Emergence and breakdown of semiclassical picture in quasiparticle states

Jiaju Zhang (张甲举)

Center for Joint Quantum Studies, Tianjin University
(天津大学量子交叉研究中心)

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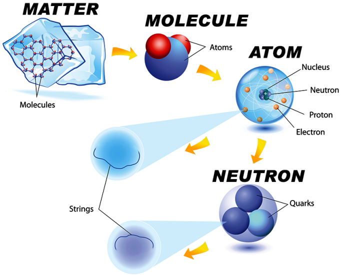
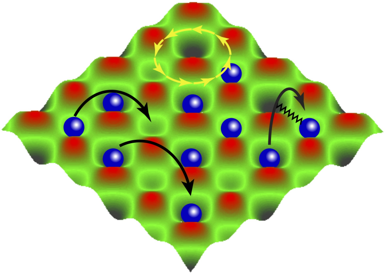
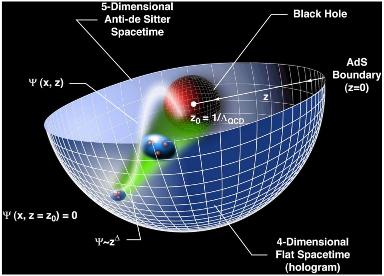
Outline

Background



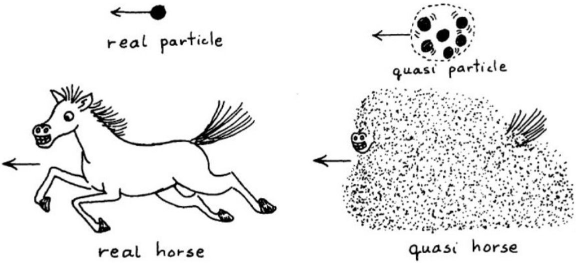
- Philosopher: what is the essence of the world?
- The world is complex and emergent (涌现或层展)!
- According to string theory, matter emerges from strings
- Classical turbulence emerges from Newton mechanics





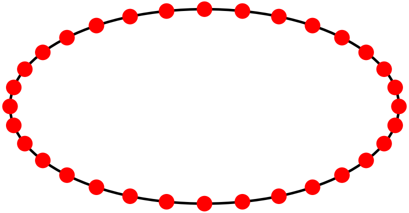
- Anderson: More is different
- Condensed matter physics emerges from quantum mechanics and classical electromagnetics
- AdS/CFT correspondence: bulk gravity emerges from a quantum field theory living in the boundary
- Wheeler: It from bit
- Modern perspective: It from qubit

Quasiparticle



- Quasiparticle: collective excitation of real particles in integrable many-body systems
- Examples: phonon (quantized sound wave in nuclear lattice), magnon (quantized spin wave in magnetic material), ...

Quasiparticles in quantum chains

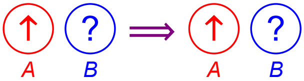


- Consider a circular quantum chain of interacting fermions, or bosons, or Pauli matrices
- Quasiparticles could be fermions, or bosons, or magnons
- Question: Under what conditions and for which quantities these quasiparticles behave like classical particles?

Quantum entanglement

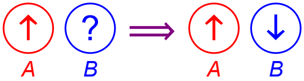
- Entanglement: key character of quantum systems
- Consider two spin-1/2 particles A and B
- Measurement of A does not affect $B \Rightarrow$ there is no entanglement

$$|\psi_{AB}\rangle = \frac{1}{2}(|\uparrow\rangle_A - |\downarrow\rangle_A) \otimes (|\uparrow\rangle_B - |\downarrow\rangle_B)$$



- Measurement of A affects $B \Rightarrow$ there exists entanglement

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle_{AB} - |\downarrow\uparrow\rangle_{AB})$$



Entanglement entropy

- Entanglement entropy: quantitative description of entanglement
- For A and its complement B in a system with a pure state $|\psi\rangle$
  - Reduced density matrix (RDM) $\rho_A = \mathrm{tr}_B(|\psi\rangle\langle\psi|)$
  - (von Neumann) entanglement entropy

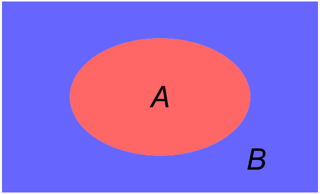
$$S_A = -\mathrm{tr}_A(\rho_A \log \rho_A)$$

  - Rényi (entanglement) entropy

$$S_A^{(n)} = -\frac{1}{n-1} \log \mathrm{tr}_A \rho_A^n$$

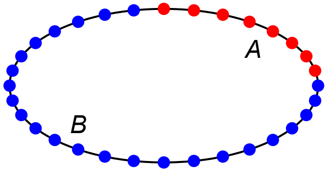
  - Replica trick

$$S_A = \lim_{n \to 1} S_A^{(n)}$$



Universal quasiparticle entanglement

Castro-Alvaredo-De Fazio-Doyon-Szécsényi 1805.04948, 1806.03247, 1904.01035, 1904.02615



- Subsystem A of ℓ consecutive sites on a circular chain of L sites in the scaling limit $L \to +\infty$, $\ell \to +\infty$ with fixed ratio $x = \frac{\ell}{L}$
- General quasiparticle excited state

$$|K\rangle = |k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s}\rangle$$

- The difference of the excited and ground state Rényi entropies

$$S_{A,K}^{(n)} - S_{A,G}^{(n)} = -\frac{1}{n-1} \log \mathcal{F}_{A,K}^{(n)}, \quad \mathcal{F}_{A,K}^{(n)} \equiv \frac{\mathrm{tr}_A \rho_{A,K}^n}{\mathrm{tr}_A \rho_{A,G}^n}$$

- Universal Rényi entropy that is independent of the model, quasiparticle momenta, and connectedness of the subsystem

$$\mathcal{F}^{(n),\mathrm{univ}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}} = \prod_{i=1}^{s} \Big\{ \sum_{p=0}^{r_i} [C_{r_i}^p x^p (1-x)^{r_i-p}]^n \Big\}$$

