How to get something from nothing?

Null state, bootstrap, Dyson-Schwinger

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Outline

- Introduction
- 2D minimal model CFT
- The null bootstrap

1. Hamiltonian
2. Lagrangian (Dyson-Schwinger)

Introduction

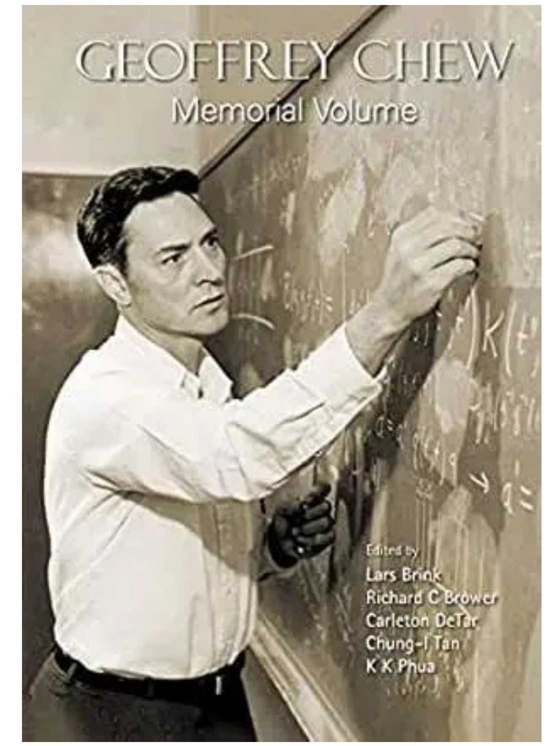
Bootstrap（自提升/自举）

Pull Yourself Up By Your Bootstraps

The term is sometimes attributed to a story in Rudolf Erich Raspe's *The Surprising Adventures of Baron Munchausen*, but in that story Baron Munchausen pulls himself (and his horse) out of a swamp by his hair (specifically, his pigtail), not by his bootstraps – and no explicit reference

Bootstrap physics

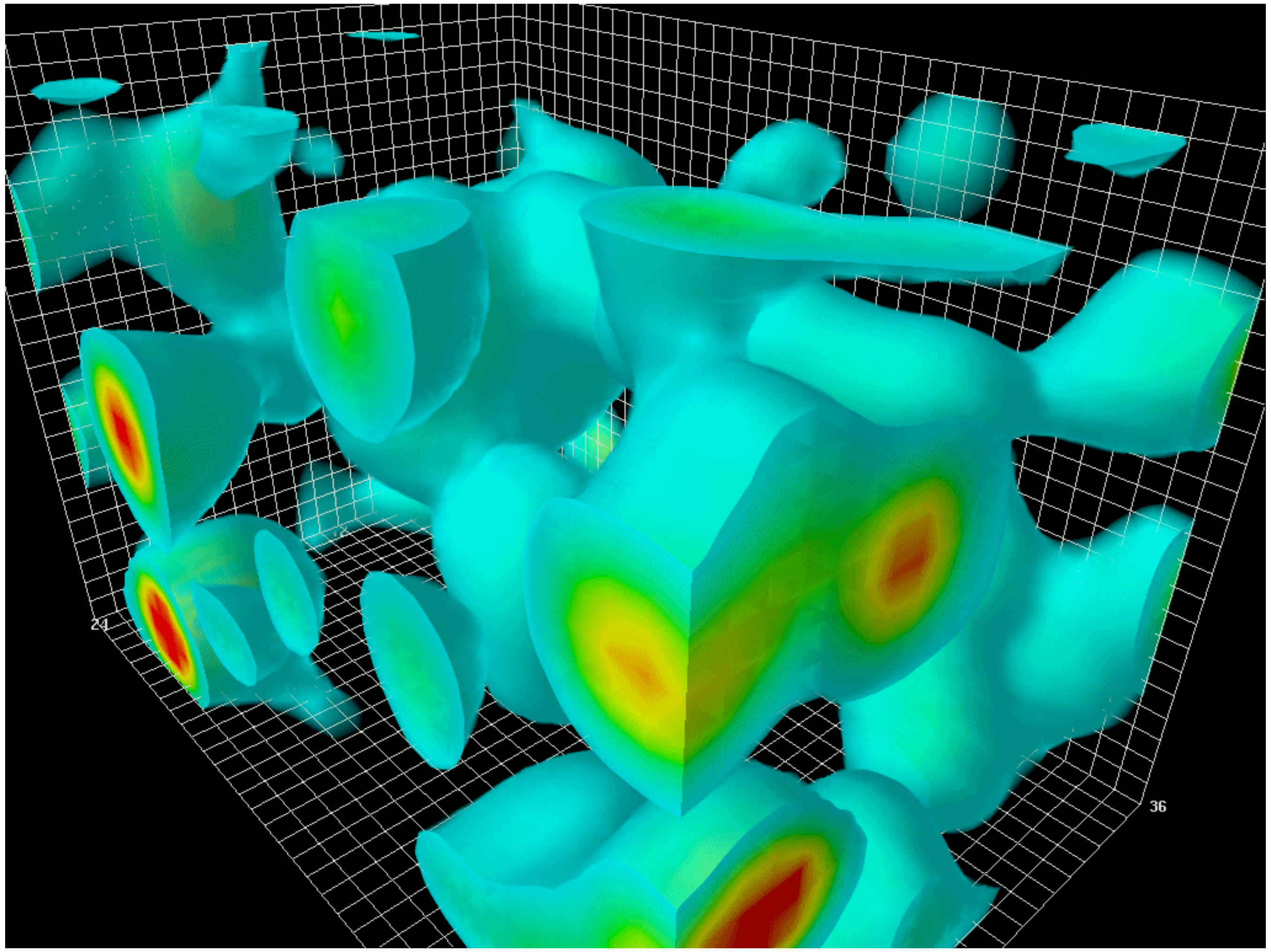
- “Nature is as it is because this is the only possible nature consistent with itself.” --Geoffrey Chew



- “... the bootstrap mechanism. it never really worked as a calculation scheme, but was extremely attractive philosophically, because it made do with very little, just the fundamental assumptions, without introducing things that we really could not know about.” --Steven Weinberg

真空不“空”

- Quantum “void” is not empty



endless, wild fluctuations

QCD vacuum fluctuations
by Derek Leinweber

Vacuum stability is a nontrivial fact !

