle Lemma)

$$\mathbf{Stab}_{\mathfrak{C}}(z) = \mathbf{Stab}_{\mathfrak{C}'}(z) \circ \mathbf{Stab}_{\mathfrak{C}/\mathfrak{C}'}(z - \hbar \det \mathsf{ind}_{\mathsf{X}^{\mathsf{A}'}}).$$

# Wall R-Matrix

If $\mathfrak{C}' = \mathfrak{C}_1 \cap \mathfrak{C}_2$ is of codimension one, then we say $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are separated by the wall $\mathfrak{C}'$, and define the **wall R-matrix**

$$\mathbf{R}_{\text{wall}} := \mathbf{R}_{\mathfrak{C}_2/\mathfrak{C}' \leftarrow \mathfrak{C}_1/\mathfrak{C}'}.$$

$\mathbf{R}_{\text{wall}}$ only involves one equivariant parameter coming from $\mathsf{A}/\mathsf{A}'$.

**Example.** Suppose that $\mathsf{A} = (\mathbb{C}^\times)^3$ and let $m_1, m_2, m_3$ be coordinates on $\mathrm{Lie}(\mathsf{A})_\mathbb{R}$. Consider two chambers:

$$\mathfrak{C}_{123} = \{m_1 < m_2 < m_3\}, \quad \mathfrak{C}_{321} = \{m_3 < m_2 < m_1\}.$$

There are two paths that connects $\mathfrak{C}_{123}$ and $\mathfrak{C}_{321}$:

1. $\mathfrak{C}_{123} \to \mathfrak{C}_{132} \to \mathfrak{C}_{312} \to \mathfrak{C}_{321}$
2. $\mathfrak{C}_{123} \to \mathfrak{C}_{213} \to \mathfrak{C}_{231} \to \mathfrak{C}_{321}$

Expansion of $\mathbf{R}_{\mathfrak{C}_{321} \leftarrow \mathfrak{C}_{123}}$ in two ways:

# Dynamical YBE

The Triangle Lemma implies:

$$\mathbf{R}_{12}(z-\hbar\det\mathsf{ind}_{3(12)})\mathbf{R}_{13}(z-\hbar\det\mathsf{ind}_{(13)2})\mathbf{R}_{23}(z-\hbar\det\mathsf{ind}_{1(23)}) = \mathbf{R}_{23}(z-\hbar\det\mathsf{ind}_{(23)1})\mathbf{R}_{13}(z-\hbar\det\mathsf{ind}_{2(13)})\mathbf{R}_{12}(z-\hbar\det\mathsf{ind}_{(12)3}),$$

where $\mathbf{R}_{ij}$ is the $ij$-wall R-matrix, and the subscript of ind means the wall and chamber for which the index bundle is taken, for example, $3(12)$ means the chamber $\{m_3 < m_1 = m_2\}$.

We get the dynamical Yang-Baxter equation.

# Quiver R-Matrix

Let $Q$ be a quiver with decomposition of nodes $Q_0 = Q_0^{\mathrm{ev}} \sqcup Q_0^{\mathrm{odd}}$.

- We choose $\zeta = \zeta_-$.

We further decompose $Q_0 = Q_0^+ \sqcup Q_0^-$, and set $c_i = \begin{cases} 1, & i \in Q_0^+, \\ -1, & i \in Q_0^-. \end{cases}$

We define the action of $\mathbb{C}_\hbar^\times$ on the doubled quiver $\overline{Q}$ by

| | $x_a$ | $\widetilde{x}_a$ | $\alpha_i$ | $\widetilde{\alpha}_i$ |
|---|---|---|---|---|
| $\mathbb{C}_\hbar^\times$-weight | $0$ | $-c_{h(a)}$ | $0$ | $-c_i$ |

We take $\mathsf{A} = (\mathbb{C}^\times)^3 \curvearrowright W$ such that $W$ decomposes into eigenspaces

$$W = u_1 W^{(1)} + u_2 W^{(2)} + u_3 W^{(3)}.$$

# Quiver R-Matrix

Define the *signed* adjacency matrix of $Q$

$$\mathsf{Q}_{ij} = c_j \cdot \#(a \in Q_1 : h(a) = j, t(a) = i),$$

and define diagonal matrices:

$$\mathsf{D}_{ii} = c_i, \quad \mathsf{P}_{ii} = \begin{cases} c_i, & i \in Q_0^{\mathrm{ev}}, \\ 0, & i \in Q_0^{\mathrm{odd}}. \end{cases}$$

