On the universal behavior of TTbar-deformed CFTs

Seminar at BIMSA

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1. Review: $T\bar{T}$-deformed CFT
2. Partition function of $T\bar{T}$-deformed CFT and modular invariance
3. Universality of the partition function and spectrum at large c
4. Conclusion and outlook

$T\bar{T}$ deformation: double trace

- The $T\bar{T}$-deformation describes a one-parameter family of quantum field theories via the differential equation

$$\frac{\partial I}{\partial \mu} = 8\pi \int d^2x T\bar{T}, \quad T\bar{T} := \frac{1}{8}(T^{\alpha\beta}T_{\alpha\beta} - (T^{\alpha}_{\alpha})^2) \tag{1}$$

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- Despite being irrelevant, the $T\bar{T}$-deformation features many interesting properties: Integrability and solvability, S-matrix, modular invariance, holography. Other interesting features include connections to two-dimensional gravity and string theory, correlation functions and entanglement entropy and generalizations to higher and dimensions.

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- By definition, (1) is a double-trace deformation that can be applied to any quantum field theory with a well-defined stress tensor.

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- A symmetric product orbifold CFT, denoted by $\text{sym}^N\mathcal{M}_0 := (\mathcal{M}_0)^N/S_N$, consists of N copies of seed CFT $\mathcal{M}_0$ supplemented by the condition that all states are invariant under the symmetric group S_N. The central charge of this theory is $c = Nc_0$.

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- The single-trace $T\bar{T}$ deformation of $\text{sym}^N\mathcal{M}_0$ yields another symmetric product orbifold $\text{sym}^N\mathcal{M}_\mu$, whose seed theory $\mathcal{M}_\mu$ is the $T\bar{T}$ deformation of the seed CFT $\mathcal{M}_0$. In other words, under the single-trace deformation each copy of the seed CFT is deformed by the $T\bar{T}$ operator in that copy.

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- On the other hand, single-trace $T\bar{T}$-deformed CFTs have been argued to be dual to the long string sector of string theory on three-dimensional linear dilaton and TsT-transformed backgrounds.

- Let us consider a $T\bar{T}$-deformed CFT quantized on a cylinder of size 2π.The spectrum of the deformed theory can be written as

$$E(\mu) = -\frac{1}{2\mu}(1 - \sqrt{1 + 4\mu E(0) + 4\mu^2 J(0)^2}), \quad J(\mu) = J(0) \tag{2}$$

where $E(\mu) = E_L(\mu) + E_R(\mu)$ is the deformed energy and $J(\mu) = E_L(\mu) - E_R(\mu)$ is the deformed angular momentum.

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- When $\mu < 0$ the spectrum becomes complex for large values of the undeformed energy $E(0)$. When $\mu > 0$, the energy of the vacuum obtained by letting $E(0) = -c/12, J(0) = 0$ is complex when $\mu c > 3$.

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- We will assume that the deformation parameter satisfies $0 \leq \frac{\mu c}{3} \leq 1$ to avoid complex spectrum.

Partition function: double trace

The partition function of a $T\bar{T}$-deformed CFT on a torus is defined in terms of the deformed spectrum by

$$
\begin{aligned}
Z(\tau,\bar{\tau};\mu) &:= \mathrm{Tr}\left(q^{E_L(\mu)}\bar{q}^{E_R(\mu)}\right) = \sum_{E_L,E_R} d(E_L,E_R)e^{2\pi i\tau E_L(\mu)-2\pi i\bar{\tau}E_R(\mu)} \\
&= \int dE_L dE_R \rho(E_L,E_R)e^{2\pi i\tau E_L(\mu)-2\pi i\bar{\tau}E_R(\mu)}
\end{aligned}
\tag{3}
$$

where τ is the modular parameter and $q = e^{2\pi i\tau}$. $d(E_L,E_R)$ is the degeneracy of the spectrum and $\rho(E_L,E_R)$ is the spectrum density in the average meaning.

Modular invariance

- The partition function satisfies the following differential equation

$$\partial_\mu Z = \frac{1}{i\pi}\left[(\tau-\bar{\tau})\partial_\tau\partial_{\bar{\tau}} - \mu(\partial_\tau - \partial_{\bar{\tau}})\partial_\mu + \frac{2\mu}{\tau-\bar{\tau}}\partial_\mu\right] Z \quad (4)$$

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which is a direct consequence of the differential equation obeyed by the deformed spectrum.