“无”中生有

A vacuum state should be stable
despite quantum fluctuations

A useful principle for Bootstrap
more general than unitarity

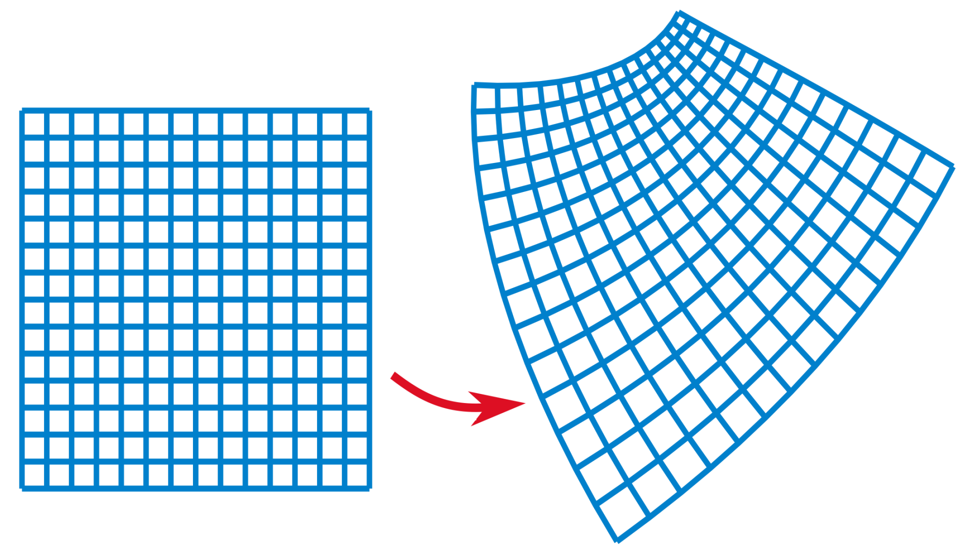
2D minimal model CFT

Conformal bootstrap

- Solve conformal field theory with

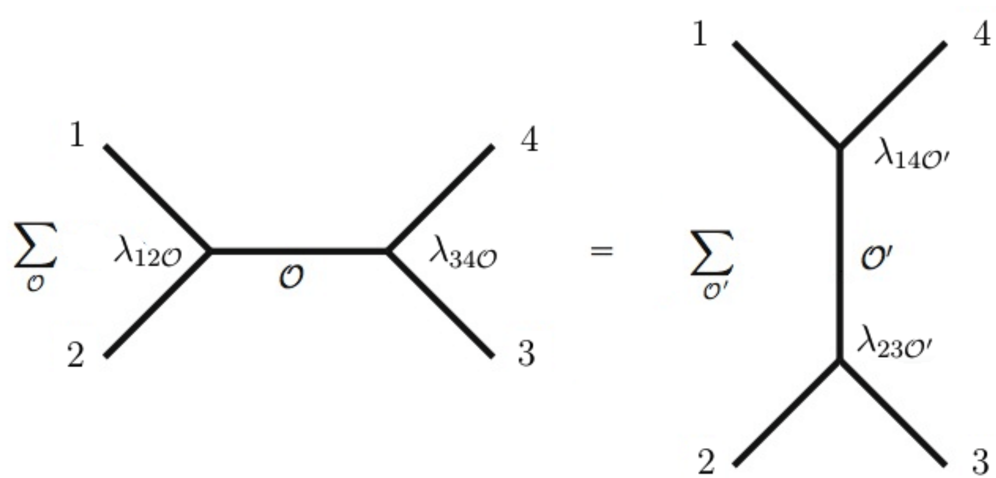
  1. Conformal symmetry

     conformal = angle-preserving
     ~ local rescaling+ rotation



  2. Consistency of operator algebra: OPE associativity

s-channel



t-channel

crossing equation for 4pt function

2D Conformal Field Theory

- In 2D, conformal symmetry is infinite dimensional

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$$

Virasoro algebra (holomorphic/anti-holomorphic)

- However, these equations are still underdetermined !
(due to infinitely many free parameters)

- We need additional constraints to close the system

- Schrödinger eq.+ boundary conditions (quantization condition)
-> bound states

2D minimal models

- Highest weight state

$$L_0|h\rangle = h|h\rangle \qquad\qquad L_n|h\rangle = 0 \qquad (n > 0)$$

- Verma module (“Virasoro conformal multiplet”)

Table 7.1. Lowest states of a Verma module.

l	$p(l)$	
0	1	$\|h\rangle$
1	1	$L_{-1}\|h\rangle$
2	2	$L_{-1}^2\|h\rangle$, $L_{-2}\|h\rangle$
3	3	$L_{-1}^3\|h\rangle$, $L_{-1}L_{-2}\|h\rangle$, $L_{-3}\|h\rangle$
4	5	$L_{-1}^4\|h\rangle$, $L_{-1}^2L_{-2}\|h\rangle$, $L_{-1}L_{-3}\|h\rangle$, $L_{-2}^2\|h\rangle$, $L_{-4}\|h\rangle$