- If quasiparticles with different momenta are independent

$$\rho^{\mathrm{univ}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}} \cong \bigotimes_{i=1}^{s} \rho^{\mathrm{univ}}_{A,k_i^{r_i}} \;\Rightarrow\; \mathcal{F}^{(n),\mathrm{univ}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}} = \prod_{i=1}^{s} \mathcal{F}^{(n),\mathrm{univ}}_{A,k_i^{r_i}}$$

- It is expected to be valid under either large gap condition or large momentum condition

$$\min\Big(\frac{1}{\Delta}, \max_{k\in K}\frac{L}{|k|}\Big) \ll \min(\ell, L-\ell)$$

- $1/\Delta$ is the correlation length of the model
- $L/|k|$ is the de Broglie wavelength of the quasiparticle with momentum k

- The condition is violated for excitations around $k = \pm\frac{L}{4}$ in XX chain
- Change the de Broglie wavelength $L/|k|$ to the Compton wavelength $1/\varepsilon_k$

$$\min\left(\frac{1}{\Delta}, \max_{k\in K}\frac{1}{\varepsilon_k}\right) \ll \min(\ell, L-\ell)$$

- Here ε_k is the energy of the quasiparticle with momentum k
- Suppose that ε_k is a continuous function of k

$$\Delta = \min_{\text{all }k} \varepsilon_k$$

- Obtain large energy condition, i.e. that sizes of the quasiparticles are much smaller than subsystem sizes JZ-Rajabpour 2109.12826

$$\max_{k\in K}\frac{1}{\varepsilon_k} \ll \min(\ell, L-\ell)$$

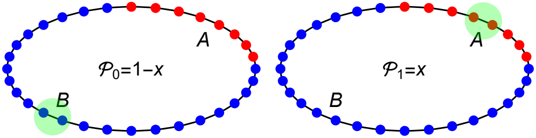
Semiclassical picture

- It could be understood from a semiclassical picture with quantum effect of distinguishability and indistinguishability
- In a single-particle state $|k\rangle$
  - ◇ The probability of find it in B is $\mathcal{P}_0 = 1 - x$
  - ◇ The probability of find it in A is $\mathcal{P}_1 = x$
  - ◇ The probability distribution $\{1 - x, x\}$
  - ◇ The effective RDM

$$\rho_{A,k}^{\text{univ}} = (1 - x)|0\rangle\langle 0| + x|k\rangle\langle k|$$

  - ◇ The universal Rényi entropy

$$\mathcal{F}_{A,k}^{(n),\text{univ}} = x^n + (1 - x)^n$$



- In the state $|k_1 k_2\rangle$ with two distinguishable particles
  - ◇ The probability of both in B is $\mathcal{P}_0 = (1-x)^2$
  - ◇ The probability of k_1 in A and k_2 in B is $\mathcal{P}_1 = x(1-x)$
  - ◇ The probability of k_2 in A and k_1 in B is $\mathcal{P}_2 = x(1-x)$
  - ◇ The probability of both in A is $\mathcal{P}_3 = x^2$
  - ◇ The probability distribution

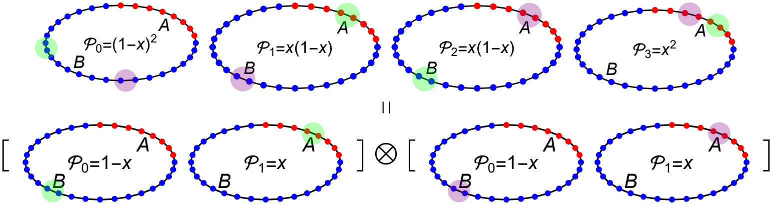
$$\{(1-x)^2, x(1-x), x(1-x), x^2\} = \{1-x, x\} \otimes \{1-x, x\}$$

  - ◇ The effective RDM

$$\begin{aligned}\rho^{\rm univ}_{A,k_1k_2} &= (1-x)^2|0\rangle\langle 0| + x(1-x)|k_1\rangle\langle k_1| + x(1-x)|k_2\rangle\langle k_2| + x^2|k_1k_2\rangle\langle k_1k_2| \\ &= [(1-x)|0\rangle\langle 0| + x|k_1\rangle\langle k_1|] \otimes [(1-x)|0\rangle\langle 0| + x|k_2\rangle\langle k_2|]\end{aligned}$$

  - ◇ The universal Rényi entropy

$$\mathcal{F}^{(n),\rm univ}_{A,k_1k_2} = [x^n + (1-x)^n]^2$$

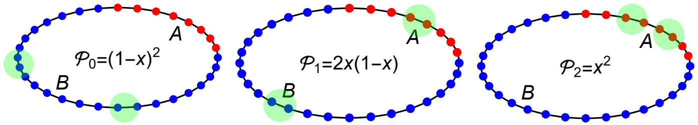


- In the state $|k^2\rangle$ with two identical particles
  - The probability of both in B is $\mathcal{P}_0 = (1-x)^2$
  - The probability of one in A and another in B is $\mathcal{P}_1 = 2x(1-x)$
  - The probability of both in A is $\mathcal{P}_2 = x^2$
  - The probability distribution $\{(1-x)^2, 2x(1-x), x^2\}$
  - The effective RDM

$$\rho^{\mathrm{univ}}_{A,k^2} = (1-x)^2|0\rangle\langle 0| + 2x(1-x)|k\rangle\langle k| + x^2|k^2\rangle\langle k^2|$$

  - The universal Rényi entropy

$$\mathcal{F}^{(n),\mathrm{univ}}_{A,k^2} = (1-x)^{2n} + [2x(1-x)]^n + x^{2n}$$



- In the state $|k^r\rangle$ with r identical particles
  - ◇ The probability of finding $p = 0, 1, \cdots, r$ quasiparticles in A

$$\mathcal{P}_p = C_r^p x^p (1-x)^{r-p}$$

  - ◇ The effective RDM

$$\rho_{A,k^r}^{\mathrm{univ}} = \sum_{p=0}^{r} C_r^p x^p (1-x)^{r-p} |k^p\rangle\langle k^p|$$