- Crucially, the partition function is invariant under modular transformations in the sense that

$$Z\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}; \frac{\mu}{|c\tau+d|^2}\right) = Z(\tau,\bar{\tau};\mu) \quad (5)$$

- Note that the deformation parameter μ does not change under modular transformations. Rather, it is the dimensionless deformation parameter μ/R^2 that changes, and the transformation of μ in (5) comes entirely from the change of the spatial circle, $R \to |c\tau + d|R$.

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- The modular $\mathcal{S}$ transformation $\tau \to -1/\tau$, together with the bound $0 \leq \mu c/3 \leq 1$ imply that

$$|\tau|^2 \geq \frac{\mu c}{3} \equiv \frac{1}{4\pi^2 T_H^2} \tag{6}$$

where T_H is the Hagedorn temperature.

Partition function: single trace

- The partition function of single trace $T\bar{T}$-deformed CFT $\text{Sym}^N\mathcal{M}_\mu$ is defined by

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- For symmetric obifold CFT $\mathrm{sym}^N\mathcal{M}_0$, a primary field is described by some representative of an orbit of S_N acting on the n-tuples $\langle\phi_1\cdots\phi_N\rangle$ of primaries ϕ_i of $\mathcal{M}_0$, together with a pair of commuting permutations $x, y \in S_N$ describing how the sheets are permuted when going around the cycle of a canonical homology basis.

- Therefore, x, y determines an N-sheeted covering of the torus which is usually not connected. Its connected components are in one-to-one correspondence with the orbits $\xi \in O(x, y)$ where $O(x, y)$ is the set of orbits of the subgroup generated by x and y. Each such connected component is itself a torus with modular parameter τ_ξ.

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- As a result, the partition function for $\mathrm{sym}^N \mathcal{M}_0$ is given by(Bantay, 97)

$$Z_N(\tau, \bar{\tau}) = \frac{1}{|S_N|} \sum_{xy=yx} \sum_{\xi \in O(x,y)} Z(\tau_\xi) \tag{8}$$

- It is difficult to apply the analogous method to determine the partition function of $\mathrm{sym}^N \mathcal{M}_\mu$ since we have an additional parameter μ and we do not know what μ_ξ is. If $\mu_\xi = \mu$, then it corresponds to the double trace deformation.

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- Since the partition function of the seed theory $\mathcal{M}_\mu$ is modular invariant, it is natural to expect the partition function of $\mathrm{sym}^N\mathcal{M}_\mu$ to be modular invariant as well. We will use this property to derive the partition function in the following.

Untwisted sector

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- A typical untwisted state Φ takes the form

$$\Phi = \text{Sym}(\otimes_{n=1}^{N}\phi^{(i_n)}) \tag{9}$$

where n labels different copies of the symmetric product and i_n labels the state on the nth copy.

For example, when $N = 3$, the untwisted states are

$$\begin{aligned} \Phi_{(i)} &:= \phi^{(i)} \otimes \phi^{(i)} \otimes \phi^{(i)} \\ \Phi_{(i,j)} &:= \mathrm{Sym}(\phi^{(i)} \otimes \phi^{(i)} \otimes \phi^{(j)}), \quad i \neq j \\ \Phi_{(i,j,k)} &:= \mathrm{Sym}(\phi^{(i)} \otimes \phi^{(j)} \otimes \phi^{(k)}), \quad i \neq j \neq k \end{aligned} \tag{10}$$

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The contribution of each of these kinds of states to the partition function of $\mathrm{sym}^3\mathcal{M}_\mu$ is

$$\begin{aligned} \Phi_{(i)} &: Z^{(3)}(\tau,\bar\tau;\mu) := Z(3\tau,3\bar\tau;\mu) \\ \Phi_{(i,j)} &: Z^{(2)}(\tau,\bar\tau;\mu) := Z(2\tau,2\bar\tau;\mu)Z(\tau,\bar\tau;\mu) - Z^{(3)}(\tau,\bar\tau;\mu) \\ \Phi_{(i,j,k)} &: Z^{(1)}(\tau,\bar\tau;\mu) := \frac{1}{3!}\left[Z(\tau,\bar\tau;\mu)^3 - 3Z^{(2)} - Z^{(3)}\right] \end{aligned} \tag{11}$$

