WKB analysis of the linear problem for modified affine Toda field equations

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Contents

Contents

Introduction

An integrable model is a Hamiltonian system with the equality between the number of degrees of freedom and the number of integrals of motion.

- Affine Toda field theory is an integrable field theory with infinite conserved quantities.
- Affine Toda field equations (EoMs) are non-linear differential equations that can be separated into two linear problems.
- The linear problems in matrix form can be converted into higher-order (pseudo) ordinary differential equations.
- It is possible to diagonalize the linear problem, where the diagonal elements turn to be classical conserved densities.

Introduction

The classical conserved densities for the $A_1^{(1)}$ Toda field theory

$$I_2(z) = \frac{T(z)}{2},$$
$$I_4(z) = \frac{\partial_z^2 T(z) - T^2(z)}{8},$$
$$I_6(z) = \frac{1}{32}\left(-5T'(z)^2 - 6T(z)u''(z) + T^{(4)}(z) + 2T(z)^3\right),$$

The WKB solutions for the $A_1^{(1)}$-type ordinary differential equation $\left(\epsilon^2\partial_z^2 + \epsilon^2 u_2(z) - p(z)\right)\psi(z,\epsilon) = 0$ with $\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon))$

$$P_0(z) = \sqrt{p(z)},$$
$$P_1(z) = -\frac{1}{2}\partial_z \ln P_0,$$
$$P_2(z) = \frac{P_0''}{16P_0^2} + \frac{u_2(z)}{2P_0} + \partial_z(\frac{3P_0'}{16P_0^2}),$$
$$P_3(z) = -\partial_z(-\frac{u_2(z)}{4P_0^2} + \frac{3P_0'^2}{16P_0^4} - \frac{P_0''}{8P_0^3}),$$

Contents

Affine Toda field equations

The action of $\hat{\mathfrak{g}}$ affine Toda field theory in $2d$ complex plane:

$$S = \int d^2z \Big\{ \frac{1}{2} \partial_z \phi \cdot \bar{\partial}_{\bar{z}} \phi + \Big(\frac{m^2}{\beta} \Big) \Big[\sum_{i=1}^{r} \exp\big(\beta \alpha_i \cdot \phi \big) + \exp\big(\beta \alpha_0 \cdot \phi \big) \Big] \Big\}. \quad (1)$$

Its equation of motion: the $\hat{\mathfrak{g}}$ affine Toda field equation is

$$\bar{\partial}_{\bar{z}} \partial_z \phi(z, \bar{z}) - \Big(\frac{m^2}{\beta} \Big) \Big[\sum_{i=1}^{r} \alpha_i \exp\big(\beta \alpha_i \cdot \phi \big) + \alpha_0 \exp\big(\beta \alpha_0 \cdot \phi \big) \Big] = 0. \quad (2)$$

$$\phi(z, \bar{z}) = \sum_{i=1}^{r} \alpha_i^{\vee} \phi_i(z, \bar{z}),$$

$\alpha_i(\alpha_i^{\vee})$: roots(coroots) of $\hat{\mathfrak{g}}$,

β : a coupling constant,

m : a mass parameter.

Affine Toda field equations

The affine Toda field equations can be separated into two Lax operators:

$$\begin{aligned}\mathcal{L} &= \partial_z + \beta \sum_{i=1}^{r} \partial_z \phi_i(z,\bar{z}) H_i + m\lambda\Lambda,\\ \bar{\mathcal{L}} &= \partial_{\bar{z}} + e^{-\beta \sum_{i=1}^{r} \phi_i H_i} (m\lambda^{-1}\bar{\Lambda})\, e^{\beta \sum_{i=1}^{r} \phi_i H_i}.\end{aligned} \tag{3}$$

λ: a spectral parameter,
$\Lambda = \sum_{i=0}^{r} E_{\alpha_i}$ and $\bar{\Lambda} = \sum_{i=0}^{r} E_{-\alpha_i}$
E_{α_i}, $E_{-\alpha_i}$, $H_i = \alpha_i^{\vee} \cdot H$ $(i = 0, \ldots, r)$: the Chevalley generators

The flatness condition

$$[\mathcal{L}, \bar{\mathcal{L}}] = 0 \tag{4}$$

is the integrability condition of the linear problem

$$\mathcal{L}\Psi = \bar{\mathcal{L}}\Psi = 0 \tag{5}$$

Affine Toda field equations

One can take the conformal transformation

$$z \to w(z), \quad \bar{z} \to \bar{w}(\bar{z}), \quad \phi \to \hat{\phi} = \phi - \rho^\vee \log(\partial_z w \partial_{\bar{z}} \bar{w}), \tag{6}$$

then the affine Toda field equations will be modified into

$$\partial_{\bar{z}}\partial_z \phi(z,\bar{z}) - [\sum_{i=1}^{r} \alpha_i \exp\left(\alpha_i \cdot \phi\right) + p(z)\bar{p}(\bar{z})\alpha_0 \exp\left(\alpha_0 \cdot \phi\right)] = 0 \tag{7}$$

with $p(z) = (\partial_z w)^h, \quad \bar{p}(\bar{z}) = (\partial_{\bar{z}} \bar{w})^h$. The modified Lax operators are

$$\begin{aligned} \mathcal{L}_m &= \partial_z + \sum_{i=1}^{r} \partial_z \phi_i(z,\bar{z}) H_i + \lambda(\sum_{i=1}^{r} E_{\alpha_i} + p(z) E_{\alpha_0}), \\ \bar{\mathcal{L}}_m &= \partial_{\bar{z}} + \lambda^{-1} e^{-\phi_i H_i}(\bar{p}(\bar{z}) E_{\alpha_0} + \sum_{i=1}^{r} E_{-\alpha_i}) e^{\phi_i H_i}. \end{aligned} \tag{8}$$

