On-shell action & Entanglement entropy of $T\bar{T}$-deformed Holographic CFTs.

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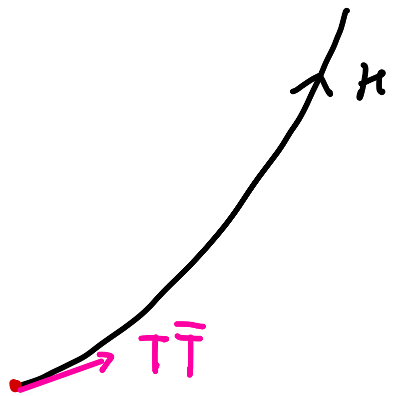
based on arXiv: 2306.01258

$T\bar{T}$ deformation

'16 Smirnov & Zamolodchikov
Cavaglià, Negro, Szécsényi & Tateo



$$\frac{dS^{(\mu)}}{d\mu} = \int d^2x \sqrt{g}\, T\bar{T}$$

$$T\bar{T} \equiv \frac{1}{8}\left[T_{\alpha\beta} T^{\alpha\beta} - (T^{\alpha}_{\ \alpha})^2 \right]$$

$$T^{\alpha\beta} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\alpha\beta}}$$

$g_{\alpha\beta}$: metric of the 2d background.

- Usually, we have factorization

$$\langle T\bar{T} \rangle \sim \langle T_{\alpha\beta} \rangle \langle T^{\alpha\beta} \rangle - \langle T^{\alpha}_{\ \alpha} \rangle^2$$

- Another definition for deformed CFT

$$\langle T^{\alpha}_{\ \alpha} \rangle = \frac{c}{24\pi} R + 2\mu \langle T\bar{T} \rangle$$

c: central charge

R: Ricci scalar

Motivation: Why $T\bar{T}$?

- Different people have different interests:
  1. It's a irrelevant deformation under control
  2. It preserves integrability
  3. It's UV complete
  4. It exhibits non-local property, like string theory does
  5. It has application in holography

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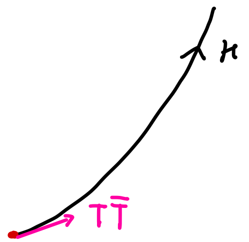
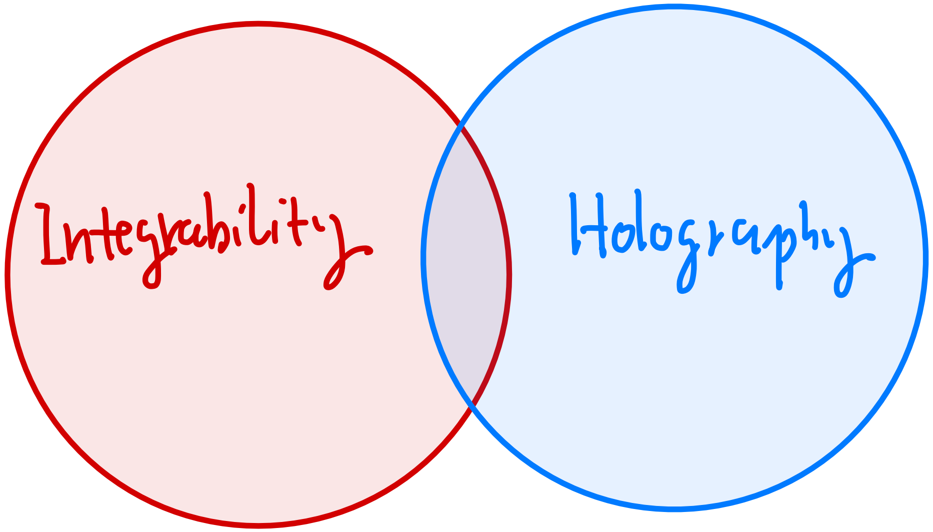


- IFT
- CFT



My interest:

- holography beyond AdS / CFT
- Deformed conformal symmetry

- Comments

1. $T\bar{T}$ deformation is universal
2. Integrability and Holography are kind of complementary since holographic CFT is chaotic

- The goal

- Solve $T\bar{T}$-deformed $\left\{\begin{array}{l}\text{IFT}\\ \boxed{\text{CFT}}\end{array}\right.$ as much as we can.

spectrum

$$\begin{cases} P^{(n)} = P^{(0)} \\ E^{(n)} = f(E^{(0)}, P^{(0)}) \end{cases}$$

correlation function

$$\langle \prod_n \Phi_n(x_n) \rangle_\lambda = \langle \prod_n \Phi_n^\lambda(x_n) \rangle_{\lambda=0}$$

'19 J. Cardy

- Correlation function (field theory analysis?)

$$\langle \prod_n \Phi_n(x_n) \rangle_\lambda = \langle \prod_n \Phi_n^\lambda(x_n) \rangle_{\lambda=0} \,.$$

$\partial_\lambda \Phi^\lambda(x) = S^b[x] \cdot \partial_b \Phi^\lambda(x)$ is non-local.

$$\sim \underbrace{\varepsilon_{ab}\, \varepsilon^{ij} \int_x^X dx'_j \, T_{ai}^\lambda (x'+\varepsilon)\, \partial_{x^b} \Phi^\lambda(x)}_{\text{quantum extension of field-dependent c.c}}$$

('19 Conti, Negro & Tateo)

- Can holography help?
  - Spectrum ✓
  - Correlation function ? $\longrightarrow$ "Simpler quantities": partition function & EE

· Partition function & EE (field theory analysis)

· Perturbative calculations.