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In total, the contribution of the untwisted states to the partition function is simply given by

$$Z_{\text{untwisted}}(\tau, \bar{\tau}; \mu) = \frac{1}{3!}\left[Z(\tau, \bar{\tau}; \mu)^3 + 3Z(2\tau, 2\bar{\tau}; \mu)Z(\tau, \bar{\tau}; \mu) + 2Z(3\tau, 3\bar{\tau}\right.$$

For arbitrary values of N, the partition function for the untwisted sector can then be written as

$$Z_{\text{untwisted}}(\tau,\bar{\tau};\mu)=\sum_{\{k_1,\cdots,k_N\}}\frac{1}{\prod_{n=1}^{N} n^{k_n}k_n!}\prod_{n=1}^{N} Z(n\tau,n\bar{\tau};\mu)^{k_n} \quad (13)$$

where $\{k_1,\cdots,k_N\}$ labels the conjugacy classes of S_N with k_n number of Z_N-cycles in each conjugacy class, i.e. the conjugacy class is $[g]=(1)^{k_1}(2)^{k_2}\cdots(N)^{k_N}$. The k_n numbers are constrained to satisfy

$$\sum_{n=1}^{N} nk_n = N \quad (14)$$

The denominator in (13) is the order of the centralizer subgroup of a permutation g in the conjugacy class.

The untwiseted partition function is not modular invariant because of each $Z(n\tau, n\bar{\tau}; \mu)$ term. In particular, although $Z(n\tau, n\bar{\tau}; \mu)$ is invariant under $\mathcal{T} : \tau \to \tau + 1$ transformation, it fails to be invariant under $\mathcal{S} : \tau \to -1/\tau$ transformations since

$$\mathcal{S} \cdot Z(n\tau, n\bar{\tau}; \mu) = Z\left(-\frac{n}{\tau}, -\frac{n}{\bar{\tau}}; \frac{\mu}{|\tau|^2}\right) = Z\left(\frac{\tau}{n}, \frac{\bar{\tau}}{n}; \frac{\mu}{n^2}\right) \qquad (15)$$

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In order to preserve the modular invariance, we need to add the contribution from the twisted sector. The twisted partition function can be determined by making each of $Z(n\tau, n\bar{\tau}; \mu)$ modular invariant by adding the modular image of (15).

When n is prime, it is not difficult to show that

$$\mathcal{S} \cdot \mathcal{T}^{\alpha+kn} \cdot Z\left(\frac{\tau}{n}, \frac{\bar{\tau}}{n}; \frac{\mu}{n^2}\right) = \mathcal{T}^{\tilde{\alpha}+kn} \cdot Z\left(\frac{\tau}{n}, \frac{\bar{\tau}}{n}; \frac{\mu}{n^2}\right) \tag{16}$$

with $\alpha, \tilde{\alpha} \in [1, n-1]$ and satisfies $\alpha\tilde{\alpha} = \tilde{k}n + 1$. As a result, the following linear combination of modular images of $Z(n\tau, n\bar{\tau}; \mu)$ is modular invariant

$$Z(n\tau, n\bar{\tau}; \mu) + \sum_{\alpha=0}^{n-1} Z\left(\frac{\tau+\alpha}{n}, \frac{\bar{\tau}+\alpha}{n}; \frac{\mu}{n^2}\right) \tag{17}$$

For any positive integer n, the sum of modular images of $Z(n\tau, n\bar{\tau}; \mu)$ is given by

$$(T'_n Z)(\tau, \bar{\tau}; \mu) = \sum_{\gamma | n} \sum_{\alpha=0}^{\gamma-1} Z\left(\frac{n\tau + \alpha\gamma}{\gamma^2}, \frac{n\bar{\tau} + \alpha\gamma}{\gamma^2}; \frac{\mu}{\gamma^2}\right) \qquad (18)$$