  - ◇ The universal Rényi entropy

$$\mathcal{F}_{A,k^r}^{(n),\mathrm{univ}} = \sum_{p=0}^{r} [C_r^p x^p (1-x)^{r-p}]^n$$

- There exist additional contributions to the universal Rényi entropy
JZ-Rajabpour 2010.13973, 2010.16348, 2109.12826

$$\mathcal{F}^{(n)}_{A,k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s}} = \mathcal{F}^{(n),\mathrm{univ}}_{A,k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s}} + \delta\mathcal{F}^{(n)}_{A,k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s}}$$

Fermionic chain

- Circular chain of spinless fermions with even integer L

$$H=\sum_{j=1}^{L}\Big[\lambda\Big(a_j^\dagger a_j-\frac{1}{2}\Big)-\frac{1}{2}(a_j^\dagger a_{j+1}+a_{j+1}^\dagger a_j)-\frac{\gamma}{2}(a_j^\dagger a_{j+1}^\dagger+a_{j+1}a_j)\Big]$$

- Related to XY chain by Jordan-Wigner transformation
- Fourier transformation with $\varphi_k=\frac{2\pi k}{L}$

$$b_k=\frac{1}{\sqrt{L}}\sum_{j=1}^{L}\mathrm{e}^{-\mathrm{i}j\varphi_k}a_j,\ \ b_k^\dagger=\frac{1}{\sqrt{L}}\sum_{j=1}^{L}\mathrm{e}^{\mathrm{i}j\varphi_k}a_j^\dagger,$$

- Bogoliubov transformation with $\mathrm{e}^{\mathrm{i}\theta_k}=\frac{\lambda-\cos\varphi_k+\mathrm{i}\gamma\sin\varphi_k}{\varepsilon_k}$

$$c_k=b_k\cos\frac{\theta_k}{2}+\mathrm{i}b_{-k}^\dagger\sin\frac{\theta_k}{2},\ \ c_k^\dagger=b_k^\dagger\cos\frac{\theta_k}{2}-\mathrm{i}b_{-k}\sin\frac{\theta_k}{2}$$

- Diagonal form with $\varepsilon_k=[(\lambda-\cos\varphi_k)^2+\gamma^2\sin^2\varphi_k]^{1/2}$

$$H=\sum_k\varepsilon_k\Big(c_k^\dagger c_k-\frac{1}{2}\Big)$$

- Consider the Neveu-Schwarz sector

$$k = -\frac{L-1}{2}, \cdots, -\frac{1}{2}, \frac{1}{2}, \cdots, \frac{L-1}{2}$$

- Ground state $|G\rangle$ defined as $c_k|G\rangle = 0, \ \forall k$
- General excited state with $|K| = s$ quasiparticles

$$|K\rangle = |k_1 k_2 \cdots k_s\rangle = c^\dagger_{k_1} c^\dagger_{k_2} \cdots c^\dagger_{k_s} |G\rangle$$

- Universal Rényi entropy Castro-Alvaredo-De Fazio-Doyon-Szécsényi 1805.04948, 1806.03247, 1904.01035, 1904.02615

$$\mathcal{F}^{(n),\mathrm{univ}}_{A,k_1 k_2 \cdots k_s} = [x^n + (1-x)^n]^s$$

- Universal entanglement entropy

$$\delta S^{\mathrm{univ}}_{A,k_1 k_2 \cdots k_s} = s[-x \log x - (1-x)\log(1-x)]$$

- We find additional contributions to the universal Rényi entropy JZ-Rajabpour 2010.13973, 2010.16348, 2109.12826

$$\mathcal{F}^{(n),\mathrm{fer}}_{A,k_1 k_2 \cdots k_s} = \mathcal{F}^{(n),\mathrm{univ}}_{A,k_1 k_2 \cdots k_s} + \delta\mathcal{F}^{(n),\mathrm{fer}}_{A,k_1 k_2 \cdots k_s}$$

Analytical calculation - subsystem mode method

- In extremely gapped limit $\lambda \to +\infty$, the model is totally free

$$H = \lambda \sum_{j=1}^{L} \left(a_j^\dagger a_j - \frac{1}{2} \right)$$

- The quasiparticle modes $c_k = b_k$, $c_k^\dagger = b_k^\dagger$
- The ground state $|G\rangle = |G_A\rangle |G_B\rangle$ defined by

$$c_k |G\rangle = a_j |G\rangle = 0 \text{ for all } k, j$$

- Vanishing ground state Rényi entropy $S_{A,G}^{(n)} = 0$
- We define subsystem modes as $c_k = c_{A,k} + c_{B,k}$, $c_k^\dagger = c_{A,k}^\dagger + c_{B,k}^\dagger$

$$c_{A,k} = \frac{1}{\sqrt{L}} \sum_{j \in A} \mathrm{e}^{-\mathrm{i} j \varphi_k} a_j, \quad c_{A,k}^\dagger = \frac{1}{\sqrt{L}} \sum_{j \in A} \mathrm{e}^{\mathrm{i} j \varphi_k} a_j^\dagger$$

$$c_{B,k} = \frac{1}{\sqrt{L}} \sum_{j \in B} \mathrm{e}^{-\mathrm{i} j \varphi_k} a_j, \quad c_{B,k}^\dagger = \frac{1}{\sqrt{L}} \sum_{j \in B} \mathrm{e}^{\mathrm{i} j \varphi_k} a_j^\dagger$$

- Anti-commutation relations

$$\{c_{A,k}, c_{A,k}^\dagger\} = x, \ \{c_{B,k}, c_{B,k}^\dagger\} = 1 - x$$

- For $k_1 \neq k_2$

$$\{c_{A,k_1}, c_{A,k_2}^\dagger\} = -\{c_{B,k_1}, c_{B,k_2}^\dagger\} = \alpha_{k_1-k_2}, \ \alpha_k \equiv \frac{1}{L}\sum_{j\in A} \mathrm{e}^{-\frac{2\pi \mathrm{i} jk}{L}}$$

