Form Factor Bootstrap & $T\bar{T}$-deformed EE for IQFT

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Contents

Motivation and Introduction

$T\bar{T}$ deformation

- The deformation preserves integrability;
- $T\bar{T}$-deformed CFT and its holographic duality: Cutoff AdS_3 & mixed boundary condition;
- 2D gravity description of $T\bar{T}$ deformation.
- Most of the global quantities in deformed theory can be computed exactly, such as spectrum, Lagrangian, S-matrix, torus partition functions, etc;
- Local quantities such as correlation functions and entanglement entropy are more challenging due to lack of non-perturbative technologies.
- The $T\bar{T}$ deformation implies intriguing new UV behaviors. We need non-perturbative technologies to study the local quantities and detect the UV behaviors.

Non-perturbative approach

- Flow equations for the $T\bar{T}$ deformed correlators were obtained by Cardy.
- A non-perturbative approach based on deformed symmetry has been proposed by Guica.
- Holography approach: worldsheet techniques and gravity.
- JT gravity formulation of the $T\bar{T}$ deformation.
- Integrability approach: *Form factor bootstrap*.

- Hilbert space of IQFT: n-particle basis states

$$|\theta_1, \theta_2, ..., \theta_n\rangle = Z^\dagger(\theta_1) Z^\dagger(\theta_2) ... Z^\dagger(\theta_n)|0\rangle$$

where the Fadeev-Zamolodchikov operators satisfy

$$\begin{aligned} Z^\dagger(\theta) Z^\dagger(\theta') &= S(\theta - \theta') Z^\dagger(\theta') Z^\dagger(\theta), \\ Z(\theta) Z(\theta') &= S(\theta - \theta') Z(\theta') Z(\theta), \\ Z(\theta) Z^\dagger(\theta') &= S(\theta - \theta') Z^\dagger(\theta') Z(\theta) + \delta(\theta - \theta'). \end{aligned}$$

- Two-point functions

$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \sum_{n=0}^{\infty} \sum_{\{\boldsymbol{\theta}\}} \langle 0|\mathcal{O}_1|\theta_1, \theta_2, ..., \theta_n\rangle \langle \theta_1, \theta_2, ..., \theta_n|\mathcal{O}_2|0\rangle$$

where the form factor is defined as

$$F_n^{\mathcal{O}}(\theta_1, \theta_2, ..., \theta_n) = \langle 0|\mathcal{O}|\theta_1, \theta_2, ..., \theta_n\rangle$$

The entangelment entropy

- The two-point functions of branch-point twist fields

$$\mathrm{Tr}_A(\rho_A^n) = \varepsilon^{4\Delta_n} \langle \mathcal{T}(0)\mathcal{T}^\dagger(r)\rangle_n$$

  where $\mathcal{T}$ is the twist operator.

- The entanglement entropy can be obtained by

$$S(r) = \lim_{n\to 1} \frac{\log(\mathrm{Tr}_A(\rho_A^n))}{1-n}$$

- The entanglement entropy for IQFT can be obtained by using the twist fields form factor.

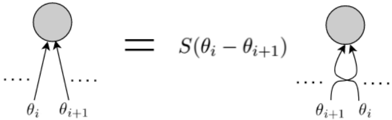
$$F_k^{\mathcal{T}|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k) = \langle 0|\mathcal{T}|\theta_1,\ldots,\theta_k\rangle_{\mu_1\ldots\mu_k}$$

- How to obtain the form factor?