(Free theories up to 2-loops
'21. He. Sun & Zhang)

1. $\mathcal{L}^{\mu} = \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \mathcal{L}^{(n)}$

$$Z^{(\mu)} = Z^{(0)} - \mu Z^{(0)} \int \langle \mathcal{L}^{(1)} \rangle + \frac{\mu^2}{2} Z^{(0)} \left(\iint \langle \mathcal{L}^{(1)} \mathcal{L}^{(1)} \rangle - \int \langle \mathcal{L}^{(2)} \rangle \right) \cdots$$

2. Using modular properties

$$Z(\tau, \bar\tau | \mu) = \sum_{p=0}^{\infty} \mu^{2p} \sum_{m=0}^{p-1} \frac{a_{p,m}}{\tau_2^{p-2(m+1)}} D^{(m)} Z_0$$

(Datta & Jiang '18)

3. 1st order correction to EE

① replica trick $\qquad \delta S_{EE} = \frac{-k\mu}{1-k} \int_{M_1} \left(\langle T\bar{T} \rangle_{M_k} - \langle T\bar{T} \rangle_{M_1} \right) \Big|_{k \to 1}$

('18 Chen, Chen & Hao)

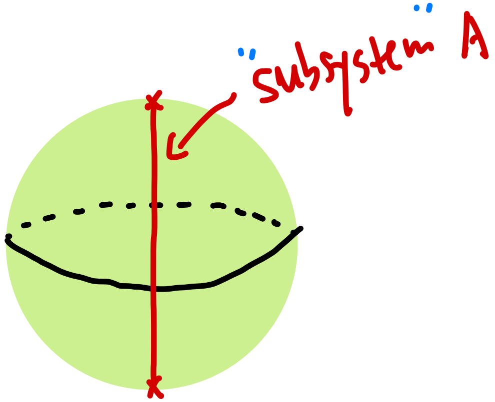
② EE 1st law $\qquad \delta S_{EE} = -\mu \langle H_{\rho_0} \int T\bar{T} \rangle$

$\downarrow$

modular Hamiltonian

. For EE, going beyond 1st order is very challenging because we lose the power of 2d conformal symmetry.

· A solvable example: ['18 Donnelly & Shyam]



CFT on S^2

$$ds^2 = r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

1. Consider a small Weyl transformation

$$g_{\alpha\beta} \longrightarrow (1+\delta\sigma)\, g_{\alpha\beta}\,, \qquad e^{\sigma} \equiv r^2$$

$$\rightarrow \begin{cases} \dfrac{\delta S_{CFT}}{\delta\sigma} = -\dfrac{1}{2}\displaystyle\int \sqrt{g}\, T^\alpha_\alpha \, d^2x \\[2ex] \underbrace{\dfrac{d}{dr}\log Z = \dfrac{1}{r}\displaystyle\int d^2x \sqrt{g}\, T^\alpha_\alpha}\; \equiv -\dfrac{dS^\mu_{CFT}}{dr} \end{cases}$$

2. For vacuum state, $T_{\alpha\beta} = \alpha\, g_{\alpha\beta}$ and α is determined by

$\langle T^\alpha_\alpha \rangle = \frac{c}{24\pi}R + 2\mu \langle T\bar{T}\rangle$, which is

$$\alpha = \frac{\sqrt{\frac{c\mu}{6\pi r^2}+4} - 2}{\mu}$$

3. Solve the ODE

$$S^{n}_{CFT} \sim -\frac{c}{6} + \frac{c}{6}\log\frac{cn}{96\pi r^2} - \frac{c^2 n}{576\pi r^2} + O(n^2)$$

$[\, S^{n}_{CFT}(r=0) = 0 \,]$

4. Use replica trick

$$S_{EE} = \lim_{n\to 1} \frac{1}{1-n}\log\frac{Z_n}{Z_1^{\,n}}$$

Z_n: $(S^2)_{replica}$

$r^2(d\theta^2 + n^2\sin^2\theta\, d\phi^2)$

$$= \frac{c}{3}\log\left(\sqrt{\frac{96\pi}{cn}}\, r\right) + \frac{c^2 n}{288\pi r^2} + \cdots$$

- Comments

1. S_{EE} is UV-finite.

2. S^{n}_{CFT} and S_{EE} do not have undeformed limit.

3. 1st order correction to EE

δS_{EE} is both UV and IR finite

$$\delta S_{EE} = \frac{-k\mu}{1-k}\int_{M_1}\left(\langle T\bar{T}\rangle_{M_k} - \langle T\bar{T}\rangle_{M_1}\right) < \infty$$

('18 Chen, Chen & Hao)

Holography

Holographic dual of $T\bar{T}$

"Single trace"