2D minimal models

- Inner product

Hermitian conjugate: $L_m^\dagger = L_{-m}$

ex
$$\langle h|L_n L_{-n}|h\rangle = \langle h|\left(L_{-n}L_n + 2nL_0 + \frac{1}{12}cn(n^2-1)\right)|h\rangle$$
$$= [2nh + \frac{1}{12}cn(n^2-1)]\langle h|h\rangle$$

- Kac determinant

$$\det M^{(l)} = \alpha_l \prod_{\substack{r,s \geq 1 \\ rs \leq l}} [h - h_{r,s}(c)]^{p(l-rs)}$$

l: level

Gram matrix $M_{ij} = \langle i|j\rangle$

2D minimal models

- Roots of the Kac determinant

$$h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \qquad c = 1 - \frac{6}{m(m+1)}$$

zero-norm state at level- rs

- If the zero-norm state is orthogonal to all states

-> quotient representation $\quad V_{r,s} = \dfrac{V_{\Delta_{r,s}}}{V_{\Delta_{r,s}+rs}}$

null state at level- rs (quantization condition)

2D minimal models

- OPE truncation

$$\phi_{(r_1,s_1)} \times \phi_{(r_2,s_2)} = \sum_{\substack{k=1+|r_1-r_2| \\ k+r_1+r_2=1 \bmod 2}}^{k=r_1+r_2-1} \; \sum_{\substack{l=1+|s_1-s_2| \\ l+s_1+s_2=1 \bmod 2}}^{l=s_1+s_2-1} \phi_{(k,l)}$$

example

$$\phi_{(1,2)} \times \phi_{(r,s)} = \phi_{(r,s-1)} + \phi_{(r,s+1)}$$

$$\phi_{(2,1)} \times \phi_{(r,s)} = \phi_{(r-1,s)} + \phi_{(r+1,s)}$$

- If the central charge c is generic
-> generalized minimal models
operator algebra is still infinite-dimensional

Zamolodchikov, 2005

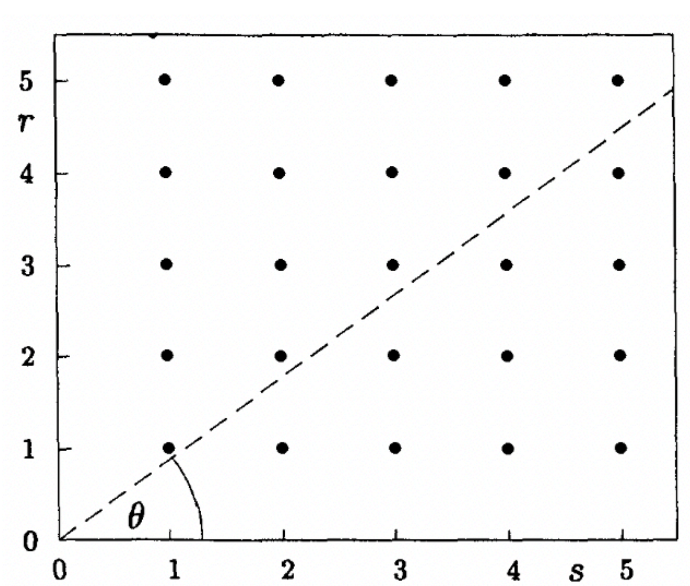
2D minimal models

- For central charge $$c = 1 - 6\frac{(p-p')^2}{pp'}$$

  one finds the periodicity relation

  $$\Delta_{r+p,s+p'} = \Delta_{r,s}$$ (difference is a null state)

- Operator algebra is truncated & finite-dimensional



$$h_{r,s} = h_0 + \frac{1}{4}\delta^2(\alpha_+^2 + \alpha_-^2)$$

δ is the Cartesian distance

For rational slope $p\alpha_- + p'\alpha_+ = 0$

2D minimal models

- For central charge $$c = 1 - 6\frac{(p-p')^2}{pp'}$$

  operator algebra is truncated and finite-dimensional
  due to the periodicity relation

- (p, p’) examples

(5, 2)	= Yang-Lee edge singularity	Cardy
(4, 3)	= Ising	BPZ
(5, 4)	= tricritical Ising	Friedan-Qiu-Shenker
(6, 5)	= three-state Potts	Dotsenko

2D minimal models

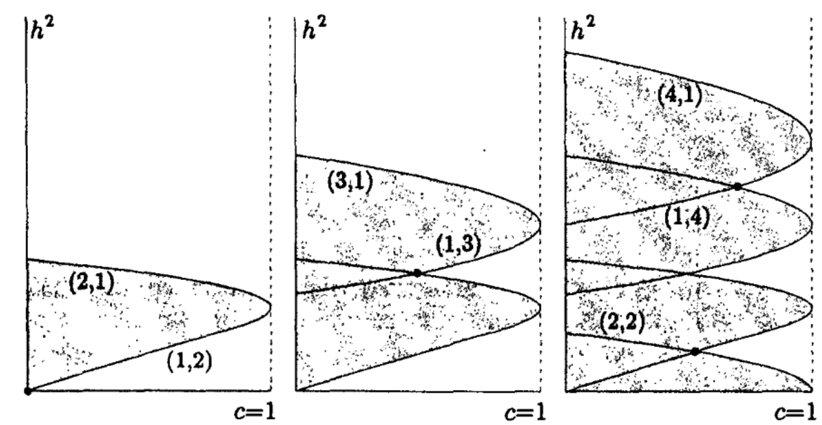
- For central charge

$$c = 1 - 6\frac{(p-p')^2}{pp'}$$

operator algebra is truncated and finite-dimensional due to the periodicity relation