Form factor axioms [Smirnov 1992]:

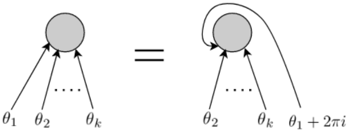
- Watson's equation

$$F_k^{\mathcal{O}}(..., \theta_i, \theta_{i+1}, ...) = S(\theta_i - \theta_{i+1}) F_k^{\mathcal{O}}(..., \theta_{i+1}, \theta_i, ...);$$



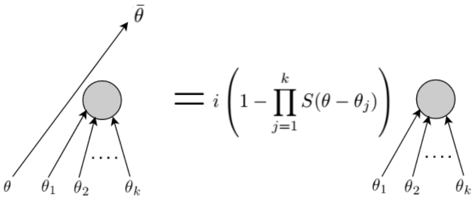
- Cyclicity

$$F_k^{\mathcal{O}}(\theta_1 + 2\pi i, \theta_2, ..., \theta_k) = F_k^{\mathcal{O}}(\theta_2, ..., \theta_k, \theta_1);$$



- Kinematic poles

$$\lim_{\bar{\theta}\to\theta}(\bar{\theta}-\theta)F^{\mathcal{O}}_{k+2}(\bar{\theta}+i\pi,\theta,\theta_1,\ldots,\theta_k)$$
$$=i\left(1-\prod_{j=1}^{k}S(\theta-\theta_j)\right)F^{\mathcal{O}}_{k}(\theta_1,\ldots,\theta_k);$$



- If the spectrum of the model contains different kinds of particles and bound states, the form factors have additional constraints.

- General solution of the form factor

$$F_k^{\mathcal{O}}(\theta_1, \ldots, \theta_k) = H_k Q_k^{\mathcal{O}}(\theta_1, \ldots, \theta_k) \prod_{i<j} \frac{F_{\min}(\theta_{ij})}{e^{\theta_i} + e^{\theta_j}}$$

- The essential step for the computation the form factors is to the determinate the minimal factor, which just satisfies the Waston's equation and cyclicity.

$$F_{\min}(\theta) = S(\theta) F_{\min}(-\theta) = F_{\min}(2\pi i - \theta).$$

- Example: Two-particle form factor for Ising model

$$\text{S-matrix} : S(\theta) = -1,$$

$$\text{Trace of stress tensor FF} : F_2^{\Theta}(\theta) = -2\pi i m^2 \sinh\left(\frac{\theta}{2}\right)$$

- Given an IQFT resulting from the massive perturbation of a known CFT, the Δ-sum rule reads

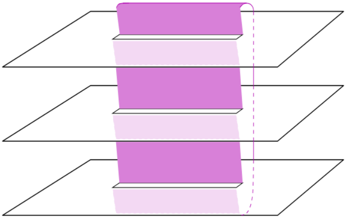
$$\Delta_{\mathcal{O}}^{UV} - \Delta_{\mathcal{O}}^{IR} = -\frac{1}{4\pi\langle\mathcal{O}\rangle}\int_0^{\infty} dr \langle\Theta(0)\mathcal{O}(r)\rangle_c,$$

  If the theory is massive, the IR scaling dimension becomes vanishing. These relations can also be used to distinguish the different operators.
- Comparing with scaling dimension of primary fields in CFT

$$\langle T(w)\Phi(u)\rangle = \left(\frac{\Delta}{(w-u)^2} + \frac{1}{w-u}\frac{\partial}{\partial u}\right)\langle\Phi(u)\rangle.$$

EE and twist operator Form Factor Bootstrap

- We compute the entanglement entropy by the replica trick. We first make n-copies of the theory and compute the partition function on a n-sheet Riemann surface where the branch points correspond to the end points of the interval A



- The n-sheet partition function can be computed by the twist operator correlation function

$$Z[\mathcal{M}_n] \propto \varepsilon^{4\Delta_n^a} \langle \mathcal{T}(0) \tilde{\mathcal{T}}(r) \rangle.$$

- We consider n-copies of a known integrable theory possessing a single particle spectrum and no bound states (such as the Ising model). We have therefore n kinds of particles, which we will denote by indices $1, 2, ..., n$. The S-matrix between particles i and j with rapidities θ_i and θ_j is given by

$$S_{ij}(\theta) = \begin{cases} S(\theta) & i = j \quad i = 1, \ldots, n \\ 1 & i \neq j \quad i, j = 1, \ldots, n \end{cases}$$