- Single-particle state $|k\rangle$ density matrix and RDM

$$\rho_k = c_k^\dagger|G\rangle\langle G|c_k = (c_{A,k}^\dagger + c_{B,k}^\dagger)|G\rangle\langle G|(c_{A,k} + c_{B,k})$$

$$\rho_{A,k} = c_{A,k}^\dagger|G_A\rangle\langle G_A|c_{A,k} + \langle c_{B,k}c_{B,k}^\dagger\rangle_G|G_A\rangle\langle G_A|$$

- Trace of RDM product

$$\mathrm{tr}_A\rho_{A,k}^n = \langle c_{A,k}c_{A,k}^\dagger\rangle_G^n + \langle c_{B,k}c_{B,k}^\dagger\rangle_G^n = x^n + (1-x)^n$$

- No additional contributions to single-particle universal Rényi entropy

$$\mathcal{F}_{A,k}^{(n),\mathrm{fer}} = \mathcal{F}_{A,k}^{(n),\mathrm{univ}} = x^n + (1-x)^n$$

- Double-particle state $|k_1k_2\rangle$ density matrix and RDM $\alpha \equiv \alpha_{k_1-k_2}$

$$\rho_{k_1k_2} = (c^\dagger_{A,k_1} + c^\dagger_{B,k_1})(c^\dagger_{A,k_2} + c^\dagger_{B,k_2})|G\rangle\langle G|(c_{A,k_2} + c_{B,k_2})(c_{A,k_1} + c_{B,k_1})$$

$$\begin{aligned}\rho_{A,k_1k_2} = {} & [(1-x)^2 - |\alpha|^2]|G_A\rangle\langle G_A| + (1-x)c^\dagger_{A,k_1}|G_A\rangle\langle G_A|c_{A,k_1} \\ & + \alpha c^\dagger_{A,k_1}|G_A\rangle\langle G_A|c_{A,k_2} + \bar{\alpha} c^\dagger_{A,k_2}|G_A\rangle\langle G_A|c_{A,k_1} \\ & + (1-x)c^\dagger_{A,k_2}|G_A\rangle\langle G_A|c_{A,k_2} + c^\dagger_{A,k_1}c^\dagger_{A,k_2}|G_A\rangle\langle G_A|c_{A,k_2}c_{A,k_1}\end{aligned}$$

- Orthonormal basis

$$\begin{aligned}|\psi_0\rangle &= |G_A\rangle \\ |\psi_1\rangle &= \frac{1}{\sqrt{2|\alpha|(x+|\alpha|)}}(\sqrt{\alpha}c^\dagger_{A,k_1} + \sqrt{\bar{\alpha}}c^\dagger_{A,k_2})|G_A\rangle \\ |\psi_2\rangle &= \frac{1}{\sqrt{2|\alpha|(x-|\alpha|)}}(\sqrt{\alpha}c^\dagger_{A,k_1} - \sqrt{\bar{\alpha}}c^\dagger_{A,k_2})|G_A\rangle \\ |\psi_3\rangle &= \frac{1}{\sqrt{x^2-|\alpha|^2}}c^\dagger_{A,k_1}c^\dagger_{A,k_2}|G_A\rangle\end{aligned}$$

- We have the RDM

$$\rho_{A,k_1k_2} = \mathrm{diag}[(1-x)^2-|\alpha|^2, (x+|\alpha|)(1-x+|\alpha|), (x-|\alpha|)(1-x-|\alpha|), x^2-|\alpha|^2]$$

- Universal Rényi entropy with additional contributions

$$\mathcal{F}^{(n),\mathrm{fer}}_{A,k_1k_2} = [(x+|\alpha|)^n + (1-x-|\alpha|)^n][(x-|\alpha|)^n + (1-x+|\alpha|)^n]$$

$$\mathcal{F}^{(2),\mathrm{fer}}_{A,k_1k_2} = [x^2 + (1-x)^2]^2 + 8x(1-x)|\alpha|^2 + 4|\alpha|^4$$

$$\mathcal{F}^{(3),\mathrm{fer}}_{A,k_1k_2} = [x^3 + (1-x)^3]^2 - 3(1-6x+6x^2)|\alpha|^2 + 9|\alpha|^4$$

$$\cdots$$

- It is similar for states with three and more quasiparticles
- All additional terms have the factor $\alpha = \alpha_{k_1-k_2}$
- When the additional terms could be omitted?

- For a single interval $A = [1, \ell]$

$$\alpha_k = \mathrm{e}^{-\frac{\pi \mathrm{i}(\ell+1)k}{L}} \frac{\sin \frac{\pi \ell k}{L}}{L \sin \frac{\pi k}{L}}$$

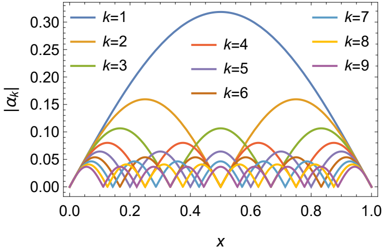
- In the scaling limit $L \to +\infty$ and $\ell \to +\infty$ with fixed $x = \frac{\ell}{L}$
- For finite $|k|$

$$|\alpha_k| \to \left| \frac{\sin(\pi k x)}{\pi k} \right|$$

- For $|k| \to +\infty$

$$|\alpha_k| \to 0$$

- Essential additional contributions for small momentum difference
- Negligible under large momentum difference condition