**Definition**

Define the **quiver R-matrix**

$$\mathbf{R}^Q(\boldsymbol{z}) := \mathbf{R}_{\mathrm{wall}}(\boldsymbol{z} + \hbar(\mathsf{P} - \mathsf{Q}^{\mathrm{t}})\mathbf{v}),$$

where $\boldsymbol{z} := \{z_i\}_{i \in Q_0}$ are the Kähler parameters corresponding to the ample line bundles $(\det \mathcal{V}_i)^{-1}$.

# Quiver R-Matrix

Let $\boldsymbol{\mu} := \mathsf{D}\mathbf{w} - \mathsf{C}\mathbf{v}$ where

$$\mathsf{C} = 2\mathsf{P} - \mathsf{Q} - \mathsf{Q}^t.$$

Then the dynamical YBE reads:

$$\mathbf{R}^{Q}_{12}(\boldsymbol{z})\mathbf{R}^{Q}_{13}(\boldsymbol{z} - \hbar\boldsymbol{\mu}^{(2)})\mathbf{R}^{Q}_{23}(\boldsymbol{z}) = \mathbf{R}^{Q}_{23}(\boldsymbol{z} - \hbar\boldsymbol{\mu}^{(1)})\mathbf{R}^{Q}_{13}(\boldsymbol{z})\mathbf{R}^{Q}_{12}(\boldsymbol{z} - \hbar\boldsymbol{\mu}^{(3)}).$$

Here we have suppressed the equivariant parameters (spectral parameters), $\mathbf{R}^{Q}_{ij}(\boldsymbol{z})$ should be $\mathbf{R}^{Q}_{ij}(u_j - u_i, \boldsymbol{z})$

# Example: $\mathfrak{sl}(1|1)$

Consider a quiver $Q$ with one odd node.

- Choose $\zeta = \zeta_-$.
- Choose $c_1 = 1$.
- Choose the framing dimensions $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} = \mathbf{w}^{(3)} = 1$.

Then

1. $\mathcal{M}^\zeta(\mathbf{w}^{(1)})$ is disjoint union of two points,
2. $\mathsf{E}_\mathsf{T}(\mathcal{M}^\zeta(\mathbf{w}^{(1)}))$ is a free module of rank two over $\mathcal{E}_x \times \mathcal{E}_\hbar \times \mathcal{E}_z$. Label the basis by

$$v_0 = [\mathcal{M}^\zeta(0,1)] \quad v_1 = [\mathcal{M}^\zeta(1,1)]$$

3. $\mathcal{M}^\zeta(\mathbf{w}^{(1)} + \mathbf{w}^{(2)}) = \mathrm{pt} \sqcup \mathrm{Tot}\left(\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1\right) \sqcup \mathbb{C}^4$.

# Example: $\mathfrak{sl}(1|1)$

The quiver R-matrix in the basis $\{v_0 \otimes v_0, v_1 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_1\}$:

$$\mathbf{R}^Q(u,z) = \left(\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & \frac{\vartheta(z)\vartheta(z\hbar^{-2})\vartheta(u)}{\vartheta(z\hbar^{-1})^2\vartheta(u\hbar^{-1})} & \frac{\vartheta(\hbar)\vartheta(uz\hbar^{-1})}{\vartheta(z\hbar^{-1})\vartheta(u^{-1}\hbar)} & 0 \\ 0 & \frac{\vartheta(\hbar)\vartheta(zu^{-1}\hbar^{-1})}{\vartheta(z\hbar^{-1})\vartheta(u^{-1}\hbar)} & \frac{\vartheta(u)}{\vartheta(u\hbar^{-1})} & 0 \\ \hline 0 & 0 & 0 & \frac{\vartheta(u\hbar)}{\vartheta(u^{-1}\hbar)} \end{array}\right),$$