- We can package the n-copy theory into one theory, which contains n kinds of particles. They also satisfy the Fadeev-Zamolodchikov algebra

$$\begin{aligned} Z_i^\dagger(\theta_i)Z_j^\dagger(\theta_j) &= S_{ij}(\theta_i - \theta_j)Z_j^\dagger(\theta_j)Z_i^\dagger(\theta_i), \\ Z_i(\theta_i)Z_j(\theta_j) &= S_{ij}(\theta_i - \theta_j)Z_j(\theta_j)Z_i(\theta_i), \\ Z_i(\theta_i)Z_j^\dagger(\theta_j) &= S_{ij}(\theta_i - \theta_j)Z_j^\dagger(\theta_j)Z_i(\theta_i) + \delta(\theta_i - \theta_j), \end{aligned}$$

- The Hilbert space for n-copy theory: the k-particle basis state

$$|\theta_1, \dots, \theta_k\rangle_{\mu_1,\dots\mu_k} = Z^\dagger_{\mu_1}(\theta_1)\dots Z^\dagger_{\mu_k}(\theta_k)|0\rangle.$$

  The index μ_i means the i-th particle is created by $Z^\dagger_{\mu_k}$.
  $(\mu_i \in \{1, 2, \dots, n\})$

- Form factors for n-copy theory:

$$F_k^{\mathcal{O}|\mu_1\dots\mu_k}(\theta_1, \dots, \theta_k) = \langle 0|\mathcal{O}|\theta_1, \dots, \theta_k\rangle_{\mu_1\dots\mu_k}$$

- The effect of twist operator

$$\begin{aligned}\Psi_i(y)\mathcal{T}(x) =& \mathcal{T}(x)\Psi_{i+1}(y) \quad x^1 > y^1,\\ \Psi_i(y)\mathcal{T}(x) =& \mathcal{T}(x)\Psi_i(y) \quad x^1 < y^1,\\ \tilde{\mathcal{T}} =& \mathcal{T}^\dagger.\end{aligned}$$

The form factor axioms for twist operator [arXiv:0706.3384]

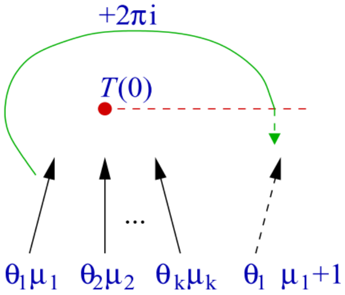
- Waston's equation

$$F_k^{\mathcal{T}|\ldots\mu_i\mu_{i+1}\ldots}(\ldots,\theta_i,\theta_{i+1},\ldots) = S_{\mu_i\mu_{i+1}}(\theta_{ii+1}) F_k^{\mathcal{T}|\ldots\mu_{i+1}\mu_i\ldots}(\ldots,\theta_{i+1},\theta_i,$$

- Cyclicity

$$F_k^{\mathcal{T}|\mu_1\mu_2\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k) = F_k^{\mathcal{T}|\mu_2\ldots\mu_n\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1);$$

where $\hat{\mu}_i = \mu_i + 1$.

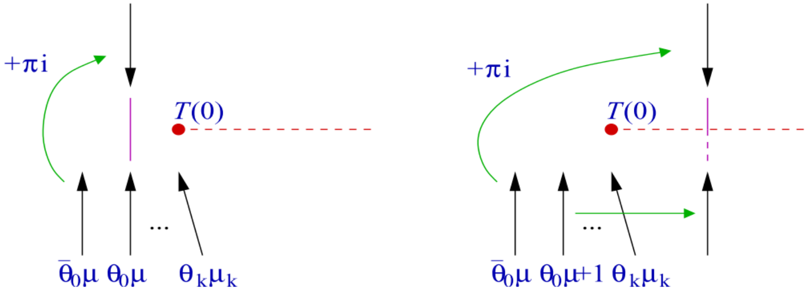


- Kinematic poles

$$\lim_{\bar\theta\to\theta}(\bar\theta\to\theta)F_{k+2}^{\mathcal{T}|\bar\mu\mu\mu_1\dots\mu_k}(\bar\theta+i\pi,\theta,\theta_1\dots,\theta_k)=iF_k^{T|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k),$$