'17 Giveon, Itzhaki and Kutasov
'20 Apolo, Detournay and Song

current-current deformation of AdS_3 worldsheet theory, which can be realized by TsT-transformation

"double trace"

hard boundary

cut-off or glue-on

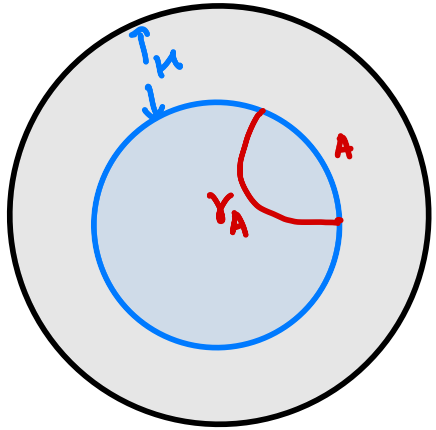
'16 McGaugh
Mezei & Verlinde
'23 Apolo, Hao & Song

mixed boundary condition

'19 Guica & Monten

- Cut-off proposal : The deformed state is dual to the cut-off spacetime

1. "conjecture"
2. Only for pure gravity.



1) The (on-shell) action

$$I_{cut\text{-}off} = -\frac{1}{16\pi G_N}\left(I_{EH} + I_{GH} + I_{ct} \right)$$

$$I_{c.t} = -2\int d^2x \sqrt{h}\,(1 + \underbrace{\mu f(x)}_{})$$

add by hand

2) E.E

RT formula $\qquad S_A = \dfrac{A_r}{4\pi G_N}$

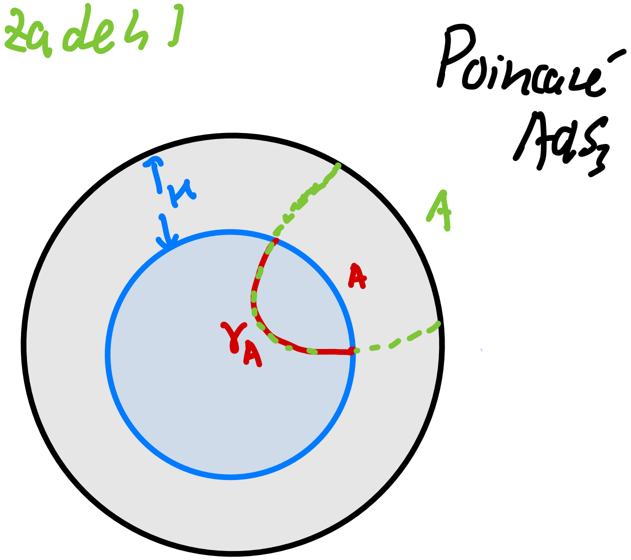
Some existing results.

- Flat 2d spacetime

① Vacuum State ['22 Allameh, Astaneh & Hassanzadeh]

Replica trick + onshell action

RT formula



$$S_{EE} = \frac{c}{3}\log\left(\sqrt{\frac{24\pi}{\mu c}}\,\ell\right) + \frac{\mu c^2}{72\pi\ell^2}$$

1. It does not match the field theory result:

$$S_{EE} = \frac{c}{3}\log\frac{\ell}{\varepsilon} + O(\mu^2)$$

2. It's different from the one obtained in the mixed boundary proposal.

② Thermal State ('18 Chen, Chen & Hao)

The bulk dual is the BTZ black hole.

· The holography EE

$$S_{EE} = \frac{c}{3}\left[\log\left(\frac{\beta r_c \sinh\left(\frac{\pi l}{\beta}\right)}{\pi}\right) - \frac{2\pi^2 l \cosh\left(\frac{\pi l}{\beta}\right)}{\beta^2 r_c} + \underbrace{\frac{\pi}{\beta r_c \sinh\frac{\pi l}{\beta}}}\right]$$

· The field theory

$$S_{EE} = \frac{c}{3}\left[\log \frac{\beta}{\pi\epsilon}\sinh\frac{\pi l}{\beta} - \frac{2\pi^2 l \cosh\left(\frac{\pi l}{\beta}\right)}{\beta^2 r_c}\right]$$

$$r_c^2 = \frac{6}{\mu\pi c}$$

- **Conclusion:** In the cut-off proposal, the RT formula is not correct even in the zero order of κ.

- **Comments:** 1) In field theory, EE is defined up to some constant (related to the ambiguity in I_{ct}).

  2) Even we use this ambiguity to fix the zeroth result, the leading order correction still does not match the one from perturbative result in field theory.

- Mixed-boundary proposal.