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$$Z_N(\tau, \bar{\tau}; \mu) = \sum_{\{k_1, \cdots, k_N\}} \frac{1}{\prod_{n=1}^{N} n^{k_n} k_n!} \prod_{n=1}^{N} (T'_n Z)(\tau, \bar{\tau}; \mu)^{k_n} \tag{19}$$

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We can write the generating functional of Z_N which sometimes is called the grand canonical partition function as

$$\mathcal{Z}(\tau, \bar{\tau}; \mu, p) := \sum_{N=0}^{\infty} p^N Z_N(\tau, \bar{\tau}; \mu) = \exp\left(\sum_{n=1}^{\infty} \frac{p^n}{n} (T'_n)(\tau, \bar{\tau}; \mu)\right) \tag{20}$$

The spectrum of twisted states

Using the definition of the partition function of the seed theory, the nth Hecke transformed partition function $T_n'Z$ can be expanded as

$$(T_n'Z)(\tau,\bar{\tau};\mu) = \sum_{\gamma|n} \sum_{\alpha=0} \sum_{E_L,E_R} \gamma d(E_L,E_R) q^{\frac{n}{\gamma^2} E_L(\frac{\mu}{\gamma^2})} \bar{q}^{\frac{n}{\gamma^2} E_L(\frac{\mu}{\gamma^2})} \delta^{\gamma}_{J(0)}$$

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For $\gamma = n$, above equation implies that for each state with $E_{L,R}(\mu)$ in the seed $\mathcal{M}_\mu$, there are n twisted states with energies

$$E^{(n)}_{L,R} := \frac{1}{n} E_{L,R}(\mu/n^2) \qquad (21)$$

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They are related to the spectrum of the twisted states of the undeformed symmetric orbifold via

$$E^{(n)}_{L,R}(0) = E^{(n)}_{L,R}(\mu) + \frac{2\mu}{n} E^{(n)}_{L}(\mu) E^{(n)}_{R}(\mu) \tag{22}$$

This matches the spectrum of perturbative strings on TsT-transformed backgrounds.

Modular invariance and universality

- Modular invariance is an equality between the partition function at high and low temperature. For the undeformed CFTs, this can help us approximate the high temperature partition function by its vacuum contribution. Working out the entropy, one gets the Cardy formula which holds in the Cardy limit with c fixed and $E_{L,R} \to \infty$.

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- Similar analysis has been performed for the $T\bar{T}$-deformed CFTs and the entropy exhibits the Hagedorn behavior for high energy states.
- For a holographic CFT, the Cardy formula is expected to hold in a semiclassical limit where $c \to \infty$ and $E_{L,R} \sim c$. This motivates Hartman, Keller and Stoica to extend the range of validity of the Cardy formula where they showed that if the spectrum for light state is sparse, then the Cardy formula holds for large c at any temperature.

Partition function of $T\bar{T}$-deformed CFT at large c

- We would like to apply HKS's analysis to the $T\bar{T}$-deformed CFT to study its universal behavior at large c. The crucial point is to find a proper sparseness condition such that the partition function is dominated by the vacuum state.

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- In order to estimate the partition function and extract the density of states, it is convenient to consider the Lorentzian torus that is obtained by setting $(\tau, \bar{\tau}) = (i\beta_L, -i\beta_R)$, where $\beta_{L,R}$ are two independent and strictly positive real numbers. Then the $\mathcal{S}$ invariance reads

$$Z(\beta_L, \beta_R) = \mathrm{Tr}\left(e^{-2\pi\beta_L E_L - 2\pi\beta_R E_R}\right) = Z(\beta'_L, \beta'_R) \qquad (23)$$

where $\beta'_{L,R} = 1/\beta_{L,R}$.

We start by considering the simplest case where $\beta_L = \beta_R = \beta > 1$ and define the light and heavy states as

$$L := \{E_L(\mu) + E_R(\mu) \leq \epsilon\}, H := \{E_L(\mu) + E_R(\mu) > \epsilon\} \qquad (24)$$

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Then we can show $\beta' E(\mu') - \beta E(\mu) < 0$ for heavy states $E(\mu) > \epsilon$. Then we have

$$Z[H] := \mathrm{Tr}_H e^{-2\pi\beta E} < \alpha Z'[H], 0 < \alpha < 1 \tag{25}$$