$$\lim_{\bar\theta\to\theta}(\bar\theta\to\theta)F_{k+2}^{T|\bar\mu\hat\mu\mu_1\dots\mu_k}(\bar\theta+i\pi,\theta,\theta_1\dots,\theta_k)$$

$$=-i\prod_{i=1}^{k}S_{\mu\mu_i}(\theta-\theta_i)F_k^{T|\mu_1\dots\mu_k}(\theta_1,\dots,\theta_k).$$



Solving the twist operator form factor axioms

- The minimal form factors

$$F_{\text{min}}^{\mathcal{T}|kj}(\theta, n) = F_{\text{min}}^{\mathcal{T}|jk}(-\theta, n) S_{kj}(\theta) = F_{\text{min}}^{\mathcal{T}|jk+1}(2\pi i - \theta, n) \quad \forall \quad j, k$$

- Repeated use of the above equations leads to the following constraints

$$F_{\text{min}}^{\mathcal{T}|ii+k}(\theta, n) = F_{\text{min}}^{\mathcal{T}|jj+k}(\theta, n) \quad \forall \quad i, j, k$$
$$F_{\text{min}}^{\mathcal{T}|1j}(\theta, n) = F_{\text{min}}^{\mathcal{T}|11}(2\pi(j-1)i - \theta, n) \quad \forall \quad j \neq 1.$$

- The form factor $F_{\text{min}}^{T|11}(\theta, n)$ is enough to determine all minimal form factors of the theory, which also satisfy

$$F_{\text{min}}^{\mathcal{T}|11}(\theta, n) = F_{\text{min}}^{\mathcal{T}|11}(-\theta, n) S(\theta) = F_{\text{min}}^{\mathcal{T}|11}(-\theta + 2\pi n i, n).$$

- Recalling the minimal form factor for local operators

$$f_{\text{min}}(\theta) = f_{\text{min}}(-\theta) S(n\theta) = f_{\text{min}}(-\theta + 2\pi i)$$

This relation allows us to identify the solution

$$F_{\text{min}}^{\mathcal{T}|11}(\theta, n) = f_{\text{min}}(\theta/n).$$

- The kinematic poles for $F_2^{\tilde{\mathcal{T}}|11}(\theta, n)$ are located at $\theta = i\pi$ and $\theta = i\pi(2n-1)$
- The full two-particle form factor is

$$F_2^{\mathcal{T}|jk}(\theta) = \frac{\langle \mathcal{T} \rangle \sin\left(\frac{\pi}{n}\right)}{2n \sinh\left(\frac{i\pi(2(j-k)-1)+\theta}{2n}\right) \sinh\left(\frac{i\pi(2(k-j)-1)-\theta}{2n}\right)} \frac{F_{\text{min}}^{\mathcal{T}|jk}(\theta, n)}{F_{\text{min}}^{\mathcal{T}|jk}(i\pi, n)}$$

where the normalization has been chosen so that the kinematical residue equation gives

$$F_0^{\mathcal{T}} = \langle \mathcal{T} \rangle.$$

- For the twist operator $\tilde{\mathcal{T}}$, we have

$$F_2^{\mathcal{T}|ij}(\theta, n) = F_2^{\tilde{\mathcal{T}}|(n-i)(n-j)}(\theta, n).$$

These property also leads to

$$F_2^{\tilde{\mathcal{T}}|11}(\theta, n) = F_2^{\mathcal{T}|11}(\theta, n),$$
$$F_2^{\tilde{\mathcal{T}}|1j}(\theta, n) = F_2^{\mathcal{T}|11}(\theta + 2\pi i(j-1), n).$$

Example: Ising model

- setup

$$S(\theta) = -1, F_{\min}^{\mathcal{T}|11}(\theta) = -i\sinh\left(\frac{\theta}{2n}\right), F_2^{\Theta}(\theta) = -2\pi i m^2 \sinh\left(\frac{\theta}{2}\right)$$