Numerical calculation - correlation matrix method

- The $\ell \times \ell$ correlation matrix $C_{A,K}$ in state $|K\rangle$

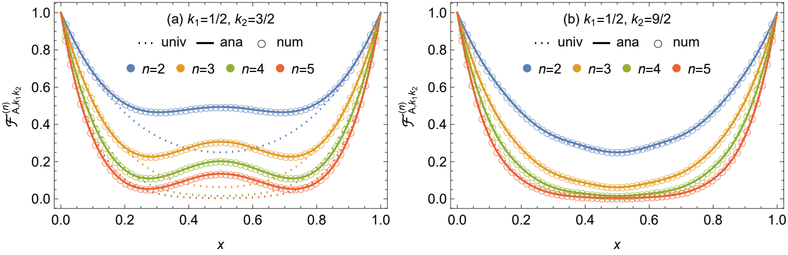
$$[C_{A,K}]_{j_1 j_2} = \langle a^\dagger_{j_1} a_{j_2} \rangle_K = h^K_{j_2-j_1},\ j_1, j_2 \in A,\ \ h^K_j = \frac{1}{L} \sum_{k \in K} \mathrm{e}^{-\mathrm{i} j \varphi_k}$$

- Rényi entropy and entanglement entropy Vidal-Latorre-Rico-Kitaev 0211074, Peschel 0212631, Alba-Fagotti-Calabrese 0909.1999, Alcaraz-Berganza-Sierra 1101.2881, ...

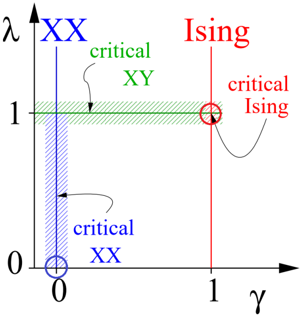
$$\mathcal{F}^{(n),\mathrm{fer}}_{A,K} = \det[(C_{A,K})^n + (1 - C_{A,K})^n]$$

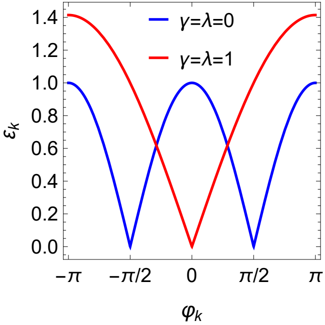
$$S^{\mathrm{fer}}_{A,K} = \mathrm{tr}_A[-C_{A,K} \log C_{A,K} - (1 - C_{A,K}) \log(1 - C_{A,K})]$$

- Single-interval Rényi entropy in double-particle state



Slightly gapped and critical chains

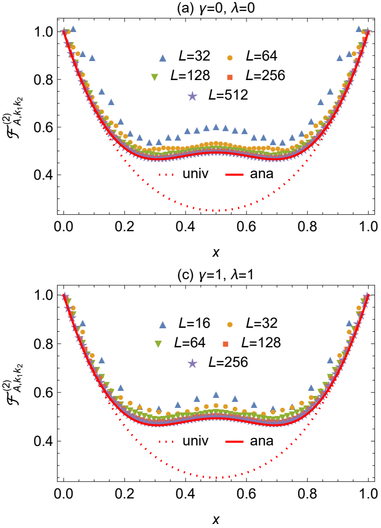




- Phase diagram of the fermionic chain (XY chain)
- Dispersion relation

$$\varepsilon_k = \sqrt{(\lambda - \cos\varphi_k)^2 + \gamma^2 \sin^2\varphi_k}$$

- The corrected Rényi entropy is still valid in slightly gapped and critical chains under the large energy condition
- For single interval in double-particle state with $(k_1, k_2) = (\frac{1}{2}, \frac{3}{2}) + \frac{L}{8}$



Bosonic chain

- Bosonic chain with local interactions (discrete Klein-Gordan theory)

$$H = \frac{1}{2}\sum_{j=1}^{L}\left[p_j^2 + m^2 q_j^2 + (q_j - q_{j+1})^2\right]$$

- Fourier transformation

$$q_j = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{\frac{2\pi \mathrm{i} jk}{L}}\varphi_k, \;\; p_j = \frac{1}{\sqrt{L}}\sum_k \mathrm{e}^{\frac{2\pi \mathrm{i} jk}{L}}\pi_k,$$

- Bosonic ladder operators with $\varepsilon_k = (m^2 + 4\sin^2\frac{\pi k}{L})^{1/2}$

$$c_k = \sqrt{\frac{\varepsilon_k}{2}}\Big(\varphi_k + \frac{\mathrm{i}}{\varepsilon_k}\pi_k\Big), \;\; c_k^\dagger = \sqrt{\frac{\varepsilon_k}{2}}\Big(\varphi_k^\dagger - \frac{\mathrm{i}}{\varepsilon_k}\pi_k^\dagger\Big)$$

- Diagonal form

$$H = \sum_k \varepsilon_k\Big(c_k^\dagger c_k + \frac{1}{2}\Big)$$

- Even integer L and periodic boundary conditions

$$k = 1 - \frac{L}{2}, \cdots, -1, 0, 1, \cdots, \frac{L}{2} - 1, \frac{L}{2}$$

- Ground state $|G\rangle$ defined as $c_k|G\rangle = 0, \ \forall k$
- General excited state with $|K| = \sum_{i=1}^{s} r_i$ quasiparticles

$$|k_1^{r_1} k_2^{r_2} \cdots k_s^{r_s}\rangle = \frac{(c_{k_1}^\dagger)^{r_1}(c_{k_2}^\dagger)^{r_2} \cdots (c_{k_s}^\dagger)^{r_s}}{\sqrt{r_1! r_2! \cdots r_s!}}|G\rangle$$

- Universal Rényi entropy Castro-Alvaredo-De Fazio-Doyon-Szécsényi 1805.04948, 1806.03247, 1904.01035, 1904.02615

$$\mathcal{F}^{(n),\mathrm{univ}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}} = \prod_{i=1}^{s}\Big\{\sum_{p=0}^{r_i}[C_{r_i}^p x^p (1-x)^{r_i-p}]^n\Big\}$$

- Corrected universal Rényi entropy JZ-Rajabpour 2010.13973, 2010.16348, 2109.12826

$$\mathcal{F}^{(n),\mathrm{bos}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}} = \mathcal{F}^{(n),\mathrm{univ}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}} + \delta\mathcal{F}^{(n),\mathrm{bos}}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}}$$