- Unitary minimal models

$$p' = p + 1$$



- Landau-Ginzburg effective action (diagonal, p=m)

$$V_m(\Phi) = \Phi^{2(m-1)} \qquad \mathcal{L} = \int d^2z \left\{ \frac{1}{2}(\partial\Phi)^2 + V(\Phi) \right\}$$

multi-critical Ising fixed point

2D minimal models

- For central charge

  $$c = 1 - 6\frac{(p-p')^2}{pp'}$$

  operator algebra is truncated and
  finite-dimensional due to periodicity relation

- Multi-critical Yang-Lee fixed point

  Lencsés-Miscioscia-Mussardo-Takács, 2022

  $$p = 2\,, \quad p' = 3 + 2n$$

  deformation of multi-critical Ising model by imaginary magnetic field

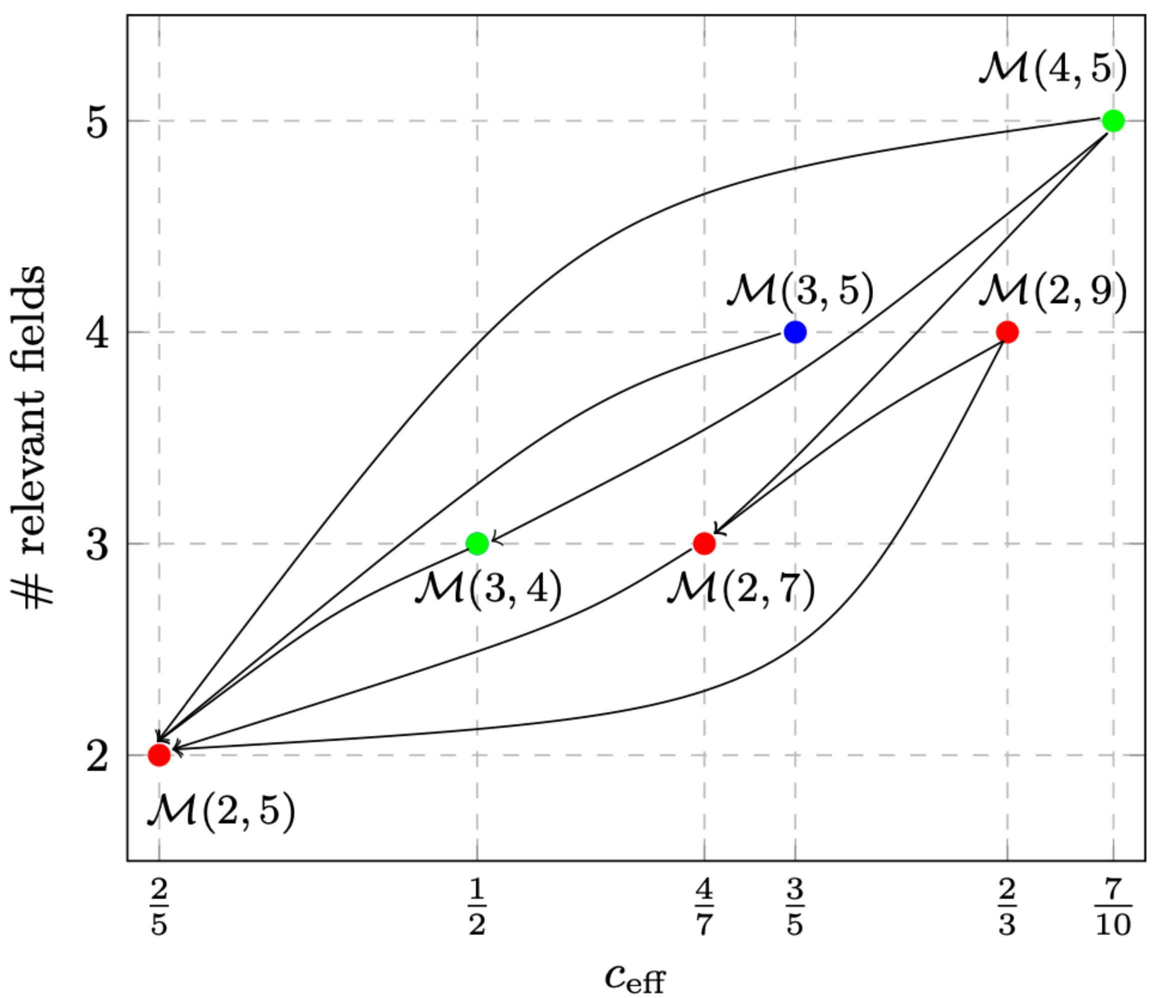
- Landau-Ginzburg effective action

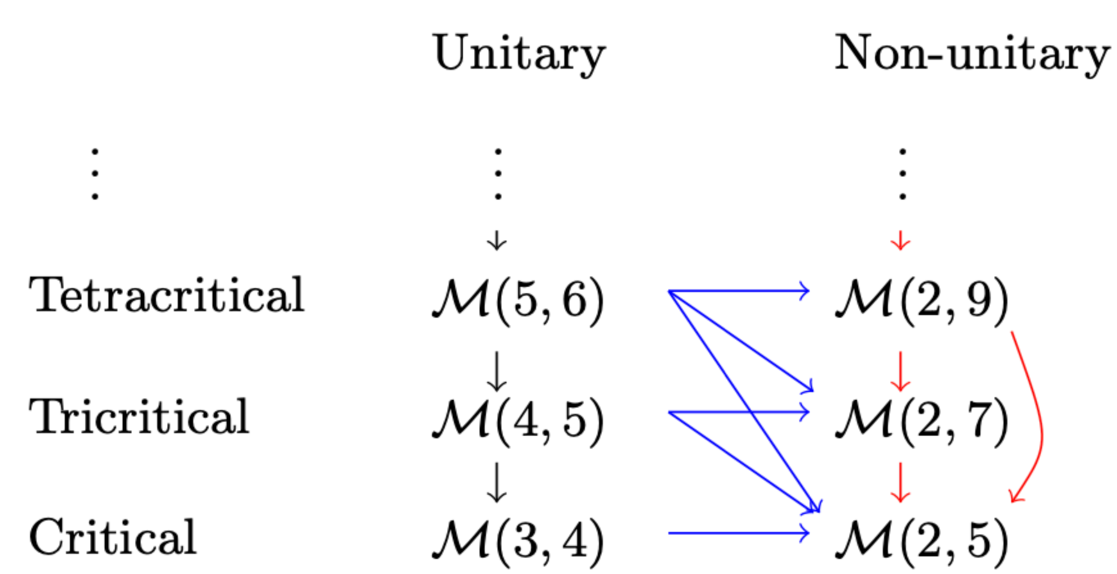
  $$\mathcal{L}_{MF} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho + \gamma'\rho^{2+1/n}$$

  Non-unitary due to imaginary coupling constant

2D minimal models

RG flows between minimal models





Lencsés-Miscioscia-Mussardo-Takács, 2023

2D minimal models

- Belavin–Polyakov–Zamolodchikov differential equation
- Example: null state at level-2

$$\left\{\mathcal{L}_{-2} - \frac{3}{2(2h+1)}\mathcal{L}_{-1}^2\right\}\langle\phi(z)X\rangle = 0$$

More explicitly