- In the UV limit, the conformal dimension of an operator is related to a special correlation function, which is the so-called Δ-sum rule

$$\Delta^{\mathcal{T}} = \Delta^{\tilde{\mathcal{T}}} = -\frac{1}{2\langle\mathcal{T}\rangle}\int_0^\infty r\langle\Theta(r)\tilde{\mathcal{T}}(0)\rangle dr$$

- For two particle approximate, we have

$$\Delta^{\mathcal{T}} \approx -\frac{n}{2\langle\mathcal{T}\rangle}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\theta_1 d\theta_2 F_2^{\Theta|11}(\theta_{12})F_2^{\mathcal{T}|11}(\theta_{12},n)^*}{2(2\pi)^2 m^2\left(\cosh\theta_1+\cosh\theta_2\right)^2} = \frac{1}{48}\left(n-\frac{1}{n}\right)$$

where we used

$$F_2^{\Theta|ij}(\theta) = 0 \quad \forall \quad i \neq j.$$

- The twist operator two-point function in the two-particle approximation is given by

$$
\begin{aligned}
&\langle\mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle \\
\approx&\langle\mathcal{T}\rangle^2\left(1+\sum_{i,j=1}^{n}\int_{-\infty}^{\infty}\frac{d\theta}{(2\pi)^2}\left|F_2^{\mathcal{T}|ij}(\theta,n)\right|^2K_0\left(2mr\cosh\frac{\theta}{2}\right)\right) \\
=&\langle\mathcal{T}\rangle^2\left(1+\frac{n}{4\pi^2}\int_{-\infty}^{\infty}d\theta f(\theta,n)K_0\left(2mr\cosh\frac{\theta}{2}\right)\right)
\end{aligned}
$$

where the function $f(\theta,n)$ satisfy

$$\lim_{n\to1}\frac{\partial}{\partial n}f(\theta,n)=\frac{\pi^2}{2}\delta(\theta),\quad \lim_{n\to1}f(\theta,n)=0,$$

- Finally, we obtain the entanglement entropy

$$S_A(rm)=-\frac{c}{3}\log(m\epsilon)+U-\frac{1}{8}K_0(2rm)+...$$

$T\bar{T}$/CDD Factor deformation

- The IQFT is characterized by the factorizable of S-matrix.
- There are general constraints of analyticity, crossing symmetry, and unitarity for the two-particle S-matrix, which together fix the S-matrix up to the so-called CDD ambiguity.
- The deformations of IQFT preserving integrability must generate deformations of the factorizable S-matrix.
- It turns out that the $T\bar{T}$ deformation actually leads to a CDD factor[Smirnov,Zamolodchikov, 2016]

$$S_\alpha(\theta) = S_{\mathbf{0}}(\theta)\Phi_\alpha(\theta), \quad \Phi_{\boldsymbol{\alpha}}(\theta) = \exp\left(-i\sum_{s\in\mathcal{S}} \alpha_s m^{s+1}\sinh(s\theta)\right)$$

- For standard $T\bar{T}$ deformation, we just consider one parameter deformation

$$\Phi_{\boldsymbol{\alpha}}(\theta) = \exp\left(-i\alpha m^2 \sinh\theta\right)$$

Local operator form factor axioms becomes

- Watson's equation

$$F_k^{\mathcal{O}}(...,\theta_i,\theta_{i+1},...;\boldsymbol{\alpha}) = S_{\boldsymbol{\alpha}}(\theta_i-\theta_{i+1})F_k^{\mathcal{O}}(...,\theta_{i+1},\theta_i,...;\boldsymbol{\alpha});$$

- Cyclicity

$$F_k^{\mathcal{O}}(\theta_1+2\pi i,\theta_2,...,\theta_k;\boldsymbol{\alpha}) = F_k^{\mathcal{O}}(\theta_2,...,\theta_k,\theta_1;\boldsymbol{\alpha});$$