Analytical calculation - subsystem mode method

- In the extremely gapped limit $m \to +\infty$

$$c_k = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} \mathrm{e}^{-\mathrm{i}j\varphi_k} a_j, \;\; c_k^\dagger = \frac{1}{\sqrt{L}} \sum_{j=1}^{L} \mathrm{e}^{\mathrm{i}j\varphi_k} a_j^\dagger$$

with local modes

$$a_j = \sqrt{\frac{m}{2}} \Big(q_j + \frac{\mathrm{i}}{m} p_j \Big), \;\; a_j^\dagger = \sqrt{\frac{m}{2}} \Big(q_j - \frac{\mathrm{i}}{m} p_j \Big)$$

- Calculations are similar to those in the fermionic chain: anti-commutation relations $\Rightarrow$ commutation relations
- Rényi entropy in double-particle state $|k_1 k_2\rangle$

$$\mathcal{F}^{(n),\mathrm{bos}}_{A,k_1k_2} = (x^2+|\alpha|^2)^n+[(1-x)^2+|\alpha|^2]^n+(x+|\alpha|)^n(1-x-|\alpha|)^n+(x-|\alpha|)^n(1-x+|\alpha|)^n$$

$$\mathcal{F}^{(2),\mathrm{bos}}_{A,k_1k_2} = [x^2+(1-x)^2]^2+4(1-2x)^2|\alpha|^2+4|\alpha|^4$$

$$\mathcal{F}^{(3),\mathrm{bos}}_{A,k_1k_2} = [x^3+(1-x)^3]^2+3(1-2x)^2(1+2x-2x^2)|\alpha|^2-3(1-8x+8x^2)|\alpha|^4$$

$$\cdots$$

Numerical calculation - wavefunction method

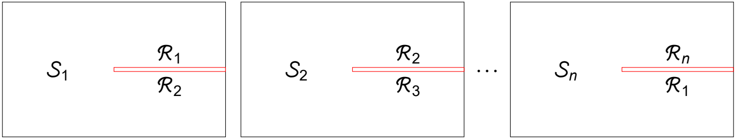
Castro-Alvaredo-De Fazio-Doyon-Szécsényi 1805.04948, 1806.03247

- Wavefunction in coordinate basis $\langle \mathcal{Q}|K\rangle$ with $\mathcal{Q}=\{q_1,q_2,\cdots,q_L\}$
- Decompose $\mathcal{Q}=(\mathcal{R},\mathcal{S})$ with $\mathcal{R}\in A$ and $\mathcal{S}\in B$
- Trace of RDM product

$$\mathrm{tr}_A\rho_A^n=\int \mathrm{D}^n\mathcal{R}\mathrm{D}^n\mathcal{S}\langle\mathcal{R}_1,\mathcal{S}_1|\rho|\mathcal{R}_2,\mathcal{S}_1\rangle\langle\mathcal{R}_2,\mathcal{S}_2|\rho|\mathcal{R}_3,\mathcal{S}_2\rangle\cdots\langle\mathcal{R}_n,\mathcal{S}_n|\rho|\mathcal{R}_1,\mathcal{S}_n\rangle$$

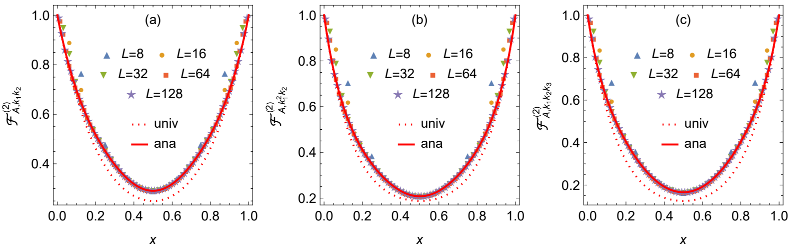
- Numerical calculation for a finite gap
- A permanent (积合式) formula in the extremely gapped limit
- The $n|K|\times n|K|$ matrix $\Omega^{(n)}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}}$ JZ-Rajabpour 2010.16348

$$\mathcal{F}^{(n)}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}}=\frac{\mathrm{per}\,\Omega^{(n)}_{A,k_1^{r_1}k_2^{r_2}\cdots k_s^{r_s}}}{\prod_{i=1}^s(r_i!)^n}$$



Slightly gapped chain

- The corrected Rényi entropy is still valid in slightly gapped chains under the large energy condition
- For single interval in double-particle $|k_1 k_2\rangle$ and triple-particle states $|k_1^2 k_2\rangle$ and $|k_1 k_2 k_3\rangle$ with $(k_1, k_2, k_3) = (1, 2, 3) + \frac{L}{8}$, $m = 10^{-4}$



XXX chain

- Spin-1/2 XXX chain with transverse field $h > 0$

$$H = -\frac{1}{4}\sum_{j=1}^{L}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z) - \frac{h}{2}\sum_{j=1}^{L}\sigma_j^z$$

- It is a real interacting model!
- Unique ferromagnetic ground state for $h > 0$

$$|G\rangle = |\uparrow\uparrow\cdots\uparrow\rangle$$

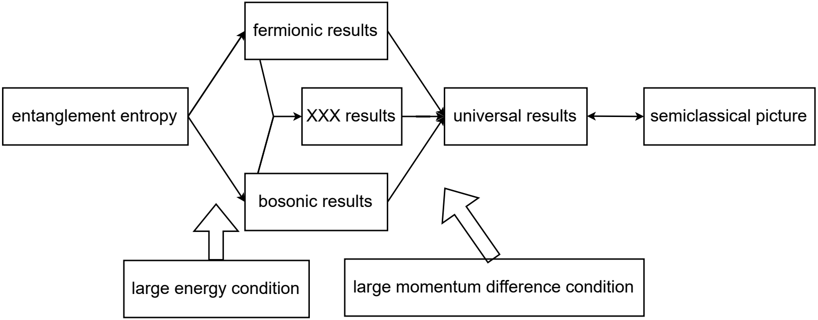
- Low-lying excited states are magnons
- Eigenstates from the coordinate Bethe ansatz
- In the scaling limit, the entanglement entropy in the XXX chain is a proper combination of certain fermionic and bosonic results