$A_r^{(1)}$ affine Toda field equations

Let us focus on the holomorphic part: $\mathcal{L}_m \Psi = 0$ and take $\bar{z}$ for a constant

$$\left\{ \partial_z + \begin{pmatrix} \partial_z\phi_1 & \lambda & & & \\ & (\partial_z\phi_2 - \partial_z\phi_1) & \lambda & & \\ & & \ddots & & \\ & & & (\partial_z\phi_r - \partial_z\phi_{r-1}) & \lambda \\ \lambda p(z) & & & & -\partial_z\phi_r \end{pmatrix} \right\} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_r \\ \psi_{r+1} \end{pmatrix} = 0.$$

We can find the higher order ODE for $\psi_1(z, \bar{z})$ by eliminating other components.

$$(-\lambda)^{-h}(\partial_z - \partial_z\phi_r)(\partial_z + \partial_z\phi_r - \partial_z\phi_{r-1}) \ldots (\partial_z + \partial_z\phi_2 - \partial_z\phi_1)(\partial_z + \partial_z\phi_1)\psi_1 = p(z)\psi_1$$

with the Coxeter number $h = r + 1$.

Comment

From the spectral determinant of the solution ψ_1, it is possible to construct integrable Q-system/T-systerm, which implies a nontrivial correspondence between quantum/classical affine Toda field theory.

Generalized to other affine Lie algebras

The ordinary differential equations obtained in [Ito, Locke (2015)]

$$A_r^{(1)}:\ (-\lambda)^{-h}(\partial_z - \partial_z\phi_r)(\partial_z - \partial_z\phi_{r-1} + \partial_z\phi_r)$$
$$\cdots(\partial_z + \partial_z\phi_1)\psi(z,\epsilon) = p(z)\psi(z,\epsilon)$$
$$A_{2r-1}^{(2)}:\ \lambda^{-(2r-1)}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - \partial_z\phi_r + \partial_z\phi_{r-1})(\partial_z + \partial_z\phi_r - \partial_z\phi_{r-1})$$
$$\cdots(\partial_z + \partial_z\phi_1)\psi + 2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0$$
$$B_r^{(1)}:\ \lambda^{-2r}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - 2\partial_z\phi_r + \partial_z\phi_{r-1})\partial_z(\partial_z + 2\partial_z\phi_r - \partial_z\phi_{r-1})$$
$$\cdots(\partial_z + \partial_z\phi_1)\psi - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0$$
$$D_{r+1}^{(2)}:\ \lambda^{-(2r+2)}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - 2\partial_z\phi_r + \partial_z\phi_{r-1})\partial_z(\partial_z + 2\partial_z\phi_r - \partial_z\phi_{r-1})$$
$$\cdots(\partial_z + \partial_z\phi_1)\psi - 4p(z)\partial_z^{-1}p(z)\psi = 0$$
$$D_r^{(1)}:\ \lambda^{-(2r-2)}(\partial_z - \partial_z\phi_1)\cdots(\partial_z - \partial_z\phi_r - \partial_z\phi_{r-1} + \partial_z\phi_{r-2})\partial_z^{-1}$$
$$(\partial_z + \partial_z\phi_r + \partial_z\phi_{r-1} - \partial_z\phi_{r-2})\cdots(\partial_z + \partial_z\phi_1)\psi - 4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi = 0$$

- The ordinary differential equations can be solved with the conventional WKB method for $A_r^{(1)}$, $A_{2r-1}^{(2)}$, $B_r^{(1)}$.
- There exists a ∂_z^{-1} operator for $D_{r+1}^{(2)}$, $D_r^{(1)}$, where the method can not be applied.

$A_r^{(1)}$ affine Toda field equations

The linear problem for $A_r^{(1)}$ type turns out to be diagonalizable [Drinfeld, Sokolov (1984)]

$$
\begin{aligned}
\mathcal{L}_{\text{diag}} &= \partial_z + \lambda\Lambda_{\text{diag}} + \sum_{i=0}^{\infty} \lambda^{-i} I_i(z) \Lambda_{\text{diag}}^{-i} \\
&= \partial_z + \lambda\Lambda_{\text{diag}} + \begin{pmatrix} I(z, e^{\frac{2\pi i}{r+1}}\lambda) & & & & \\ & \ddots & & & \\ & & I(z, e^{\frac{2\pi i(r-1)}{r+1}}\lambda) & & \\ & & & I(z, e^{\frac{2\pi i r}{r+1}}\lambda) & \\ & & & & I(z,\lambda) \end{pmatrix},
\end{aligned}
\tag{9}
$$

where $I(z,\lambda)$ is the classical conserved densities $\partial_{\bar{z}} I_i + \partial_z \mathcal{A}_i = 0$. However, there was no efficient way to find $I(z,\lambda)$. We provide the WKB method to realize it.