$$\left\{\sum_{i=1}^{N}\left[\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{(z-z_i)^2}\right] - \frac{3}{2(2h+1)}\frac{\partial^2}{\partial z^2}\right\}\langle\phi_{(2,1)}(z)\phi_1(z_1)\phi_2(z_2)\cdots\rangle = 0$$

2D minimal models

- The null state condition

  1. Fix the scaling dimensions, central charge

  2. Restrict the possible intermediate states in OPE

  3. Lead to differential equations for correlation functions

The null bootstrap
1. Hamiltonian

Hamiltonian Bootstrap

- Hamiltonian eigenstates

$$\langle\psi_E|\mathcal{O}H|\psi_E\rangle = E\langle\psi_E|\mathcal{O}|\psi_E\rangle = \langle\psi_E|H\mathcal{O}|\psi_E\rangle$$

- Bootstrap??

  given an explicit Hamiltonian
  determine the observables using consistency relations
  without knowing the wave functions

- Classify and solve the representations of operator algebra
  state = linear functional = representation (GNS construction)

Hamiltonian Bootstrap

- For quartic anharmonic oscillator

Han-Hartnoll-Kruthoff, 2020

$$H = p^2 + x^2 + gx^4$$

- Expectation values are not arbitrary

$$4tE\langle x^{t-1}\rangle + t(t-1)(t-2)\langle x^{t-3}\rangle$$
$$-4(t+1)\langle x^{t+1}\rangle - 4g(t+2)\langle x^{t+3}\rangle = 0$$