We start by considering the simplest case where $\beta_L = \beta_R = \beta > 1$ and define the light and heavy states as

$$L := \{E_L(\mu) + E_R(\mu) \leq \epsilon\}, H := \{E_L(\mu) + E_R(\mu) > \epsilon\} \quad (24)$$

Then we can show $\beta' E(\mu') - \beta E(\mu) < 0$ for heavy states $E(\mu) > \epsilon$. Then we have

$$Z[H] := \mathrm{Tr}_H e^{-2\pi\beta E} < \alpha Z'[H], 0 < \alpha < 1 \quad (25)$$

This, together with the modular invariance $Z[L] + Z[H] = Z'[L] + Z'[H]$ implies that the partition function is dominated by the contribution of the light states

$$\log Z[L] < \log Z(\beta; \mu) < \log Z[L] - \log(1 - \alpha) \quad (26)$$

To have the vacuum dominance, we only need to let the contribution from vacuum dominates over light states. It is not difficult to show that the sparseness condition reads

$$\log \rho(E_L, E_R) \leq 2\pi[E_L(\mu) + E_R(\mu) - E_{vac}(\mu)], \quad E_L(\mu) + E_R(\mu) \leq \epsilon \tag{27}$$

where $E_{vac}(\mu)$ is the vacuum energy of the deformed CFT

$$E_{vac}(\mu) = -\frac{1 - \sqrt{1 - \mu c/3}}{2\mu} \tag{28}$$

Consequently, under above sparseness condition for the light states, we have

$$\log Z(\beta; \mu) \approx -2\pi\beta E_{vac}(\mu), \beta > 1 \tag{29}$$

Although the vacuum dominance (29) holds for $\beta_L = \beta_R = \beta > 1$, it is ready to extend it to the region $\beta_{L,R} > 1$.

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$$\rho(E_L, E_R) < Z(\beta_L, \beta_R) e^{2\pi\beta_L E_L(\mu) + 2\pi\beta_R E_R(\mu)} \tag{30}$$

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$$Z(\beta_L, \beta_R; \mu) = \int \rho(E_L, E_R) e^{-2\pi\beta_L E_L - 2\pi\beta_R E_R} < \alpha' e^{-\pi(\beta_L + \beta_R) E_{vac}(\mu)} \tag{31}$$

where α' is some numerical constant. Consequently, vacuum dominance holds in this case, i.e.

$$\log Z(\beta_L, \beta_R; \mu) \approx -\pi(\beta_L + \beta_R) E_{vac}(\mu), \beta_{L,R} > 1 \tag{32}$$

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$$E_L(\mu) < 0, \quad \text{or} \quad E_R(\mu) < 0 \tag{33}$$

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In contrast, the heavy states satisfy $E_{L,R}(\mu) > 0$. The appropriate sparseness condition is given by

$$\log \rho(E_L, E_R) \leq 4\pi \sqrt{\left(E_L(\mu) - \frac{1}{2}E_{vac}(\mu)\right)\left(E_R(\mu) - \frac{1}{2}E_{vac}(\mu)\right)} \tag{34}$$

Partition function of $T\bar{T}$-deformed CFT at large c

- Using induction method, we can prove that the partition function is found to be universal

$$\log Z \approx \max\left\{\pi i(\tau-\bar{\tau})E_{vac}(\mu), -\pi i\left(\frac{1}{\tau}-\frac{1}{\bar{\tau}}\right)E_{vac}(\mu')\right\}, |\tau|^2 \neq 1 \quad (35)$$

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- The logarithm of the asymptotic density of states which is the entropy is given by the $T\bar{T}$ analog of the Cardy formula

$$S(E_L, E_R) \approx 2\pi\left(\sqrt{\frac{c}{6}E_L(1+2\mu E_R)} + \sqrt{\frac{c}{6}E_R(1+2\mu E_L)}\right) \tag{36}$$

with the range of validity being

$$\frac{E_L E_R}{1+2\mu(E_L+E_R)} > \frac{E_{vac}(\mu)^2}{4(1+2\mu E_{vac}(\mu))} \tag{37}$$

Universality in single-trace $T\bar{T}$-deformed CFTs

- Previous discussion about the universality is only for double-trace deformation. In the following, we consider the single-trace $T\bar{T}$-deformed CFTs.