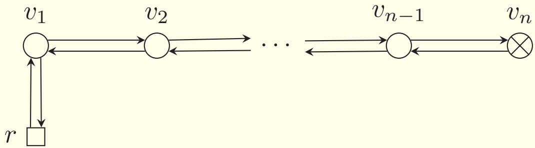
DYBE reads (variables written multiplicatively):

$$\mathbf{R}^Q_{12}(u,z)\mathbf{R}^Q_{13}(uw,z/\hbar)\mathbf{R}^Q_{23}(w,z) = \mathbf{R}^Q_{23}(w,z/\hbar)\mathbf{R}^Q_{13}(uw,z)\mathbf{R}^Q_{12}(u,z/\hbar),$$

which is the DYBE for $\mathfrak{sl}(1|1)$ in the fundamental representation.

# Example: $\mathfrak{sl}(n|1)$

Consider the quiver:




- Choose $\zeta = \zeta_-$.
- Choose $c_i = 1$ for all $i \in Q_0$.
- Choose $\mathbf{w}^{(1)} = \mathbf{w}^{(2)} = \mathbf{w}^{(3)} = (1, 0, \cdots, 0)$

# Example: $\mathfrak{sl}(n|1)$

Then

1. $\mathcal{M}^\zeta(\mathbf{w}^{(1)})$ is the disjoint union of $n+1$ points.
2. $\mathsf{E}_\mathsf{T}(\mathcal{M}^\zeta(\mathbf{w}^{(1)}))$is a free module of rank $n+1$ over $\mathcal{E}_x \times \mathcal{E}_\hbar \times \mathcal{E}_{\boldsymbol{z}}$, basis labelled by $\{v_\alpha\}_{\alpha=0}^n$.
3. $\mathsf{C}_{i,j} = 2\delta_{i,j}(1-\delta_{i,n}) - \delta_{i+1,j} - \delta_{i-1,j}$, $(1 \le i, j \le n)$, which is the symmetric Cartan matrix of $\mathfrak{sl}(n|1)$.
4. $\boldsymbol{\mu}(v_\alpha)$ is the weight of $v_\alpha$ in the fundamental rep of $\mathfrak{sl}(n|1)$.
5. $\mathcal{M}^\zeta(\mathbf{w}^{(1)} + \mathbf{w}^{(2)})$ is disjoint union of the following four kinds of varieties:

$$\mathrm{pt}, \quad T^*\mathbb{P}^1, \quad \mathrm{Tot}\left(\mathcal{O}(-1)^{\oplus 2} \to \mathbb{P}^1\right), \quad \mathbb{C}^4.$$

# Example: $\mathfrak{sl}(n|1)$

The quiver R-matrix is

$$\mathbf{R}^Q(u,\boldsymbol{z})(v_\alpha \otimes v_\beta) =$$

$$= \begin{cases} v_\alpha \otimes v_\beta, & \alpha = \beta < n, \\ D(u) v_\alpha \otimes v_\beta, & \alpha = \beta = n, \\ C(u) v_\alpha \otimes v_\beta + B(u, \hbar^{-\delta_{\beta,n}} \prod_{i=\alpha+1}^{\beta} z_i) v_\beta \otimes v_\alpha, & \alpha < \beta, \\ \substack{A(u, \hbar^{-\delta_{\alpha,n}} \prod_{i=\beta+1}^{\alpha} z_i) v_\alpha \otimes v_\beta \\ + B(u, \hbar^{\delta_{\alpha,n}} \prod_{i=\beta+1}^{\alpha} z_i^{-1}) v_\beta \otimes v_\alpha}, & \beta < \alpha. \end{cases}$$

where

$$A(u,z) = \frac{\vartheta(z\hbar)\vartheta(z\hbar^{-1})\vartheta(u)}{\vartheta(z)^2\vartheta(u\hbar^{-1})}, \qquad B(u,z) = \frac{\vartheta(\hbar)\vartheta(uz)}{\vartheta(z)\vartheta(u^{-1}\hbar)},$$

$$C(u) = \frac{\vartheta(u)}{\vartheta(u\hbar^{-1})}, \qquad D(u) = \frac{\vartheta(u\hbar)}{\vartheta(u^{-1}\hbar)},$$