  JZ-Rajabpour 2109.12826

$$\lim_{L\to+\infty} S_{A,K}^{\mathrm{XXX}} = \sum_{K'} S_{A,K'}^{\mathrm{fer}} + \sum_{K''} S_{A,K''}^{\mathrm{bos}}$$

Summary for entanglement

- Different fermionic and bosonic results under large energy condition
- Results in XXX chain written as combination of fermionic and bosonic results
- Universal entanglement when both large energy condition and large momentum difference condition are satisfied
- Semiclassical picture for the Universal entanglement



Subsystem distance

- It is important to distinguish quantitatively the RDMs of a subsystem: distance in the space of RDMs
- Mathematically, a distance should: Nielsen-Chuang 2000
  - be nonnegative $d(\rho,\sigma) \geq 0$
  - be equal to zero if and only if its two inputs are exactly the same $d(\rho,\sigma) = 0 \Leftrightarrow \rho = \sigma$
  - be symmetric in its inputs $d(\rho,\sigma) = d(\sigma,\rho)$
  - obeys the triangular inequality $d(\rho,\sigma) + d(\sigma,\lambda) \geq d(\rho,\lambda)$
- An infinite number of definitions n-distance with $n \geq 1$ Watrous 2018

$$D_n(\rho,\sigma) = \frac{1}{2^{1/n}} \|\rho - \sigma\|_n$$

- n-norm $\|\Lambda\|_n = (\sum_i \lambda_i^n)^{1/n}$ with λ_i being the singular values of Λ
- D_1 trace distance, D_2 Frobenius distance or Hilbert-Schmidt distance, D_∞ operator distance or spectral distance

- Trace distance is special $D_1(\rho,\sigma)=\frac{1}{2}\mathrm{tr}|\rho-\sigma|$
  JZ-Calabrese-Dalmonte-Rajabpour 2003.00315
- It provides an upper bound for the difference between expectation values of observables in different states

$$|\mathrm{tr}[(\rho-\sigma)\mathcal{O}]|\leq 2D_1(\rho,\sigma)s_{\mathcal{O}}$$

  where $s_{\mathcal{O}}$ is the largest singular value of $\mathcal{O}$
- RDMs are Hermitian: $\|\rho-\sigma\|_{n_e}^{n_e}=\mathrm{tr}(\rho-\sigma)^{n_e}$ for an even integer n_e
- Replica trick for trace distance JZ-Ruggiero-Calabrese 1901.10993, 1907.04332
  - ◇ Calculate n_e-distance D_{n_e} for an arbitrary even integer n_e
  - ◇ Take $n_e\to 1$ limit to get the trace distance
- It is UV cutoff-independent and scale invariant in CFT
- Average subsystem trace distance as indicator of quantum many-body integrability and chaos Khasseh-JZ-Heyl-Rajabpour 2301.13218

Subsystem distance between quasiparticle states

JZ-Rajabpour 2202.11448

- The same picture as that of entanglement entropy
- Different fermionic and bosonic results under large energy condition

$$D_1^{\mathrm{fer}}(\rho_{A,G},\rho_{A,k_1k_2}) = x(2-x)+|\alpha_{k_1-k_2}|^2$$

$$D_1^{\mathrm{bos}}(\rho_{A,G},\rho_{A,k_1k_2}) = x(2-x)-|\alpha_{k_1-k_2}|^2$$

- XXX results written in terms of fermionic and bosonic results
- Universal entanglement when both large energy condition and large momentum difference condition are satisfied

$$D_1^{\mathrm{univ}}(\rho_{A,G},\rho_{A,k_1k_2}) = x(2-x)$$

- Semiclassical picture for the universal subsystem distance

Shannon entropy

- Shannon entropy for any probability distribution $\{p_i\}$

$$H = -\sum_i p_i \log p_i$$

- It measures the randomness or uncertainty of the probability distribution
- Entropy for tossing of a fair coin: $\{\frac{1}{2}, \frac{1}{2}\} \Rightarrow H = \log 2$
- Entropy for rolling of a fair dice: $\{\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\} \Rightarrow H = \log 6$

Shannon entropy in quasiparticle states Ye-JZ 2303.14132

- A pure state $|\psi\rangle = \sum_i c_i |i\rangle$ or a mixed state $\rho = \sum_{i,j} \rho_{ij} |i\rangle\langle j|$ in any orthonormal basis $\{|i\rangle\}$
- Probability distribution of measuring results $\{p_i = |c_i|^2 = |\langle i|\psi\rangle|^2\}$ or $\{p_i = \rho_{ii} = \langle i|\rho|i\rangle\}$
- Shannon entropy for both the total system $H(L)$ and subsystem $H(\ell)$
- It depends on the chosen measuring basis!
- Choose the basis of product of local quasiparticle number eigenstates
- Consider the double-particle state $|k_1 k_2\rangle$ with $k_{12} = k_1 - k_2 \neq 0$
- Total system Shannon entropy in limit $|k_{12}| \ll L$ of free fermionic and bosonic chains

$$H^{\rm fer}_{k_1 k_2}(L) = H^{\rm bos}_{k_1 k_2}(L) = H^{\rm univ}_{k_1 k_2}(L) = 2 \log L - 1$$