$A_r^{(1)}$ affine Toda field equations

To apply the WKB method, we will replace the spectral parameter λ with WKB order parameter ϵ (Planck constant)

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}. \tag{10}$$

The higher-order ODE (11) now becomes

$$[(-\epsilon)^h(\partial_z-\partial_z\phi_r(z))\cdots(\partial_z+\partial_z\phi_2(z)-\partial_z\phi_1(z))(\partial_z+\partial_z\phi_1(z))-p(z)]\psi_1(z,\epsilon)=0 \tag{11}$$

$\lambda \sim \epsilon^{-1}$ implies the relations between classical IM and ODE.
One can apply the WKB ansatz to solve it.

$$\psi_1(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon)) \tag{12}$$

The equation satisfied by $P(z,\epsilon)$ is called the **Riccati equation**.

Contents

The diagonalization approach

We will diagonalize the Lax operator $\mathcal{L}_m$ with the WKB method.

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon \sum_{i=1}^{r} \partial_z \phi_i(z) H_i + \sum_{i=1}^{r} E_{\alpha_i} + p(z) E_{\alpha_0}. \tag{13}$$

One can view it as a covariant derivative with connection:

$$A(z) = \sum_{i=1}^{r} \epsilon\partial_z \phi_i(z) H_i + \sum_{i=1}^{r} E_{\alpha_i} + p(z) E_{\alpha_0} \tag{14}$$

Then the gauge transformation is given by

$$\mathbf{Gau}_T[A(z)] = T^{-1}(z) A(z) T(z) + \epsilon T^{-1}(z) \partial_z T(z). \tag{15}$$

The diagonalization approach

The transformation matrix T can be decomposed into

$$T(z) = T_d T_{d-1} \dots T_3 T_2 T_1. \tag{16}$$

d is the representation dimension and $T_i(z)$ are $d \times d$ matrices satisfying

$$T_i(z)_{ab} = \begin{cases} 1, & \text{if} \quad a = b, \\ g_{i,b}(z,\epsilon), & \text{if} \quad a = i, \quad b \neq i, \quad 1 \leq b \leq d, \\ 0, & \text{otherwise.} \end{cases}$$

The decomposition means we diagonalize the connection row by row from the bottom to the top. For instance

$$T_d = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ g_{d,1} & g_{d,2} & \cdots & g_{d,d-1} & 1 \end{pmatrix},$$

The diagonalization approach

The final diagonalized connection $A_{\text{diag}}(z)$ is given by

$$A_{\text{diag}}(z) = \mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2} \ldots \mathbf{Gau}_{T_{d-2}} \circ \mathbf{Gau}_{T_{d-1}} \circ \mathbf{Gau}_{T_d}[A(z)]. \tag{17}$$

For each step of the gauge transformation $\mathbf{Gau}_{T_i}$, we fix $g_{i,b}(z)$ such that the connection $A'(z)$ satisfies

$$A'_{ij} = 0, \quad 1 \le j \le d, \quad j \neq i. \tag{18}$$

There are finally $d-1$ constraints to diagonalize the i-th row in $A(z)$ and fix the diagonal elements perturbatively. For instance, the connection after first gauge transformation:

$$A'(z) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 1 & \\ \mathbf{Gau}_{T_d}[A]_{d,1} & \mathbf{Gau}_{T_d}[A]_{d,2} & \cdots & \mathbf{Gau}_{T_d}[A]_{d,d-1} & \mathbf{Gau}_{T_d}[A]_{d,d} \end{pmatrix},$$

Contents

The diagonalization of $A_1^{(1)}$

The holomorphic part of the modified Lax operator in $A_1^{(1)}$ is of the form

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon\partial_z\phi(z)H + E_\alpha + p(z)E_{-\alpha} \tag{19}$$

with

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We decompose the transformation matrix: $T(z) = T_1 T_2$ by

$$T_2(z) = \begin{pmatrix} 1 & 0 \\ g_{2,1}(z,\epsilon) & 1 \end{pmatrix}, \quad T_1(z) = \begin{pmatrix} 1 & g_{1,2}(z,\epsilon) \\ 0 & 1 \end{pmatrix}. \tag{20}$$

$T_2(z)$ is determined to diagonalize the second row

$$\mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} g_{2,1} + \epsilon\phi' & 1 \\ -2\epsilon g_{2,1}\phi' + \epsilon g'_{2,1} - g_{2,1}^2 + p & -g_{2,1} - \epsilon\phi' \end{pmatrix}. \tag{21}$$

It gives the condition for $g_{2,1}(z)$:

$$g_{2,1}^2(z,\epsilon) + 2\epsilon g_{2,1}(z,\epsilon)\phi'(z,\bar{z}) - \epsilon g'_{2,1}(z,\epsilon) - p(z) = 0. \tag{22}$$

The diagonalization of $A_1^{(1)}$

Diagonal element $f(z,\epsilon) := -g_{2,1}(z,\epsilon) - \epsilon\phi'(z)$ satisfies Riccati equation

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0. \tag{23}$$

$u_2(z) = \phi'(z)^2 - \phi''(z)$ is the classical energy-momentum tensor in sine-Gordon theory, where the subscript denotes the spin.