# K-Theory and Cohomology Limit

In the $q \to 0$ limit, the $\mathbb{E} = \mathbb{C}^\times / q^{\mathbb{Z}}$ degenerates to a nodal compactification of $\mathbb{C}^\times$, and $\mathrm{Ell}_{\mathsf{T}}(\mathsf{X})$ degenerates to $K_{\mathsf{T}}(\mathsf{X}) \otimes \mathbb{C}$

- Because of half-periodicity property of $\vartheta$-function: $\vartheta(e^{2\pi i}x) = -\vartheta(x)$, their $q \to 0$ limit is defined on the double cover of $\mathbb{C}^\times$.
- For the limit of elliptic stable envelope, one should take $q \to 0$ limit in a twisted way:

$$\mathsf{Stab}^{\mathsf{s}}_{\mathfrak{C}} := \lim_{q \to 0} \left[ (\det \mathrm{Pol}_{\mathsf{X}})^{-\frac{1}{2}} \circ \mathbf{Stab}_{\mathfrak{C}} \circ (\det \mathrm{Pol}_{\mathsf{X}^{\mathsf{A}}})^{\frac{1}{2}} \right] |_{z \mapsto z q^{\mathsf{s}}} \in K_{\mathsf{T}}(\mathsf{X} \times \mathsf{X}^{\mathsf{A}}).$$

  where $\mathsf{s} \in \mathrm{Pic}_{\mathsf{T}}(\mathsf{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ is generic.
- A further reduction from the K-theory to cohomology can be defined by

$$\mathrm{Stab}_{\mathfrak{C}} := \text{lowest cohomological degree term in } \mathrm{ch}(\mathsf{Stab}^{\mathsf{s}}_{\mathfrak{C}}),$$

# K-Theory and Cohomology Limit

Assume that either

- the gauge group is abelian, or
- X is a quiver variety,

then

**Proposition** ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])

$\mathsf{Stab}^{\mathsf{s}}_{\mathfrak{C}}$ *is the K-theoretic stable envelope with slope* $\mathsf{s}$ *for the chamber* $\mathfrak{C}$*, and* $\mathrm{Stab}_{\mathfrak{C}}$ *is the cohomological stable envelope for the chamber* $\mathfrak{C}$*.*

# K-Theory and Cohomology Limit: $A_1$ Quiver

In the example of $A_1$ quiver with an odd node,

$$\mathsf{Stab}^{\mathsf{s}}_{\mathfrak{C}}([\mathcal{F}_p]) = \hbar^{\frac{\#(i>p(a))}{2}} \mathsf{Sym}_{S_N}\left[\left(\prod_{a=1}^{N} \mathsf{f}_{p(a)}(s_a, \boldsymbol{x}, \hbar, \boldsymbol{s})\right)\cdot\left(\prod_{a>b} \hat{\mathbf{a}}(s_a s_b^{-1})\right)\right],$$

where $\mathsf{f}_m(s, \boldsymbol{x}, \hbar, \mathsf{s})$ is

$$\mathsf{f}_m(s, \boldsymbol{x}, \hbar, \mathsf{s}) := (s x_m)^{\lfloor \mathsf{s} \rfloor} \prod_{i<m}(1 - s^{-1}x_i^{-1}) \prod_{j>m}(1 - \hbar^{-1}s^{-1}x_j^{-1}),$$

and $\hat{\mathbf{a}}(w) = \frac{1}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}}$.

# K-Theory and Cohomology Limit: $A_1$ Quiver

A further reduction to cohomology gives

$$\mathrm{Stab}_{\mathfrak{C}}([\mathcal{F}_p]) = \mathsf{Sym}_{S_N}\left[\left(\prod_{a=1}^{N} \mathrm{f}_{p(a)}(s_a, \boldsymbol{x}, \hbar)\right) \cdot \left(\prod_{a>b} \frac{1}{s_a - s_b}\right)\right],$$

where $\mathrm{f}_m(s, \boldsymbol{x}, \hbar)$ is the following function

$$\mathrm{f}_m(s, \boldsymbol{x}, \hbar) := \prod_{i<m}(s + x_i) \prod_{j>m}(s + x_j + \hbar).$$

This recovers the result of [R. Rimányi and L. Rozanky (2021)].