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- Previous discussion about the universality is only for double-trace deformation. In the following, we consider the single-trace $T\bar{T}$-deformed CFTs.
- For the single trace $T\bar{T}$-deformed CFT, the large c limit is realized by the large N limit with c_0 fixed.
- A convenient quantity for the analysis is the generating function which can be written as

$$\mathcal{Z} = (1-\tilde{p})^{-1} \prod_{n>0, E_L, E_R} \left(1-\tilde{p}^n (q\bar{q})^{-\frac{n}{2}E_{vac}(\mu)} q^{E_L^{(n)}(\mu)} \bar{q}^{E_L^{(n)}(\mu)}\right)^{-\rho(E_L, E_R)\delta^n_{J(0)}} \tag{38}$$

where $\tilde{p} = (q\bar{q})^{E_{vac}(\mu)/2} p$.

Formally, we can write $\mathcal{Z} = (1-\tilde{p})^{-1}R(\tilde{p})$. On the other hand, we have $\mathcal{Z} = \sum \tilde{p}^N \tilde{Z}_N$ with $\tilde{Z}_N = (q\bar{q})^{-NE_{vac}(\mu)/2} Z_N$. It is then easy to see that $\tilde{Z}_\infty = R(1)$.

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$$\log \tilde{Z}_\infty = -\sum_{n>0,E_L,E_R} \rho\delta^n_{J(0)} \log\left(1 - n(q\bar{q})^{-\frac{n}{2}E_{vac}(\mu)} q^{E_L^{(n)}(\mu)} \bar{q}^{E_L^{(n)}(\mu)}\right)$$

$$= \sum_{n>0,E_L,E_R} \sum_{k=1}^{\infty} \frac{1}{k} \rho\delta^n_{J(0)} e^{2\pi k[\frac{n}{2}(\beta_L+\beta_R)E_{vac}(\mu)-\beta_L E_L^{(n)}(\mu)-\beta_R E_R^{(n)}(\mu)]}$$

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It can be shown that the double sum converges in the large N limit so that $\log \tilde{Z}_\infty$ is finite. Therefore, the partition function is universal such that

$$\log Z_N \approx \max\left\{\pi i(\tau-\bar{\tau})NE_{vac}(\mu), -\pi i\left(\frac{1}{\tau}-\frac{1}{\bar{\tau}}\right)NE_{vac}(\mu')\right\}, |\tau|^2 \neq 1 \quad (39)$$

Density of states

We see that the universality of the single-trace partition function holds without assuming sparseness of the light states. This implies that the density of light states is sparse as a result of the orbifolding.

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Density of states

We see that the universality of the single-trace partition function holds without assuming sparseness of the light states. This implies that the density of light states is sparse as a result of the orbifolding.
Actually, the density of light states saturates the sparseness bound as we will show in the following. Firstly, note that the single-trace partition function Z_N and the nth Hecke transformed partition function $T_n'Z$ can be written as

$$
\begin{aligned}
Z_N(\beta_L,\beta_R;\mu) &= \int dE_L dE_R \rho_N(E_L,E_R) e^{-2\pi\beta_L E_L - 2\pi\beta_R E_R} \\
(T_n'Z)(\beta_L,\beta_R;\mu) &= \int dE_L dE_R \rho_{T_n'}(E_L,E_R) e^{-2\pi\beta_L E_L - 2\pi\beta_R E_R}
\end{aligned}
\tag{40}
$$

where the respective density ρ_N and $\rho_{T_n'}$ are introduced.

The vacuum dominance of Z_N implies that an upper bound on the density of all states

$$\log \rho_N < \min\{\log Z_N + 2\pi\beta_L E_L + 2\pi\beta_R E_R\} \\ \leq 4\pi\sqrt{(E_L - \frac{NE_{vac}}{2})(E_R - \frac{NE_{vac}}{2})} := \log \rho^*(h_L, h_R) \tag{41}$$

where $h_{L,R} = E_{L,R} - NE_{vac}/2$ is the energy above the vacuum.