The second gauge transformation $T_1(z)$ gives

$$\mathbf{Gau}_{T_1} \circ \mathbf{Gau}_{T_2}[A(z)] = \begin{pmatrix} -f(z,\epsilon) & 1 - 2g_{1,2}(z,\epsilon)f(z,\epsilon) \\ 0 & f(z,\epsilon) \end{pmatrix}. \tag{24}$$

We do not need to extract the diagonalization condition from the first row since $g_{1,2}$ is independent of the diagonal elements.

The diagonalization of $A_1^{(1)}$

Let us substitute $f(z,\epsilon)=\sum_{i=0}^{\infty}\epsilon^i f_i(z)$ into the Riccati equation. The first four orders of diagonal elements $\pm f(z,\epsilon)$ are listed below

$$
\begin{aligned}
f_0(z) &= \sqrt{p(z)}, & -f_0(z) &= -\sqrt{p(z)},\\
f_1(z) &= -\frac{1}{2}\partial_z \ln f_0, & -f_1(z) &= f_1(z)+\partial_z \ln f_0,\\
f_2(z) &= \frac{f_0''}{16f_0^2}+\frac{u_2(z)}{2f_0}+\partial_z\Big(\frac{3f_0'}{16f_0^2}\Big), & -f_2(z) &= -f_2(z),\\
f_3(z) &= \partial_z\Big(\frac{u_2(z)}{4f_0^2}-\frac{3f_0^{'2}}{16f_0^4}+\frac{f_0^{''}}{8f_0^3}\Big), & -f_3(z) &= f_3(z)-\partial_z\Big(\frac{u_2(z)}{2f_0^2}-\frac{3f_0^{'2}}{8f_0^4}+\frac{f_0^{''}}{4f_0^3}\Big).
\end{aligned}
$$

Therefore, the diagonal elements can be summarized as

$$
A_{\text{diag}}(z)=\begin{pmatrix} -f(z,-\epsilon)+d(*) & 0\\ 0 & f(z,\epsilon)\end{pmatrix}, \tag{25}
$$

where $d(*)$ denotes total derivatives.

The traceless condition implies $f_{2i+1}(z)$ are all total derivatives.

The diagonalization of $A_1^{(1)}$

Let us pay attention back to $f(z,\epsilon)$ and the Riccati equation

$$f^2(z,\epsilon) + \epsilon f'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0.$$

What about the relations between $f(z,\epsilon)$ and the WKB solutions $P(z,\epsilon)$ to the $A_1^{(1)}$ ordinary differential equation

$$\left(\epsilon^2\partial_z^2 + \epsilon^2\partial_z^2\phi(z) - \epsilon^2(\partial_z\phi)^2 - p(z)\right)\psi(z,\epsilon) = 0 \tag{26}$$

Substitute the WKB ansatz $\psi_1(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, P(z,\epsilon))$, one can obtain the Riccati equation

$$P^2(z,\epsilon) + \epsilon P'(z,\epsilon) - \epsilon^2 u_2(z) - p(z) = 0. \tag{27}$$

There exists an equivalence between the diagonal elements and the WKB solutions

$$f(z,\epsilon) = P(z,\epsilon) \tag{28}$$

The diagonalization of $A_1^{(1)}$

Let us also compare $f(z,\epsilon)$ with the conserved density $I(z,\lambda)$. Due to the existence of ϵ in front of ∂_z, $f_0(z)$ is actually the -1st term corresponding to the $\lambda\Lambda_{\text{diag}}$ term. Besides this, $f_i(z) \sim I_{i-1}(z)$.

$$\begin{aligned}\mathcal{L}_{\text{diag}} &= \partial_z + \lambda\Lambda_{\text{diag}} + \sum_{i=0}^{\infty} \lambda^{-i} I_i(z) \Lambda_{\text{diag}}^{-i} \\ &= \epsilon\partial_z + f_0\Lambda_{\text{diag}} + \sum_{i=1}^{\infty} \epsilon^i f_i(z) \Lambda_{\text{diag}}^{-i}\end{aligned} \tag{29}$$

One may worry about the total derivatives $d(*)$ in

$$A_{\text{diag}}(z) = \begin{pmatrix} -f(z,-\epsilon) + d(*) & 0 \\ 0 & f(z,\epsilon) \end{pmatrix}, \tag{30}$$

The diagonal elements are uniquely determined up to total derivatives. One can act $T_{\textbf{diag}} = \textbf{diag}\{\exp(t_1(z,\epsilon)), \ldots, \exp(t_d(x,\epsilon))\}$, $[A_{\textbf{diag}}(z)]_{ii}$ will receive a $\partial_z t_i(z)$ shift.