The holographic dictionary may be summarized as

$$Z_{T\bar{T},CFT}[\gamma_{\alpha\beta}^{[\mu]}] = Z_{grav}\left[g_{\alpha\beta}^{(0)} + \underset{\underset{\rho_c}{|||}}{\frac{\mu}{16\pi G_N}} g_{\alpha\beta}^{(2)} + \frac{\mu^2}{(16\pi G_N)^2} g_{\alpha\beta}^{(4)} = \gamma_{\alpha\beta}^{[\mu]} \right]$$

- Derivation

1. In the linear order, the deformation is just a double trace deformation

$$S_{CFT}^{[\mu]} = S_{CFT} + \mu \int d^2x \sqrt{\gamma}\, T\bar{T}$$

2. The double trace deformation does two things:

① It shifts the source by the $\langle O \rangle$

② It shifts the generation function (the on-shell action) as

$$W^{[\mu]} = W_0 - \mu \int d^2x \sqrt{\gamma}\, T\bar{T}$$

3. Using the defining property $\delta W = \frac{1}{2}\int d^2x \sqrt{\gamma}\, T_{\alpha\beta}\, \delta\gamma^{\alpha\beta}$ we can derive a flow equation

$$\frac{1}{2}\partial_\mu\left(\int d^2x \sqrt{\gamma^{[\mu]}}\, T_{\alpha\beta}^{[\mu]}\, \delta\gamma_{\alpha\beta}^{[\mu]}\right) = -\delta\left(\int d^2x \sqrt{\gamma^{[\mu]}}\, T\bar{T}^{[\mu]}\right).$$

whose solution is

$$\begin{cases} \gamma_{\alpha\beta}^{[\mu]} = \gamma_{\alpha\beta}^{[0]} + \frac{1}{2}\mu \hat{T}_{\alpha\beta}^{[0]} + \frac{1}{16}\mu^2 \hat{T}_{\alpha\rho}^{[0]} \hat{T}_{\sigma\beta}^{[0]} \gamma^{[0]\rho\sigma} \\ \hat{T}_{\alpha\beta}^{[\mu]} = \hat{T}_{\alpha\beta}^{[0]} + \frac{1}{4}\mu \hat{T}_{\alpha\rho}^{[0]} \hat{T}_{\sigma\beta}^{[0]} \gamma^{[0]\rho\sigma} \end{cases}$$

$$\boxed{\hat{T}_{\alpha\beta} \equiv T_{\alpha\beta} - \gamma_{\alpha\beta} T}$$

4. The general asympotitic AdS_3 solution can be written in the Fefferman-Graham gauge

$$ds^2 = \frac{d\rho^2}{4\rho^2} + g_{\alpha\beta}(\rho, x^\alpha)\, dx^\alpha dx^\beta, \qquad g_{\alpha\beta}(\rho, x^\alpha) = \frac{g_{\alpha\beta}^{(0)}}{\rho} + g_{\alpha\beta}^{(2)} + \rho\, g_{\alpha\beta}^{(4)}$$

and in particular

$$\hat{T}_{\alpha\beta}^{[0]} = \frac{1}{8\pi G_N} g_{\alpha\beta}^{(2)}, \quad \gamma_{\alpha\beta}^{[0]} = g_{\alpha\beta}^{(0)}$$

- How to use the dictionary?

1) Let's consider a $T\bar{T}$-deformed CFT defined on the background with metric $\gamma_{\alpha\beta}^{[\mu]}$.
Then the state with the expectation value of energy tensor being $\hat{T}_{\alpha\beta}^{[\mu]}$ is dual to a bulk geometry satisfying mixed-boundary conditions

$$\begin{cases} \gamma_{\alpha\beta}^{[\mu]} = \gamma_{\alpha\beta}^{[0]} + \frac{1}{2}\mu \hat{T}_{\alpha\beta}^{[0]} + \frac{1}{16}\mu^2 \hat{T}_{\alpha\rho}^{[0]} \hat{T}_{\sigma\beta}^{[0]} \gamma^{[0]\rho\sigma} \\ \hat{T}_{\alpha\beta}^{[\mu]} = \hat{T}_{\alpha\beta}^{[0]} + \frac{1}{4}\mu \hat{T}_{\alpha\rho}^{[0]} \hat{T}_{\sigma\beta}^{[0]} \gamma^{[0]\rho\sigma} \end{cases}$$

2) Or we can start from a bulk geometry. It can be either dual to a CFT on $g_{\alpha\beta}^{[0]}$ or dual to $T\bar{T}$-deformed CFT on $\gamma_{\alpha\beta}^{[\mu]}$!

. The on-shell action ('23 Tian)

$$I^{[\mu]}_{\text{on-shell}} = I^{[0]}_{\text{Euclidean}}\left(\gamma^{[\mu]}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + \rho_c\, g^{(2)}_{\alpha\beta} + \rho_c^2\, g^{(4)}_{\alpha\beta}\right) - \mu \int \sqrt{\gamma^{[\mu]}}\, T\bar{T}$$

$$= I^{[\mu]}_{\text{bulk}} + I^{[\mu]}_{\text{bdy.}}$$

$$I^{[0]}_{\text{Euclidean}} = \frac{1}{16\pi G_N}\left(-\int_B \sqrt{h}\,(R+2) - 2\int_{\partial B}\sqrt{\gamma}\,K + 2\int_{\partial B}\sqrt{\gamma}\right)$$

$$= -\frac{1}{16\pi G_N}\int_{\partial B}\sqrt{g^{(0)}}\cdots \quad + \underbrace{\frac{c}{6}\,\chi(\partial B)\log\delta}$$

It's divergent but universal capturing the Weyl anomaly.