- Kinematic poles

$$\begin{aligned}&\lim_{\bar\theta\to\theta}(\bar\theta-\theta)F_{k+2}^{\mathcal{O}}(\bar\theta+i\pi,\theta,\theta_1,\ldots,\theta_k;\boldsymbol{\alpha})\\ =&i\left(1-\prod_{j=1}^{k}S_{\boldsymbol{\alpha}}(\theta-\theta_j)\right)F_k^{\mathcal{O}}(\theta_1,\ldots,\theta_k;\boldsymbol{\alpha});\end{aligned}$$

- We assume that the form factors $F_k^{\mathcal{O}}(\theta_1, \ldots, \theta_k; \boldsymbol{\alpha})$ of the deformed theory factorise as the product of the original undeformed form factors and a deformation-dependent function.[2305.17068,2306.01640]

$$F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha}) = F_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n; 0)\Upsilon_n^{\mathcal{O}}(\theta_1, \ldots, \theta_n; \boldsymbol{\alpha}),$$
$$F_{\min}(\theta; \boldsymbol{\alpha}) = F_{\min}(\theta; \mathbf{0})\varphi(\theta; \boldsymbol{\alpha})$$

- The minimal form factor satisfy

$$F_{\min}(\theta; \boldsymbol{\alpha}) = S_{\boldsymbol{\alpha}}(\theta)F_{\min}(-\theta; \boldsymbol{\alpha}) = F_{\min}(2\pi i - \theta; \boldsymbol{\alpha}).$$

- The factor $\varphi(\theta; \boldsymbol{\alpha})$ satisfies the equation

$$\varphi(\theta; \boldsymbol{\alpha}) = \Phi_{\boldsymbol{\alpha}}(\theta)\varphi(-\theta; \boldsymbol{\alpha}) = \varphi(2\pi i - \theta; \boldsymbol{\alpha}),$$

whose solution is

$$\varphi(\theta; \boldsymbol{\alpha}) = \exp\left[-\frac{i\pi - \theta}{2\pi}\sum_{s\in\mathcal{S}}\alpha_s \sinh s\theta + \sum_{s\in\mathcal{S}'}\beta_s \cosh s\theta\right],$$

We just consider the case of $\beta = 0$

Twist operator form factor axioms are still holds

- Waston's equation

$$F_k^{\mathcal{T}|\ldots\mu_i\mu_{i+1}\cdots}(\ldots,\theta_i,\theta_{i+1},\ldots;\boldsymbol{\alpha}) = S^{\boldsymbol{\alpha}}_{\mu_i\mu_{i+1}}(\theta_{ii+1})F_k^{\mathcal{T}|\ldots\mu_{i+1}\mu_i\cdots}(\ldots,\theta_{i+1},$$

- Cyclicity

$$F_k^{\mathcal{T}|\mu_1\mu_2\ldots\mu_k}(\theta_1+2\pi i,\ldots,\theta_k;\boldsymbol{\alpha}) = F_k^{\mathcal{T}|\mu_2\ldots\mu_n\hat{\mu}_1}(\theta_2,\ldots,\theta_k,\theta_1;\boldsymbol{\alpha});$$

- Kinematic pole

$$\lim_{\bar{\theta}\to\theta}(\bar{\theta}\to\theta)F_{k+2}^{\mathcal{T}|\bar{\mu}\mu\mu_1\ldots\mu_k}(\bar{\theta}+i\pi,\theta,\theta_1\ldots,\theta_k;\boldsymbol{\alpha}) = iF_k^{T|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta$$

$$\lim_{\bar{\theta}\to\theta}(\bar{\theta}\to\theta)F_{k+2}^{T|\bar{\mu}\hat{\mu}\mu_1\ldots\mu_k}(\bar{\theta}+i\pi,\theta,\theta_1\ldots,\theta_k;\boldsymbol{\alpha})$$

$$= -i\prod_{i=1}^{k}S^{\boldsymbol{\alpha}}_{\mu\mu_i}(\theta-\theta_i)F_k^{T|\mu_1\ldots\mu_k}(\theta_1,\ldots,\theta_k;\boldsymbol{\alpha}).$$