The diagonalization of $A_2^{(1)}$

The equality $f(z,\epsilon) = P(z,\epsilon)$ can be generalized into $A_2^{(1)}$ types.
There are two scalar fields: $\phi_1(z)$ and $\phi_2(z)$. The modified Lax operator is

$$\mathcal{L}_m = \epsilon\partial_z + \sum_{i=1}^{2} \epsilon\partial_z\phi_i(z,\bar{z})H_i + \sum_{i=1}^{2} E_{\alpha_i} + p(z)E_{\alpha_0} \tag{31}$$

Perform the diagonalization by $T = T_3T_2T_1$.
The gauge transformation by T_3 leads to

$$\mathbf{Gau}_{T_3}[A(z)] = \begin{pmatrix} \epsilon\phi_1' & 1 & 0 \\ g_{3,1} & g_{3,2} + \epsilon(\phi_2' - \phi_1') & 1 \\ \mathbf{Gau}_{T_3}[A(z)]_{3,1} & \mathbf{Gau}_{T_3}[A(z)]_{3,2} & -g_{3,2} - \epsilon\phi_2' \end{pmatrix},$$

where

$$\mathbf{Gau}_{T_3}[A(z)]_{3,1} = -g_{3,1}\left(g_{3,2} + \epsilon\left(\phi_1' + \phi_2'\right)\right) + \epsilon g_{3,1}' + p,$$
$$\mathbf{Gau}_{T_3}[A(z)]_{3,2} = \epsilon g_{3,2}\left(\phi_1' - 2\phi_2'\right) + \epsilon g_{3,2}' - g_{3,2}^2 - g_{3,1}.$$

The diagonalization of $A_2^{(1)}$

Set $f(z,\epsilon) \equiv -g_{3,2} - \epsilon\phi_2'$, $\mathbf{Gau}_{T_3}[A(z)]_{3,1} = \mathbf{Gau}_{T_3}[A(z)]_{3,2} = 0$ gives

$$f^3 + 3\epsilon ff' - \epsilon^2 u_2 f + \epsilon^2 f'' - \epsilon^3 u_3 - p = 0. \tag{32}$$

After the second gauge transformation T_2,

$$\mathbf{Gau}_{T_2T_3}[A(z)] = \begin{pmatrix} \epsilon\phi_1' + g_{2,1} & 1 & g_{2,3} \\ \mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} & g_{3,2} + \epsilon(\phi_2' - \phi_1') - g_{2,1} & \mathbf{Gau}_{T_2T_3}[A(z)]_{2,3} \\ 0 & 0 & f \end{pmatrix}$$

where

$$\begin{aligned} \mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} &= -2\epsilon g_{2,1}\phi_1' + \epsilon g_{2,1}\phi_2' + \epsilon g_{2,1}' - g_{2,1}^2 + g_{3,2}g_{2,1} + g_{3,1} \\ \mathbf{Gau}_{T_2T_3}[A(z)]_{2,3} &= -\epsilon g_{2,3}\phi_1' + 2\epsilon g_{2,3}\phi_2' + \epsilon g_{2,3}' + 2g_{3,2}g_{2,3} - g_{2,1}g_{2,3} + 1. \end{aligned} \tag{33}$$

Set $h(z,\epsilon) \equiv g_{2,1} + \epsilon\phi_1'$, $\mathbf{Gau}_{T_2T_3}[A(z)]_{2,1} = 0$ leads to the equation

$$h^2 + fh + f^2 - \epsilon h' + \epsilon f' - \epsilon^2 u_2 = 0. \tag{34}$$

The diagonalization of $A_2^{(1)}$

$u_2(z)$ and $u_3(z)$ are classical energy-momentum tensor and $\mathcal{W}_3$ field in $A_2^{(1)}$ affine Toda field theory with

$$u_2(z) = \phi_1'(z)^2 - \phi_2'(z)\phi_1'(z) + \phi_2'(z)^2 - \phi_1''(z) - \phi_2''(z),$$
$$u_3(z) = 2\phi_2'(z)\phi_2''(z) - \phi_1'(z)\phi_2''(z) - \phi_1'(z)\phi_2'(z)^2 + \phi_1'(z)^2\phi_2'(z) - \phi_2^{(3)}(z).$$

The Riccati equation satisfied by $f(z,\epsilon)$ can also be obtained from $\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int dz\, P(z,\epsilon))$ and

$$(-\epsilon)^3(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1)(\partial_z + \partial_z\phi_2)\psi + p(z)\psi = 0. \qquad (35)$$

This is the adjoint ordinary differential equation of

$$[(-\epsilon)^h(\partial_z - \partial_z\phi_2(z))(\partial_z + \partial_z\phi_2(z) - \partial_z\phi_1(z))(\partial_z + \partial_z\phi(z)) - p(z)]\psi_1(z,\epsilon) = 0.$$

The adjoint means $\partial_z \to -\partial_z$ and $\phi_i \to \phi_{h-i}$.