- Total system Shannon entropy for classical particles

$$H^{\rm cl}_{12}(L) = 2 \log L$$

- Subsystem Shannon entropy in limit $|k_{12}| \ll L$

$$H^{\rm fer}_{k_1k_2}(\ell) = 2x\log L - 2x\log 2 - \Big[(1-x)^2 - \frac{\sin^2(\pi k_{12}x)}{\pi^2 k_{12}^2}\Big]\log\Big[(1-x)^2 - \frac{\sin^2(\pi k_{12}x)}{\pi^2 k_{12}^2}\Big]$$
$$- 4\int_0^{x/2}\mathrm{d}y\Big[(1-x) + \frac{\sin(\pi k_{12}x)\cos(2\pi k_{12}y)}{\pi k_{12}}\Big]\log\Big[(1-x) + \frac{\sin(\pi k_{12}x)\cos(2\pi k_{12}y)}{\pi k_{12}}\Big]$$
$$- 4\int_0^{k_{12}x}\frac{\mathrm{d}z}{k_{12}}\Big(x - \frac{z}{k_{12}}\Big)\sin^2(\pi z)\log\sin^2(\pi z)$$

$$H^{\rm bos}_{k_1k_2}(\ell) = 2x\log L - 2x\log 2 - \Big[(1-x)^2 + \frac{\sin^2(\pi k_{12}x)}{\pi^2 k_{12}^2}\Big]\log\Big[(1-x)^2 + \frac{\sin^2(\pi k_{12}x)}{\pi^2 k_{12}^2}\Big]$$
$$- 4\int_0^{x/2}\mathrm{d}y\Big[(1-x) - \frac{\sin(\pi k_{12}x)\cos(2\pi k_{12}y)}{\pi k_{12}}\Big]\log\Big[(1-x) - \frac{\sin(\pi k_{12}x)\cos(2\pi k_{12}y)}{\pi k_{12}}\Big]$$
$$- 4\int_0^{k_{12}x}\frac{\mathrm{d}z}{k_{12}}\Big(x - \frac{z}{k_{12}}\Big)\cos^2(\pi z)\log\cos^2(\pi z)$$

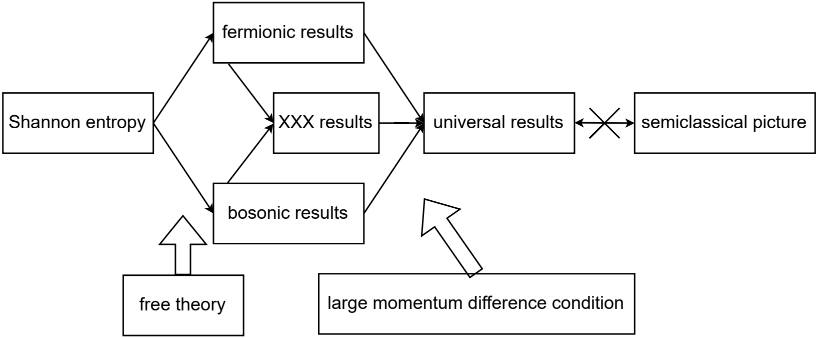
- Universal subsystem Shannon entropy in limit $1 \ll |k_{12}| \ll L$

$$H^{\rm univ}_{k_1k_2}(\ell) = 2x\log L - 2(1-x)\log(1-x) - x^2 - 2x(1-x)\log 2$$

- Subsystem Shannon entropy for classical particles

$$H^{\rm cl}_{12}(\ell) = 2x\log L - 2(1-x)\log(1-x)$$

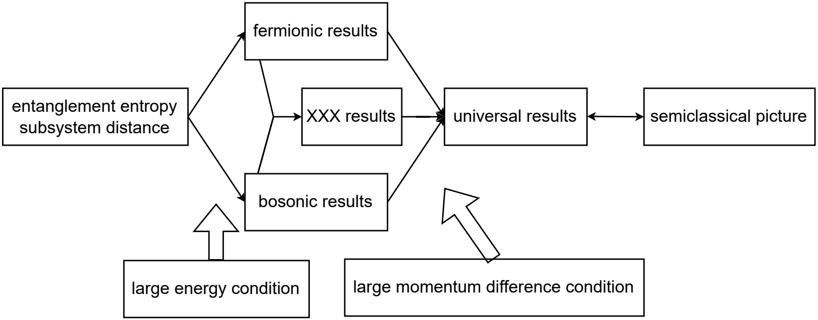
- The fermionic, bosonic and universal results $H^{\rm fer}_{k_1k_2}(\ell)$, $H^{\rm bos}_{k_1k_2}(\ell)$, $H^{\rm univ}_{k_1k_2}(\ell)$ apply to XXX chain under respective proper limits



Conclusion

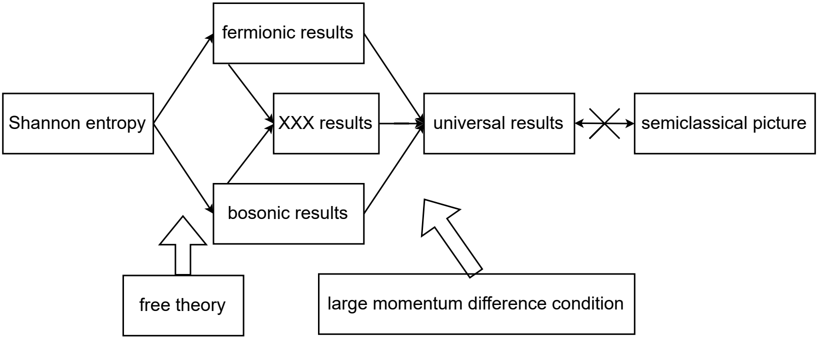
For entanglement entropy and subsystem distance

- Different fermionic and bosonic results under large energy condition
- Universal results under further large momentum difference condition
- Proper combinations of fermionic, bosonic and universal results apply to XXX chain under respective certain circumstances
- Semiclassical quasiparticle picture



For Shannon entropy

- Different results in free fermionic and bosonic chains
- Universal results under further large momentum difference condition
- Fermionic, bosonic and universal results apply to XXX chain under respective certain limits
- No semiclassical quasiparticle picture



Summary and outlook

Summary

- Scrutinizing carefully enough, one sees the difference between quasiparticles and classical particles!

Outlook

- Spinon excitations in antiferromagnetic phase of XXX chain?
- Anyonic excitations?
- Shannon entropy in slightly gapped and gapless systems?
- Semiclassical quasiparticle picture after quantum quench?
- Disjoint intervals: mutual information, entanglement negativity, reflected entropy...
- ...

Thanks for your attention!