$c = \frac{3}{2G_N}$ is the central charge

$\chi(\partial B)$ is the Euler character

. EE

RT formular matches the zeroth result but may fail in higher order

· Examples

1. Vacuum state in flat background.

· Since $\langle T_{\alpha\beta}\rangle = 0$, so the bulk geometry does not get deformed, namely

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{d\tau^2 + dx^2}{\rho^2}.$$

· The RT formula matches the field theory result

$$S_{EE} = \frac{c}{3} \log \frac{\ell}{\varepsilon}$$

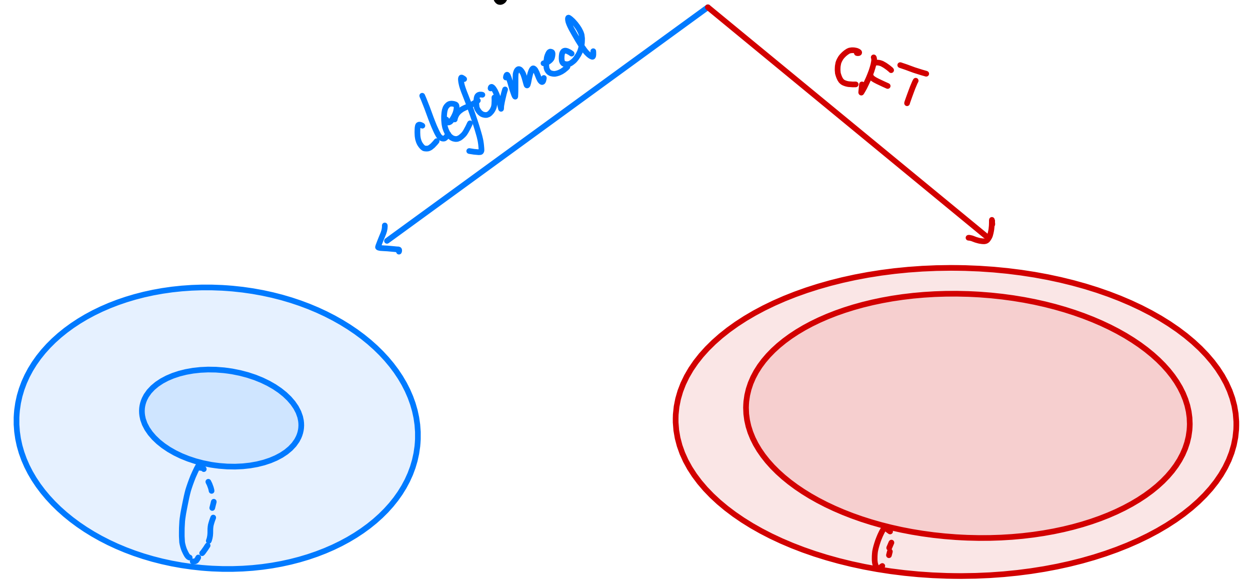
$\longrightarrow$ The entanglement entropy in ground state does not get deformed!

2. The thermal state

- Let's consider the BTZ black hole geometry,

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{(1+L_0\rho)^2 d\chi^2 + (-1+L_0\rho)^2 d\tau^2}{\rho}$$

whose boundary is a torus.



Thermal period: β

spatial period: w

$$\beta_0 = \frac{1}{2}\left(\beta + \sqrt{\beta^2 + 4\tau^2\rho_c}\right)$$

$$w_0 = \frac{1}{2} w \left(\frac{\beta}{\sqrt{\beta^2 + 4\tau^2\rho_c}} + 1\right)$$

· On-shell action

$$I_{bulk}^{[H]} = I_{Euclidean}^{[c]} = -\frac{c\lambda}{6}\frac{\omega_0}{\beta_0}$$

$$= -\frac{c\lambda}{6}\frac{\omega}{\sqrt{\beta^2+4\lambda^2\rho_c}} = -\frac{c\lambda}{6}\frac{\omega}{\beta} + \frac{c\lambda^3\omega\rho_c}{3\beta^3}$$

$$I_{bdy}^{[H]} = -h\int\sqrt{\gamma}\,\langle T\bar{T}\rangle = -\frac{c\lambda^3\omega\rho_c}{6\beta^3} + O(\rho_c^2)$$

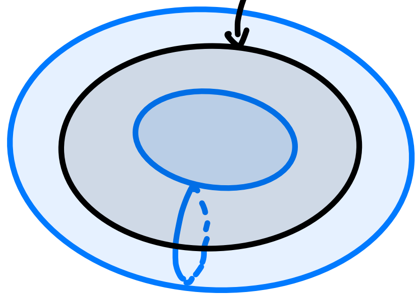
$$\Rightarrow I_{on\text{-}shell}^{[H]} = -\frac{c\lambda}{6}\frac{\omega}{\beta} + \frac{c\lambda^3\omega\rho_c}{6\beta^3}$$

$$= I_{on\text{-}shell}^{[c]} + h\int\sqrt{\gamma}\,\langle T\bar{T}\rangle + O(\rho_c^2)$$

which matches the field theory result:

$$S_{CFT} + h\int\sqrt{\gamma}\,\langle T\bar{T}\rangle$$

· EE

① 

EE of the whole spatial circle

· Replica trick

$$S_n = \frac{1}{1-n}\log\frac{Z_n}{Z_1^n} = \frac{1}{1-n}\log\frac{e^{-I^{[H]}_{\text{on-shell}}(n\beta)}}{e^{-nI^{[H]}_{\text{on-shell}}(\beta)}}$$