- The S-matrix for the n-copy theory becomes

$$S^{\boldsymbol{\alpha}}_{ij}(\theta) = \begin{cases} S_{\boldsymbol{\alpha}}(\theta) & i=j \quad i=1,\ldots,n \\ 1 & i\neq j \quad i,j=1,\ldots,n \end{cases}$$

- Here, we just consider the two-particle form factors. The deformed minimal form factor should satisfy the first two equations

$$F_{\min}^{\mathcal{T}|kj}(\theta,n;\boldsymbol{\alpha}) = F_{\min}^{\mathcal{T}|jk}(-\theta,n;\boldsymbol{\alpha})S_{kj}^{\boldsymbol{\alpha}}(\theta) = F_{\min}^{\mathcal{T}|jk+1}(2\pi i-\theta,n;\boldsymbol{\alpha})$$

Similarly to the undeformed case, we have

$$\begin{aligned} F_{\min}^{\mathcal{T}|ii+k}(\theta,n;\boldsymbol{\alpha}) &= F_{\min}^{\mathcal{T}|jj+k}(\theta,n;\boldsymbol{\alpha}) \quad \forall \quad i,j,k \\ F_{\min}^{\mathcal{T}|1j}(\theta,n;\boldsymbol{\alpha}) &= F_{\min}^{\mathcal{T}|11}(2\pi(j-1)i-\theta,n;\boldsymbol{\alpha}) \quad \forall \quad j\neq 1. \end{aligned}$$

- We assume the minimal form factor is factorizable

$$F_{\min}^{\mathcal{T}|11}(\theta,n;\boldsymbol{\alpha}) = F_{\min}^{\mathcal{T}|11}(\theta,n)\varphi^{\mathcal{T}|11}(\theta,n;\boldsymbol{\alpha}), \quad \varphi^{\mathcal{T}|11}(\theta,n;0)=1$$

Then one can get

$$\varphi^{\mathcal{T}|11}(\theta,n;\boldsymbol{\alpha}) = \varphi^{\mathcal{T}|11}(-\theta,n;\boldsymbol{\alpha})\Phi_{\boldsymbol{\alpha}}(\theta) = \varphi^{\mathcal{T}|11}(-\theta+2\pi n i,n;\boldsymbol{\alpha})$$

the solution is

$$\varphi^{\mathcal{T}|11}(\theta,n;\boldsymbol{\alpha}) = \exp\left[-\frac{i\pi-\theta/n}{2\pi}\sum_{s\in\mathcal{S}}\alpha_s\sinh s\theta\right].$$

- One can also find

$$\varphi^{\mathcal{T}|jj}(\theta, n; \boldsymbol{\alpha}) = \varphi^{\mathcal{T}|11}(\theta, n; \boldsymbol{\alpha}),$$
$$\varphi^{\mathcal{T}|1j}(\theta, n; \boldsymbol{\alpha}) = \exp\left[\frac{i(n-j+1)}{n}\sum_{s\in\mathcal{S}}\alpha_s \sinh s\theta\right]\varphi^{\mathcal{T}|11}(\theta, n; \boldsymbol{\alpha}),$$

These relations allow us to determine all the factors $\varphi^{\mathcal{T}|jk}$.

- The main results for the deformed form factor

$$F_2^{\mathcal{O}}(\theta; \boldsymbol{\alpha}) = F_2^{\mathcal{O}}(\theta)\varphi(\theta; \boldsymbol{\alpha}),$$
$$F_2^{\mathcal{T}|jk}(\theta; \boldsymbol{\alpha}) = F_2^{\mathcal{T}|jk}(\theta)\varphi^{\mathcal{T}|jk}(\theta, n; \boldsymbol{\alpha})$$