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where $h_{L,R} = E_{L,R} - NE_{vac}/2$ is the energy above the vacuum. While for the high energy states, their density is approximated by the entropy

$$\log \rho \approx 2\pi \left[\sqrt{\frac{c}{6}E_L(1+\frac{2\mu}{N}E_R)} + \sqrt{\frac{c}{6}E_R(1+\frac{2\mu}{N}E_L)}\right] := S_N \tag{42}$$

which is valid for $\frac{E_L(\mu)E_R(\mu)}{1+\frac{2\mu}{N}(E_L(\mu)+E_R(\mu))} > \frac{N^2E_{vac}^2(\mu)}{4(1+2\mu E_{vac}(\mu))}$.

Recall that we have the following identity

$$\sum_{N=0}^{\infty} p^N Z_N = \exp\left(\sum_{n=1}^{\infty} \frac{p^n}{n}(T_n' Z)\right) \tag{43}$$

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Expanding the right hand side and comparing the coefficient of p^N, we get

$$\rho_N(h_L, h_R) \geq \sum_{n=1}^{N} \frac{1}{n} \rho_{T'_n}(h_L, h_R) \tag{44}$$

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for non vacuum states.Since $T_n' Z$ is also modular invariant, it features vacuum dominance and we have

$$\log \rho_{T_n'} \approx S_n(h_L, h_R) \tag{45}$$

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Interestingly, $S_n(h_L, h_R)$ reaches its maximal value when $n = n_*(h_L, h_R)$ such that the maximal value happens to be the upper bound on $\log \rho_N$. Then we have

$$\log \rho_N \geq \log \rho_{T'_{n_*}} = \log \rho^*(h_L, h_R) \quad (46)$$

The condition $n_*(h_L, h_R) \leq N$ is shown to be equivalent to the low energy regime which is complementary to the high energy regime where the entropy formula applies.

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$$\log \rho(E_L, E_R) \approx 4\pi \sqrt{(E_L - \frac{NE_{vac}}{2})(E_R - \frac{NE_{vac}}{2})} \qquad (47)$$

while for states with $\frac{E_L(\mu)E_R(\mu)}{1+\frac{2\mu}{N}(E_L(\mu)+E_R(\mu))} > \frac{N^2 E_{vac}^2(\mu)}{4(1+2\mu E_{vac}(\mu))}$, the state density is given by the entropy

$$\log \rho = 2\pi \left[\sqrt{\frac{c}{6} E_L (1 + \frac{2\mu}{N} E_R)} + \sqrt{\frac{c}{6} E_R (1 + \frac{2\mu}{N} E_L)} \right] \qquad (48)$$

- We study universal properties of the torus partition function of $T\bar{T}$-deformed CFTs under the assumption of modular invariance for both double-trace and single-trace versions.

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- In the double-trace case, we specify sparseness conditions for the light states for which the partition function is dominated by the vacuum. Using modular invariance, this implies a universal density of high energy states, in analogy with the behavior of holographic CFTs.
- For the single-trace case, we use the modular invariance to determined the partition function which matched holographic calculations in previous literature.
- The density of states in the single-trace deformed theory is universal when $c \sim N \to \infty$.

- Using modular covariance, the partition function for single-trace $J\bar{T}$-deformed partition function can be similarly determined which matches with the result in string theory.

- Using modular covariance, the partition function for single-trace $J\bar{T}$-deformed partition function can be similarly determined which matches with the result in string theory.
- String theory analysis implies that for the general single-trace $T\bar{T} + J\bar{T} + \bar{J}T$-deformed CFT, its partition function is defined in an analogous way to the $T\bar{T}$ case with the generalized Hecke operator given by

$$\begin{aligned} &(T_n' Z)(\tau, \bar{\tau}, \nu, \bar{\nu}; \mu_0, \mu_+, \mu_-) \\ = \;\; &\sum_{\gamma|n} \sum_{\alpha=0}^{\gamma-1} Z\left(\frac{n\tau+\alpha\gamma}{\gamma^2}, \frac{n\bar{\tau}+\alpha\gamma}{\gamma^2}, \frac{n\nu}{\gamma}, \frac{n\bar{\nu}}{\gamma}; \frac{\mu_0}{\gamma^2}, \frac{\mu_+}{\gamma}, \frac{\mu_-}{\gamma}\right) \end{aligned} \tag{49}$$

However, for such general case, the partition function has no good modular property. It would be interesting to derive this result in the field theory side.

Thanks for your attention!