E is energy

- Probability is non-negative

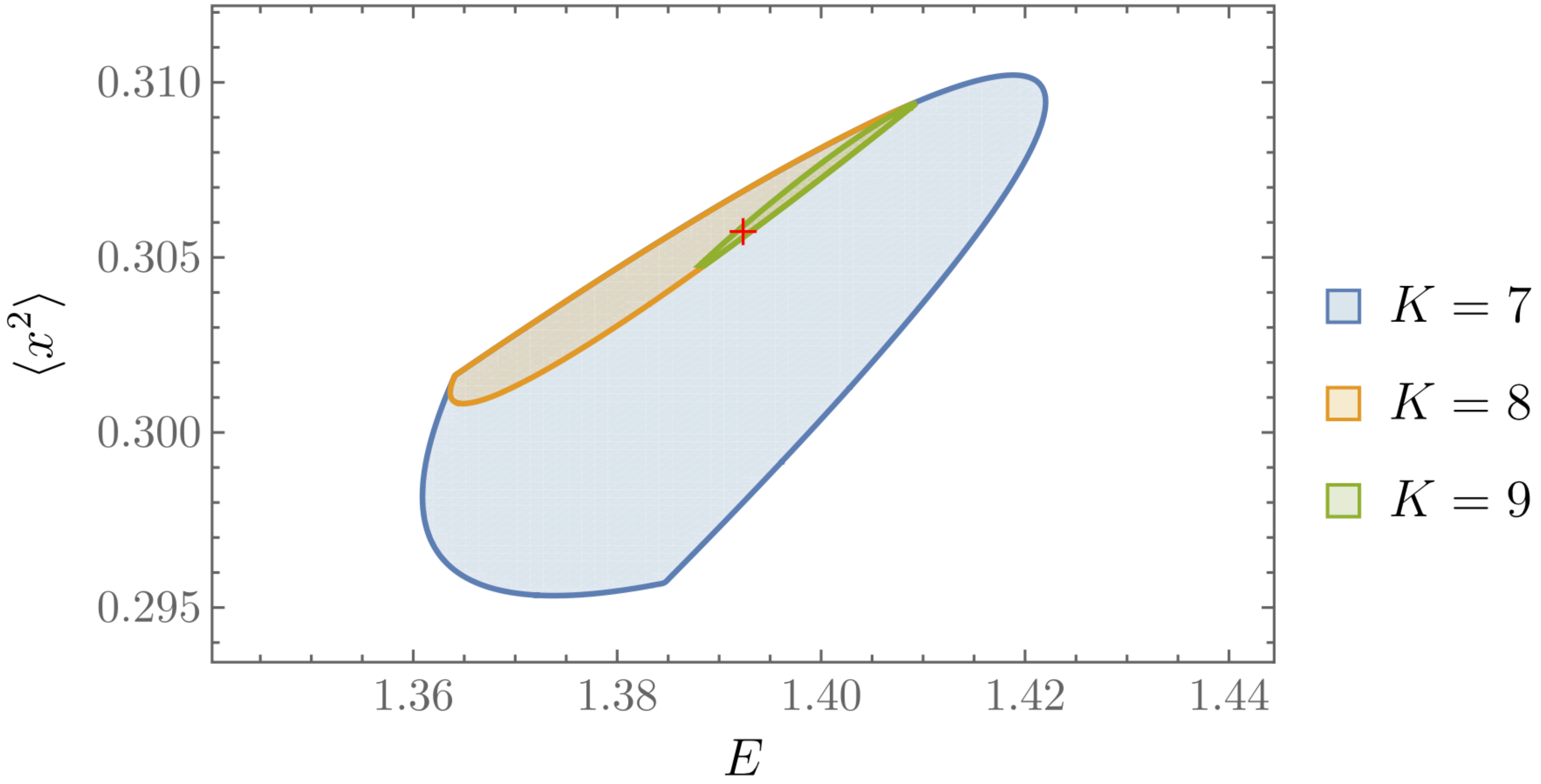
$$\langle \mathcal{O}^\dagger \mathcal{O}\rangle \geq 0, \quad \forall \ \mathcal{O} = \sum_{i=0}^{K} c_i x^i$$

Positivity constraints

- Positive semidefinite matrix

Han-Hartnoll-Kruthoff, 2020

$$\mathcal{M}_{ij} = \langle x^{i+j} \rangle \qquad \langle \mathcal{O}^\dagger \mathcal{O} \rangle \geq 0, \quad \forall \;\; \mathcal{O} = \sum_{i=0}^{K} c_i x^i$$



Beyond Hermitian?

- More QM bootstrap based on positivity

  Berenstein-Hulsey, Bhattacharya-Das-Das-Jha-Kundu, Aikawa-Morita-Yoshimura, Tchoumakov-Florens, Du-Huang-Zeng, Lawrence, Bai, Nakayama, Khan-Agarwal-Tripathy-Jain, Blacker-Bhattacharyya-Banerjee, Nancarrow-Yin, Lin, ...

- Can we solve the QM bootstrap without using positivity?

- Why? Non-Hermitian physics is also rich and interesting

  Yang-Lee edge singularity, Gribov's Reggeon field theory, open system, ultracold atoms, non-Hermitian band theory (exceptional points/lines, non-Hermitian skin effect) ...

- PT symmetric non-Hermitian theory has a real spectrum

  Bender-Boettcher, 1998

How to bootstrap without positivity?

- Harmonic oscillator $$H = p^2 + x^2$$

- Solve the energy spectrum $$\langle \psi_E | \mathcal{O} H | \psi_E \rangle = E \langle \psi_E | \mathcal{O} | \psi_E \rangle = \langle \psi_E | H \mathcal{O} | \psi_E \rangle$$

$$\langle \psi_{\text{test}} | (H - E_k) | \psi_k \rangle = \langle \mathcal{O}_{\text{test}} (H - E_k) L_k \rangle_E = 0$$