Deformed Ising model

- Deformed form factor

$$F_{\min}^{\mathcal{T}|jk}(\theta) = -\, i \sinh\left(\frac{\theta}{2n}\right) \varphi^{\mathcal{T}|jk}(\theta, n; \boldsymbol{\alpha}),$$

$$F_2^{\Theta}(\theta) = -\, 2\pi i m^2 \sinh\left(\frac{\theta}{2}\right) \varphi(\theta; \boldsymbol{\alpha})$$

- In the UV limit, the conformal dimension of twist operator is related to a special correlation function, which is the so-called Δ-sum rule

$$\begin{aligned}\Delta^{\mathcal{T}} &= -\frac{1}{2\langle \mathcal{T} \rangle} \int_0^\infty r \langle \Theta(r) \tilde{\mathcal{T}}(0) \rangle dr \\ &= -\, \frac{n}{2\langle \mathcal{T} \rangle} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2 F_2^{\Theta|11}(\theta_{12}) F_2^{\mathcal{T}|11}(\theta_{12}, n)^*}{2(2\pi)^2 m^2 \left(\cosh\theta_1 + \cosh\theta_2\right)^2} \varphi(\theta; \boldsymbol{\alpha}) \varphi^{\mathcal{T}|jk}(\theta, n; \boldsymbol{\alpha} \\ &= \frac{1}{48}\left(n - \frac{1}{n}\right) + \alpha(an + b) - \frac{\alpha^2 (n+1)^2 (2(n-1)n + 1)}{8n}.\end{aligned}$$

- The twist operator two-point function in the two-particle approximation is given by

$$
\begin{aligned}
&\langle\mathcal{T}(r)\tilde{\mathcal{T}}(0)\rangle \\
\approx&\langle\mathcal{T}\rangle^2\left(1+\frac{n}{4\pi^2}\int_{-\infty}^{\infty}d\theta f(\theta,n)|\varphi^{\mathcal{T}|jk}(\theta,n;\boldsymbol{\alpha})|^2 K_0\left(2mr\cosh\frac{\theta}{2}\right)\right)
\end{aligned}
$$

where the function $f(\theta,n)$ satisfy

$$
\lim_{n\to1}\frac{\partial}{\partial n}f(\theta,n)=\frac{\pi^2}{2}\delta(\theta),\quad \lim_{n\to1}f(\theta,n)=0,
$$

- Finally, we obtain the deformed entanglement entropy

$$
S_A(rm)=-\frac{c_{\text{eff}}}{3}\log(m\epsilon)(m\epsilon)^{d_1^\alpha}+U-\frac{1}{8}K_0(2rm)+\ldots
$$

$$
d_n^\alpha=2\Delta^{\mathcal{T}},\quad c_{\text{eff}}=2c\frac{\partial d_n^\alpha}{\partial n}\Big|_{n=1}.
$$

- This is consistent with the fact that the deformation is irrelevant and does not change the IR physics. The effect of the deformation shows up in the UV limit. We see that first of all the divergence is no longer logarithmic.
- For $\alpha = 0$, we recover

$$S_A(rm, 0) = -\frac{1}{6}\log(m\epsilon) + U - \frac{1}{8}K_0(2rm)\,.$$

- For the small α, our result can reproduce the perturbation result

$$S_A(rm, \alpha) = -\frac{1}{6}\log(m\epsilon) - \frac{1}{3}\alpha\log(m\epsilon)(6a + (a+b)\log(m\epsilon)) + O(\alpha^2)$$

- There is a log-square divergence for the Ising field theory. This is consistent with the recent perturbative calculations in [2302.06688]

Conclusion and outlook

- We have focused on the diagonal scattering theory with a single particle and no bound states. The deformed form factor axioms for twist operator have been solved up to two particle.
- Using the deformed form factor, we computed the deformed Entanglement entropy $T\bar{T}$-deformed IQFTs.
- We find that the UV behavior of the deformed entanglement entropy is modified, but the IR behavior does not change.
- There are many future directions to pursue. for example: Symmetry resolved entanglement entropy, the high particle contributions, theory with multi-particles scattering and bound states.
- Compute the correlation function and consider the conformal limit.