The diagonalization of $A_2^{(1)}$

Expand $f = \sum_{n=0}^{\infty} f_n \epsilon^n$ and $h = \sum_{n=0}^{\infty} h_n \epsilon^n$. The first four terms are

$$f_0(z) = p^{\frac{1}{3}}, \qquad h_0(z) = e^{-\frac{2\pi i}{3}} f_0,$$

$$f_1(z) = -\frac{f_0'}{f_0}, \qquad h_1(z) = f_1(z) + 2\partial_z(\ln f_0),$$

$$f_2(z) = \frac{f_0''}{6f_0^2} + \frac{u_2(z)}{3f_0} + \partial_z(\frac{f_0'}{2f_0^2}), \qquad h_2(z) = e^{\frac{2\pi i}{3}} f_2(z),$$

$$f_3(z) = -\frac{u_3(z)}{3f_0^2} + \frac{f_0' u_2(z)}{3f_0^3} - \partial_z(-\frac{f_0^{'2}}{2f_0^4} + \frac{f_0^{''}}{3f_0^3} - \frac{u_2(z)}{3S_0^2}), \qquad h_3(z) = e^{\frac{4\pi i}{3}}(f_3(z) - \partial_z(\frac{u_2}{3f_0^2})).$$

The diagonal connection is summarized as

$$A_{\text{diag}}(z) = \begin{pmatrix} e^{-\frac{i2\pi}{3}} f(z, e^{\frac{i2\pi}{3}}\epsilon) + d(*) & 0 & 0 \\ 0 & e^{-\frac{i4\pi}{3}} f(z, e^{\frac{i4\pi}{3}}\epsilon) + d(*) & 0 \\ 0 & 0 & f(z,\epsilon) \end{pmatrix}. \quad (36)$$

The traceless condition implies $f_{1+3i}(z)$ are total derivatives.
We also observe that $f_{1+2i}(z)$ are also total derivatives when $u = 0$.

The canonical Lax operator

u_2 and u_3 can be obtained from the generalized Miura transformation

$$(\partial_z - \partial_z\phi_1)(\partial_z - \partial_z\phi_2 + \partial_z\phi_1)(\partial_z + \partial_z\phi_2) = \partial_z^3 - \sum_{i=0}^{2} u_{3-i}\partial_z^i \tag{37}$$

The equivalence between the following two Lax operators is proved in [Drinfeld, Sokolov (1984)]

$$\mathcal{L}_m = \epsilon\partial_z + \sum_{i=1}^{2} \epsilon\partial_z\phi_i(z,\bar{z})H_i + \sum_{i=1}^{2} E_{\alpha_i} + p(z)E_{\alpha_0}$$

$$\mathcal{L}_{\text{can}} = \epsilon\partial_z + \sum_{i=1}^{2} \epsilon^{i+1}u_{1+i}(z)e_{i+1,1} + \sum_{i=1}^{2} E_{\alpha_i} + p(z)E_{\alpha_0}$$

The same Ricaati equation and diagonal elements will be given after applying the same diagonalization approach.

Contents

Generalized to other affine Lie algebras

The diagonalized Lax operators for other affine Lie algebras are also predicted in [Drinfeld, Sokolov (1984)]

$$\mathcal{L}_0 = \partial_z + \lambda\Lambda + H(z,\lambda) \tag{38}$$

with

$$H(z,\lambda) = \sum_{i=0}^{\infty} \lambda^{-(2i+1)} I_i(z)\Lambda^{-(2i+1)}, \quad \text{for } B_r^{(1)},\ D_{r+1}^{(2)},\ A_{2r-1}^{(2)},$$
$$H(z,\lambda) = \sum_{i=0}^{\infty} \lambda^{-(2i+1)} I_i(z)\Lambda^{-(2i+1)} + \sum_{i=0}^{\infty} \lambda^{-(2i+1)} J_i(z) F, \quad \text{for } D_r^{(1)}.$$

- $\Lambda^{-(2i+1)} = \Lambda^{-(2i+1)+kh}$ with sufficiently large k for $B_r^{(1)}$, $A_{2r-1}^{(2)}$ and $D_r^{(1)}$.
- I_i, J_i turn out to be conserved densities.
- There is an extra matrix F commuting with Λ, which generates J_i in $D_r^{(1)}$.
- The terms proportional to Λ^{-2i} are missing (up to total derivatives).