# Rational R-Matrix of $\mathfrak{sl}(n|1)$

Reduction to cohomology for the elliptic dynamical R-matrix of $\mathfrak{sl}(n|1)$ gives

$$\mathrm{R}(u) = \mathrm{P}\left(\frac{u}{u-\hbar}\Pi - \frac{\hbar}{u-\hbar}\mathbf{1}\right).$$

- $\mathrm{P}$ is the usual swapping-tensor operator: $\mathrm{P}(v_\alpha \otimes v_\beta) = v_\beta \otimes v_\alpha$,
- $\Pi$ is the super swapping-tensor operator: $\Pi(v_\alpha \otimes v_\beta) = (-1)^{|v_\alpha|\cdot|v_\beta|} v_\beta \otimes v_\alpha$

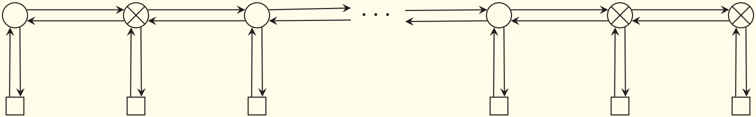
We can rewrite

$$(u-\hbar)\Pi \mathrm{P} \mathrm{R}(u) = u\mathbf{1} - \hbar\Pi,$$

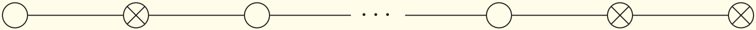
the RHS is the rational R-matrix for the fundamental representation of $\mathfrak{sl}(n|1)$ in the literature [E. Ragoucy and G. Satta, (2007)].

# Representation Theoretic Perspective

Consider a finite or affine type $A$ quiver



This reminds us of Kac-Dynkin diagram



We found that the dynamical shifts $\boldsymbol{\mu}$ in DYBE are weights in a certain highest-weight module of $\widehat{\mathfrak{sl}}(m|n)$ or $\mathfrak{sl}(m|n)$.

# Representation Theoretic Perspective

**Conjecture** ([S. F. Moosavian, N. Ishtiaque, and Y. Z. (2023)])

*Let $Q$ be a finite or affine type A quiver with nodes decorated as above, and let $\mathfrak{g}_Q$ be the Lie superalgebra associated with the corresponding Kac-Dynkin diagram. The there exist actions*

$$E_{\hbar,\tau}(\mathfrak{g}_Q) \curvearrowright \mathsf{E}_{\mathsf{T}}(\mathcal{M}^{\zeta}(\mathbf{w}))$$

$$\mathfrak{U}_{\hbar}(\hat{\mathfrak{g}}_Q) \curvearrowright K_{\mathsf{T}}(\mathcal{M}^{\zeta}(\mathbf{w}))$$

$$\mathsf{Y}_{\hbar}(\mathfrak{g}_Q) \curvearrowright H_{\mathsf{T}}(\mathcal{M}^{\zeta}(\mathbf{w}))$$

*Moreover, all the actions factor through the corresponding Maulik-Okounkov quantum groups constructed via the stable envelopes for $\mathcal{M}^{\zeta}(\mathbf{w})$.*

# Representation Theoretic Perspective

**Theorem** ([M. Yamazaki and Y.Z. In progress])

*The above conjecture is true.*

- We anticipate that the conjecture is still true when $Q$ is not of type $A$, the corresponding algebra should be replaced by the quiver BPS algebra studied by Gelakhov-Li-Yamazaki.
- If $Q_0^{\mathrm{odd}} = \emptyset$, then the corresponding statement is a result of A. Negut.

# Future Directions

1. Relation to integrable systems.
2. Stable envelopes from 4d Chern-Simons theory.
3. Compare the modules of the quiver BPS algebra coming from stable envelope with the ones studied by Gelakhov-Li-Yamazaki.
4. It will be nice if there is a 3d $\mathcal{N} = 2$ mirror symmetry for the stable envelope, generalizing the mirror symmetry in the 3d $\mathcal{N} = 4$ setting.

# The End

**Thank You!**