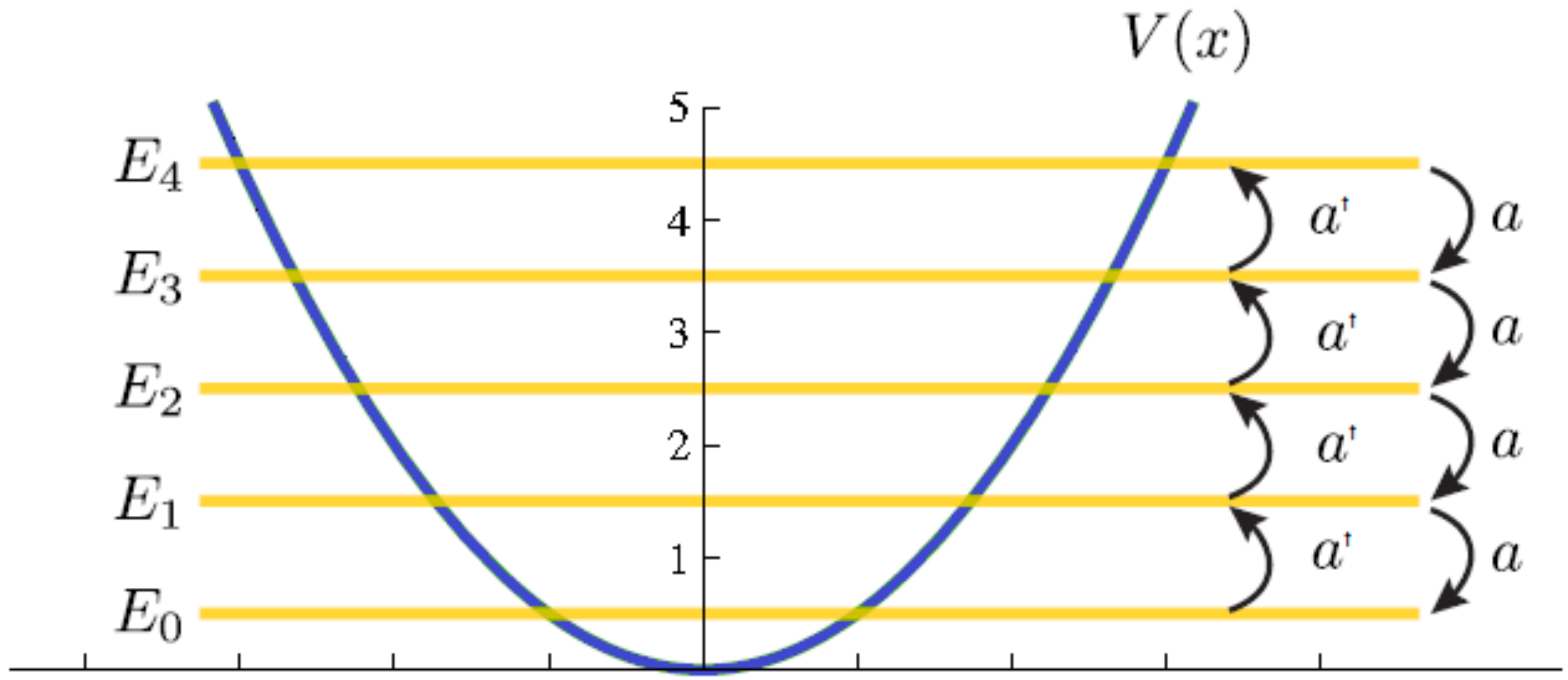
- Energy spectrum $$E_k = E + 2k$$

- Lowering operator $$L_{-n} = (x + ip)^n$$

- Underdetermined system: E is a free parameter

Stability

- Stability: energy should be bounded from below



- Ground state should be annihilated by lowering operator

- Highest weight representation of operator algebra

Null state condition

- A null state should have zero-norm

$$\langle \psi_E | \mathcal{O} H | \psi_E \rangle = E \langle \psi_E | \mathcal{O} | \psi_E \rangle = \langle \psi_E | H \mathcal{O} | \psi_E \rangle$$

⬇

$$\langle (x - ip)^n (x + ip)^n \rangle_E = \langle 1 \rangle_E \prod_{k=0}^{n-1} (E - 2k - 1)$$

- The null state condition gives

$$E_n = 2n+1 \text{ with } n = 0, 1, 2, \cdots$$

- Not using any positivity constraint

Operator algebra perspective of the null bootstrap

Below, we will set $\hbar$ to one. Mathematically, a representation of an abstract operator algebra can be induced by a state

$$\rho : \quad \mathcal{A} \to \mathbb{C} \,, \tag{1.2}$$

which is a linear functional mapping the elements of the operator algebra to complex numbers. Then one may construct the space of states as a representation of $\mathcal{A}$ on $\mathcal{H}$

$$\pi : \quad \mathcal{A} \to \mathrm{End}(\mathcal{H}) \,, \tag{1.3}$$

and show the existence of a vector $\psi_\rho \in \mathcal{H}$ with

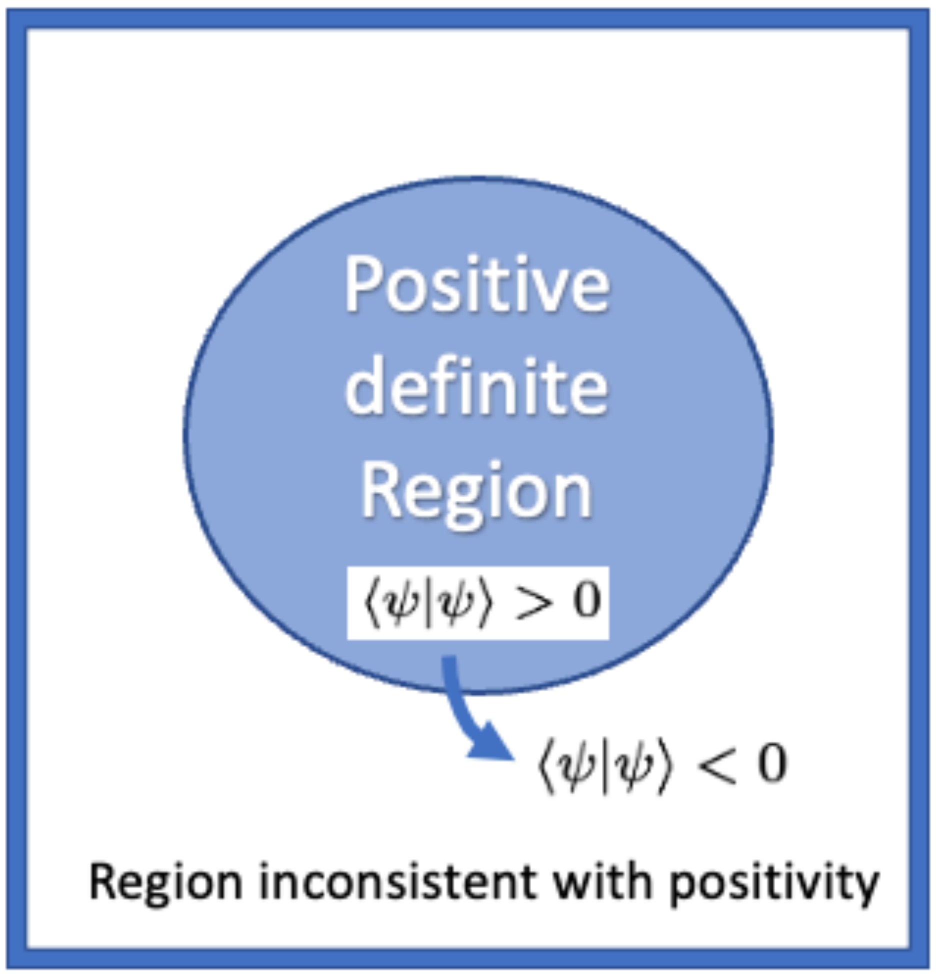
$$\rho(A) = \langle \psi_\rho | A | \psi_\rho \rangle := \langle \psi_\rho, \pi(A)\, \psi_\rho \rangle \,, \tag{1.4}$$