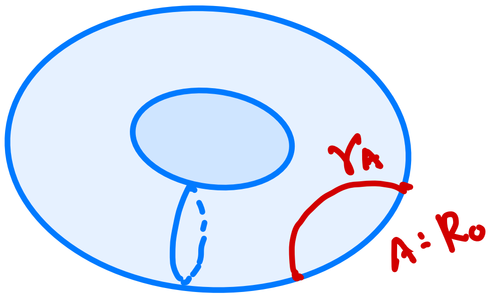
$$= \frac{c\pi w}{3\sqrt{\beta^2+4\pi^2\rho_c}} + \frac{(n-1)\, c\pi w\beta^2}{6(4\pi^2\rho_c+\beta^2)^{3/2}} + O((n-1)^2)$$

· RT formula

$$S_{EE} = \frac{\gamma_H}{4G_N} = \frac{c\pi w}{3\sqrt{\beta^2+4\pi^2\rho_c}}$$

✓

② EE of a subsystem.



· RT formula

$$\begin{cases} \beta_0 = \frac{1}{2}\left(\beta + \sqrt{\beta^2 + 4\pi^2 \rho_c}\right) \\ R_0 = \frac{R}{2}\left(1 + \frac{\beta}{\sqrt{\beta^2 + 4\pi^2 \rho_c}}\right) \end{cases}$$

$$S_{EE} = \frac{c}{3} \log\left(\frac{\beta_0}{\pi \epsilon_0} \sinh \frac{2\pi R_0}{\beta_0} \right)$$

$$= \frac{c}{3} \log\left(\frac{\beta + \sqrt{\beta^2 + 4\pi^2 \rho_c}}{2\pi \epsilon_0} \sinh \frac{2\pi R}{\sqrt{\beta^2 + 4\pi^2 \rho_c}} \right)$$

$$= \underbrace{\frac{c}{3} \log\left(\frac{\beta}{\pi \epsilon_0} \sinh \frac{2\pi R}{\beta} \right) - \frac{4c\pi^3 R \coth \frac{2\pi R}{\beta}}{3\beta^3}}_{\text{The field theory result}} + \frac{c\pi^2}{3\beta^2} \rho_c$$

• It's still off by $\frac{c\pi^2}{3\beta^2}\rho_c$. Fortunately, it does not dependent on the detail of the subsystem! So it can be absorbed into the UV cut-off, for example by defining

$$\frac{1}{\varepsilon_0} = \frac{1}{\varepsilon}\,\frac{1}{2}\left(1+\frac{\beta}{\sqrt{\beta^2+4\pi^2\rho_c}}\right)$$

$$\hookrightarrow \quad S_{EE} = \frac{c}{3}\log\left(\frac{\left(\beta+\sqrt{\beta^2+4\pi^2\rho_c}\right)^2}{4\pi\varepsilon\sqrt{\beta^2+4\pi^2\rho_c}}\,\sinh\frac{2\pi R}{\sqrt{\beta^2+4\pi^2\rho_c}}\right)$$

We conjecture this is an exact result.

- Another interesting check is to consider the EE of the interval with length $2R_0^t$ along the thermal direction. (Effectively we are considering the EE of a CFT on a finite region)

$$R_0^t = \frac{R^t}{2\beta}\left(\beta + \sqrt{\beta^2 + 4\pi^2\rho_c}\right)$$

$$S_{EE} = \frac{c}{3}\log\left(\frac{\beta_0}{\pi\epsilon_0}\sinh\frac{2\pi R_0^t}{\beta_0}\right)$$

$$= \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\frac{2\pi R^t}{\beta}\right) + \mathcal{O}(\rho_c^2).$$

which also matches the result in field theory: $\delta S_{EE} = 0$ is the leading order.

- Properties of EE

1. To have a well-defined EE, we find that ρ_c is bounded:

$$1-\rho_c^2 L_0^2 > 0 \quad \Rightarrow \quad 0 \leq \rho_c^2 < \frac{1}{L_0} .$$