We follow the same diagonalization steps to obtain $A_{\mathrm{diag}}(z)$. First, the modified Lax operator for an affine Lie algebra $\hat{\mathfrak{g}}$:

$$\mathcal{L}_m = \epsilon\partial_z + \epsilon\sum_{i=1}^{r}\partial_z\phi_i(z)H_i + \sum_{i=1}^{r}E_{\alpha_i} + p(z)E_{\alpha_0}.$$

Denote $f(z,\epsilon)$ as the bottom component of diagonal elements of $A_{\mathrm{diag}}(z)$

$$f(z,\epsilon) = \sum_{i=0}^{\infty}\epsilon^i f_i(z).$$

with the WKB ansatz

$$\psi(z,\epsilon) = \exp(\frac{1}{\epsilon}\int^z dz\, f(z,\epsilon))$$

Generalized to other affine Lie algebras

The diagonalization can be generalized to other affine Lie algebras.

$$A_r^{(1)}: \ \textbf{Diag}\{e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i r}{h}} f(z, e^{\frac{2\pi i r}{h}}\epsilon) + d(*), f(z,\epsilon)\}$$

$$A_{2r-1}^{(2)}: \textbf{Diag}\{h(z,\epsilon) + d(*), e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i(h-1)}{h}} f(z, e^{\frac{2\pi i(h-1)}{h}}\epsilon) + d(*), f(z,\epsilon)\}$$

$$B_r^{(1)}: \ \textbf{Diag}\{d(*), e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i(2r-1)}{h}} f(z, e^{\frac{2\pi i(2r-1)}{h}}\epsilon) + d(*), f(z,\epsilon)\}$$

$$D_{r+1}^{(2)}: \ \textbf{Diag}\{e^{-\frac{2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i(2r+1)}{h}} f(z, e^{\frac{2\pi i(2r+1)}{h}}\epsilon) + d(*), f(z,\epsilon)\}$$

$$D_r^{(1)}: \ \textbf{Diag}\{e^{\frac{-2\pi i}{h}} f(z, e^{\frac{2\pi i}{h}}\epsilon) + d(*), \ldots, e^{\frac{-2\pi i(r-1)}{h}} f(z, e^{\frac{2\pi i(r-1)}{h}}\epsilon) + d(*), \ e^{i\pi(r-1)} K(z,-\epsilon) + d(*),$$
$$K(z,\epsilon), \ e^{-\frac{2\pi i r}{h}} f(z, e^{\frac{2\pi i r}{h}}\epsilon) + d(*), \ldots, e^{-\frac{2\pi i(2r-1)}{h}} f(z, e^{\frac{2\pi i(2r-1)}{h}}\epsilon) + d(*), \ f(z,\epsilon)\}$$

- $f(z,\epsilon)$ satisfy the Riccati equation from the ordinary differential equations with WKB ansatz for $A_r^{(1)}$, $A_{2r-1}^{(2)}$, $B_r^{(1)}$.
- $f(z,\epsilon)$ should also lead to the solutions to the pseudo differential equations for $D_{r+1}^{(2)}$, $D_r^{(1)}$.
- $f_{1+2i}(z)$ are all total derivatives in the affine Lie algebras except $A_r^{(1)}$.
- $K(z,\epsilon)$ is another conserved density corresponding to matrix F in $\mathcal{L}_0$.

Generalized to other affine Lie algebras

The adjoint ordinary differential equations satisfied by $f(z,\epsilon)$ are

$$A_r^{(1)}:\ (-\epsilon)^h(\partial_z-\partial_z\phi_1)(\partial_z-\partial_z\phi_2+\partial_z\phi_1)\cdots(\partial_z+\partial_z\phi_r)\psi(z,\epsilon)=p(z)\psi(z,\epsilon)$$

$$A_{2r-1}^{(2)}:\ \epsilon^{(2r-1)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-\partial_z\phi_r+\partial_z\phi_{r-1})(\partial_z+\partial_z\phi_r-\partial_z\phi_{r-1})\cdots(\partial_z+\partial_z\phi_1)\psi-2\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi=0$$

$$B_r^{(1)}:\ \epsilon^{2r}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-2\partial_z\phi_r+\partial_z\phi_{r-1})\partial_z(\partial_z+2\partial_z\phi_r-\partial_z\phi_{r-1})\cdots(\partial_z+\partial_z\phi_1)\psi-4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi=0$$

$$D_{r+1}^{(2)}:\ \epsilon^{(2r+2)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-2\partial_z\phi_r+\partial_z\phi_{r-1})\partial_z(\partial_z+2\partial_z\phi_r-\partial_z\phi_{r-1})\cdots(\partial_z+\partial_z\phi_1)\psi-4p(z)\partial_z^{-1}p(z)\psi=0$$

$$D_r^{(1)}:\ \epsilon^{(2r-2)}(\partial_z-\partial_z\phi_1)\cdots(\partial_z-\partial_z\phi_r-\partial_z\phi_{r-1}+\partial_z\phi_{r-2})\partial_z^{-1}(\partial_z+\partial_z\phi_r+\partial_z\phi_{r-1}-\partial_z\phi_{r-2})\cdots(\partial_z+\partial_z\phi_1)\psi-4\sqrt{p(z)}\partial_z\sqrt{p(z)}\psi=0$$

- The $D_r^{(1)}$-type ODE is reduced to $B_{r-1}^{(1)}$ type for $\partial_z\phi_r+\partial_z\phi_{r-1}-\partial_z\phi_{r-2}=0$...