2. When $\rho_c > 0$, then it has an upper limit

$$0 < \rho_c < \frac{1}{\sqrt{L_0}} = \frac{\beta_0}{\pi}$$

3. The effect of this positive deformation is

   ① shrink the thermal circle

   ② expand the spatial circle

   ③ increase the energy scale $\rightsquigarrow$ an opposite RG flow

3. Excited state in flat spacetime

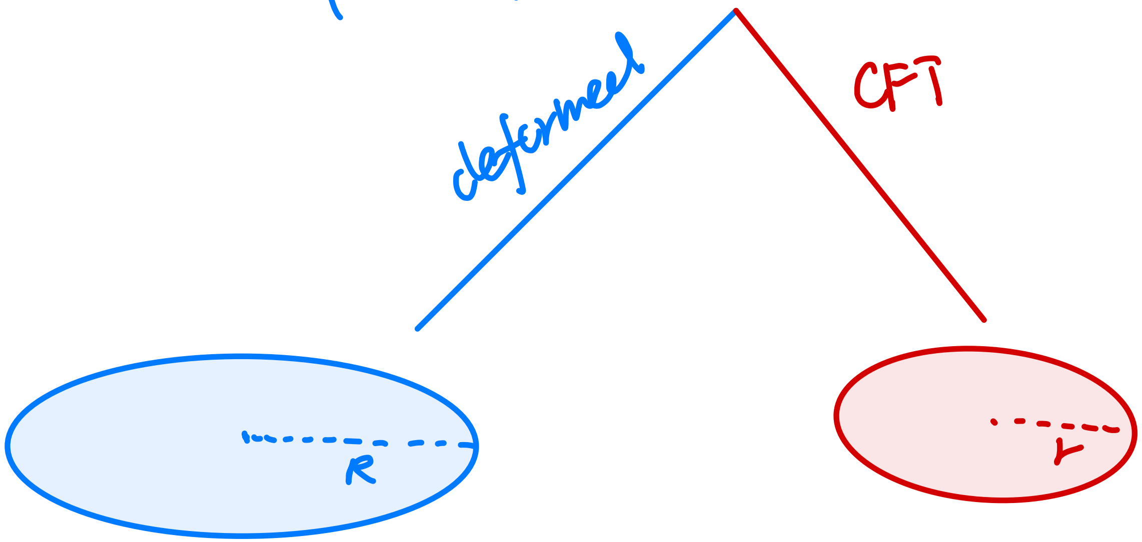
· Let us start from the Poincaré AdS_3 metric

$$ds^2 = \frac{dw\,d\bar{w} + dz^2}{z^2}$$

and consider the conformal transformation $w = y^n$, $\bar{w} = \bar{y}^n$ at the boundary. Using the Bañados map, we can a bulk solution which is dual to a primary state.

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho}\left(\left(1 + \frac{a^2\rho}{r}\right)^2 dr^2 + \left(r - \frac{a^2\rho}{r}\right)^2 d\theta^2 \right).$$

$$\left[a^2 \equiv \frac{n^2-1}{4} \right]$$



$$r = \frac{1}{2}\left(R + \sqrt{R^2 + 4a^2\rho_c}\right)$$

· On-shell action

$$I^{[h]}_{\text{bulk}} = I^{(0)}_{\text{Euclidean}} = -\frac{ca^2}{3h}\log\frac{\Lambda_0}{\varepsilon_0}$$

$$= -\frac{ca^2}{3h}\log\frac{\Lambda+\sqrt{4a^2\rho_c+\Lambda^2}}{\varepsilon+\sqrt{4a^2\rho_c+\varepsilon^2}}$$

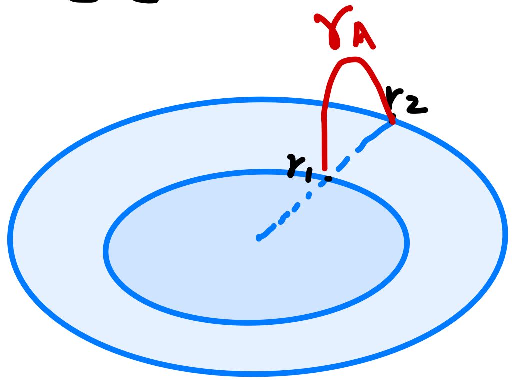
$$I^{[h]}_{\text{bdy}} = \frac{2a^4c\rho_c}{3h}\left(\frac{1}{(\Lambda+\sqrt{4a^2\rho_c+\Lambda^2})^2}-\frac{1}{(\varepsilon+\sqrt{4a^2\rho_c+\varepsilon^2})^2}\right)$$

$$I^{[h]}_{\text{on-shell}} = \underbrace{-\frac{ca^2}{3h}\log\frac{\Lambda}{\varepsilon}}_{} + \frac{a^4c\rho_c}{6h}\left(\frac{1}{\varepsilon^2}-\frac{1}{\Lambda^2}\right) + \mathcal{O}(\rho_c^2)$$

$$= I^{[0]}_{\text{on-shell}} + \kappa\int\sqrt{\gamma}\,T\bar{T} +$$

Weyl anomaly due to the cut-off boundary at $r=\varepsilon,\Lambda$.

· EE



RT formula:

$$S_{EE} = \underbrace{\frac{c}{6}\log\frac{(r_1 r_2)^{(1-n)}(r_1^n - r_2^n)^2}{\delta_0^2 n^2} + \frac{c\,n(n^2-1)(r_1^n + r_2^n)}{24(r_1^n - r_2^n)}\left(\frac{1}{r_1^2} - \frac{1}{r_2^2}\right)\rho_c}_{\text{field theory result}}$$

$$+ \frac{c(n^2-1)(r_1^2 + r_2^2)}{24 r_1^2 r_2^2}\rho_c + \mathcal{O}(\rho_c^2)$$

- The extra terms depends on r_1 and r_2, so it can not be absorbed into the UV cut-off. It's negligible only when

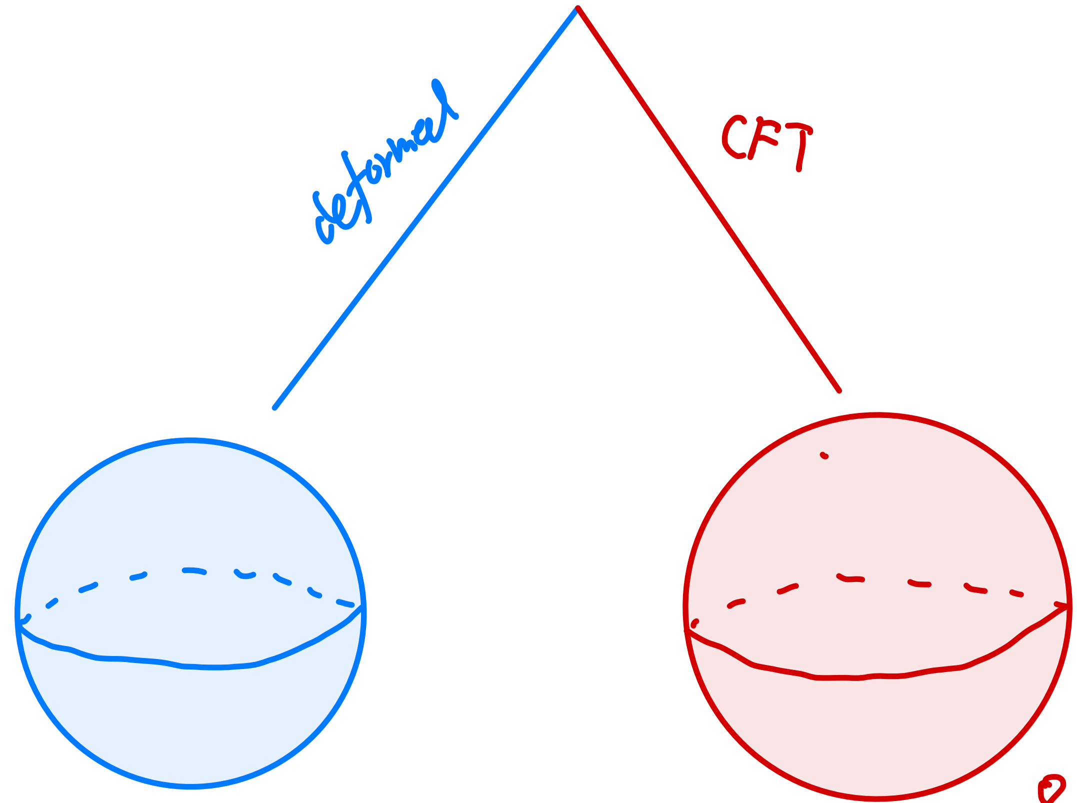
$$n \gg 1, \quad r_2 \gg r_1$$

- Suggestion: RT formula needs modification for excited states

4. Vacuum state in sphere background

- Let us consider bulk metric

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{(8+\Lambda_0\rho)^2}{(-8\Lambda_0)\,\rho}\;\frac{dk^2 + k^2 d\varphi^2}{(1+k^2)^2}$$



$-R = \Lambda > 0$

$$-R = \Lambda_0 = -\frac{8\left(\rho_c\Lambda - 4 + 2\sqrt{4 - 2\rho_c\Lambda}\right)}{\rho_c^2\Lambda}$$

· On-shell action

$$I^{[H]}_{\text{on-shell}} = -\frac{c}{6}\left(1+\log\frac{8}{-\Lambda}\right) + \frac{c\Lambda}{48}\rho_c + O(\rho_c^2)$$

comparing with field theory result

$$I^{[H]}_{\text{on-shell}} - S^{H}_{CFT} = \frac{c}{6}\log\frac{24\pi}{cH}$$

⟶ Instead of choosing $S^{H}_{CFT}(r=0)=0$, we chose $S^{H}_{CFT}(r=0)=\frac{c}{6}\log\frac{24\pi}{cH}$.

· Shifting the on-shell action will shift EE

$$\begin{cases} I_{\text{on-shell}} \to I_{\text{on-shell}} + \alpha \\ S_{EE} \to S_{EE} - \alpha \end{cases}$$

- RT formula

$$S_{EE} = \frac{c}{6}\log\frac{8\ell}{-\Lambda\delta^2} - \frac{c^2\Lambda\mu}{576\pi} + \mathcal{O}(\mu^2)$$

which will match the UV-finite field theory result if we take accounts of the shift due to Weyl anomaly ($\frac{c}{6}\chi(\partial B)\log\delta$) and the shift

($I^{LW}_{\text{on-shell}} - S^{\mu}_{CFT} = \frac{c}{6}\log\frac{24\pi}{c\mu}$).

Conclusion: In $T\bar{T}$-deformed holography, RT formula needs certain modification!

Future works: 1. Consider more examples, specilly the cases with matters or excited states.

2. More general curved backgrounds.

3. Compute correlation functions

4. Derive RT formula directly

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