Contents

Continuity equations from the KdV hierarchies

Recall the $A_r^{(1)}$ Toda field equations in terms of x and t

$$(-\epsilon)^h(\partial_x - \partial_x\phi_1)(\partial_x - \partial_x\phi_2 + \partial_x\phi_1)\cdots(\partial_x + \partial_x\phi_r)\psi(x,\epsilon) = p(x)\psi(x,\epsilon)$$

The left-hand side is the corresponding scalar Lax operator for the modified KdV hierarchies with linear problem $L\psi(x) = \lambda\psi(x)$

$$L_{\text{scalar}} = (\partial_x - \partial_x\phi_1)\cdots(\partial_x + \partial_x\phi_2 - \partial_x\phi_1)(\partial_x + \partial_x\phi_r),$$

Both the Lie algebraic and the scalar Lax operators satisfy the Lax equation for parameters t_i $(i = 1, 2, \dots)$ with $t_1 = t$.

$$\partial_{t_i} L = [A_i, L]$$

with $A_i = (L^{\frac{i}{h}})_+$, where $(A)_+$ denotes the non-negative part in ∂_x of the differential operator A.

Continuity equations from the KdV hierarchies

Act ∂_{t_i} on $L\psi(x) = \lambda\psi(x)$, one can obtain $(L-\lambda)(\partial_{t_i}\psi - A_i\psi) = 0$, which implies, for some function $g(t_i)$

$$\partial_{t_i}\psi(x) - A_i\psi(x) = g(t_i)\psi(x).$$

Substitute the WKB expansion $\psi(x,\epsilon) = \exp(\frac{1}{\epsilon}\int dx\, P(x,\epsilon))$

$$\partial_{t_i}P(x,\epsilon) - \partial_x a_i(x) = 0,$$

The integrable hierarchies defined by $\mathcal{L}$ with $\mathcal{A}(x) = \sum_{i=0}^{\infty}\mathcal{A}_i(x)(\lambda\Lambda)^{-i}$ is

$$\partial_t\mathcal{L} = [\mathcal{A}, \mathcal{L}],$$

After the diagonalization $\mathcal{L}_{diag} = T\mathcal{L}T^{-1}$, $\partial_t\mathcal{L}_{\mathrm{diag}} = [\mathcal{A}', \mathcal{L}_{\mathrm{diag}}]$

$$\partial_t f_i + \partial_x\mathcal{A}'_i = 0.$$

It implies the equality $f(x,\epsilon) = P(x,\epsilon)$ up to total derivatives.

Contents

Conserved density vs. WKB solution

The classical conserved densities for the sine-Gordon equations

$$
\begin{aligned}
I_2(z) &= \frac{T(z)}{2}, \\
I_4(z) &= \frac{\partial_z^2 T(z) - T^2(z)}{8}, \\
I_6(z) &= \frac{1}{32}\left(-5T'(z)^2 - 6T(z)u''(z) + T^{(4)}(z) + 2T(z)^3\right),
\end{aligned}
\tag{39}
$$

The WKB solutions for the $A_1^{(1)}$-type ordinary differential equation

$$
\begin{aligned}
f_0(z) &= \sqrt{p(z)}, \\
f_1(z) &= -\frac{1}{2}\partial_z \ln f_0, \\
f_2(z) &= \frac{f_0''}{16f_0^2} + \frac{u_2(z)}{2f_0} + \partial_z\left(\frac{3f_0'}{16f_0^2}\right), \\
f_3(z) &= -\partial_z\left(-\frac{u_2(z)}{4f_0^2} + \frac{3f_0'^2}{16f_0^4} - \frac{f_0''}{8f_0^3}\right),
\end{aligned}
\tag{40}
$$

Conserved density vs. WKB solution

Recall the appearance of $p(z)$: the conformal transformation $z \to w(z)$

$$dw = \sqrt{p(z)}dz, \quad \hat{u}_2\big(w(z)\big) = \frac{1}{p(z)}\left[u_2(z) + \frac{4pp'' - 5p'^2}{16p^2}\right] \tag{41}$$

After the conformal transformation,

$$\begin{aligned}\hat{f}_0(w) &= 1,\\ \hat{f}_2(w) &= \frac{\hat{u}_2(w)}{2},\\ \hat{f}_4(w) &= \frac{\partial_w^2 \hat{u}_2(w) - \hat{u}_2^2(w)}{8},\end{aligned} \tag{42}$$

They are nothing but the commonly conserved densities. In conclusions, the quantum period Π_i and conserved charges $\mathcal{Q}_i$ are related as follows:

$$\Pi_i \equiv \oint dz\ f_i(z) = \oint dz\ \sqrt{p(z)}\hat{f}_i(z) = \oint dw\ \hat{f}_i(w) \equiv \mathcal{Q}_i. \tag{43}$$

Contents

Summary

- A WKB method is found to diagonalize the linear problem. The diagonal elements are the WKB solutions to the adjoint higher-order (pseudo) ordinary differential equations.
- There is a relation via the conformal transformation between the conserved densities and the WKB solutions.

Futhre work

- It is possible to take exact WKB analysis on the conserved densities $f(z,\epsilon)$.
- Apply the diagonalization results to the quantum SW curve in the Argyres-Douglas theory.
- Combine the diagonalization approach with $T\bar{T}$-deformation.

Thank you for watching.