for all $A \in \mathcal{A}$. Typically, $\mathcal{H}$ is a quotient vector space

$$\mathcal{H} := \mathcal{A}/N \,, \tag{1.5}$$

where N is a left ideal in $\mathcal{A}$, corresponding to the subspace of null states. The null subspace plays a crucial role in the null bootstrap program [32], which aims to classify physical solutions and extracts concrete predictions by the null states. From the algebraic perspective, this can be viewed as a classification program based on the ideals in operator algebra.

The positive bootstrap vs **The null bootstrap**



$$\langle\psi|\psi\rangle_{\text{boundary}} = 0$$

↓ **Hermitian Hamiltonian**

$$|\psi\rangle_{\text{boundary}} = 0$$

Null constraints for the bootstrap

$$\langle\psi|\psi\rangle = \sum_n \langle\psi|\phi_n\rangle\langle\phi_n|\psi\rangle = \sum_n |\langle\phi_n|\psi\rangle|^2 = 0$$

η minimization

- Finite-dimensional search space

- Overdetermined system
  more null constraints than free parameters

- Measure the violation of the null state condition

$$\eta = \sqrt{\sum_{m=0}^{L} \sum_{n=0}^{L-m} \left| \frac{1}{m!n!} \frac{\partial \langle \psi_{\text{test}}^{(L)} | \psi_{\text{null}}^{(K)} \rangle}{\partial b_{mn}} \right|^2}$$

least square

η minimization in conformal bootstrap

- Truncation approach (Gliozzi, 2013, PRL)

- Minimize the errors in the crossing constraints (Li, 2017)

- AI minimization: reinforcement learning

  Ko̧ntor-Niarchos-Papageorgakis, 2021
  (PRL, Editors' suggestion)

- Stochastic minimization: Monte Carlo method

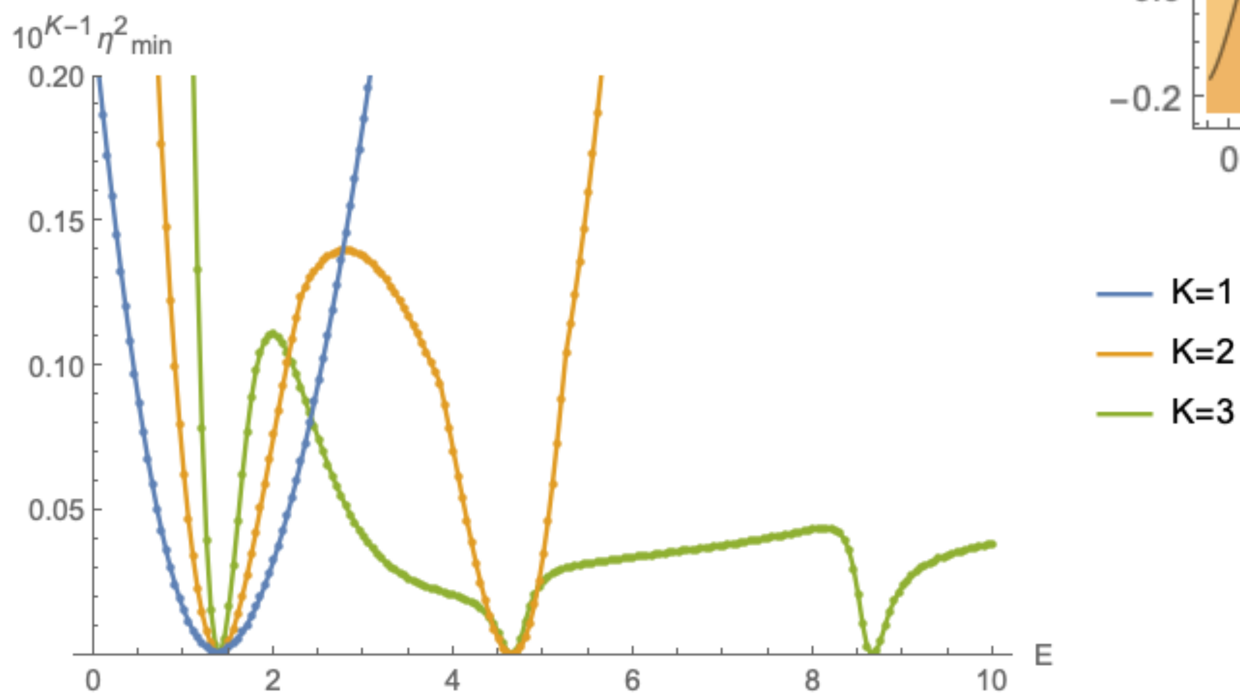
  Laio-Valenzuela-Serone, 2022

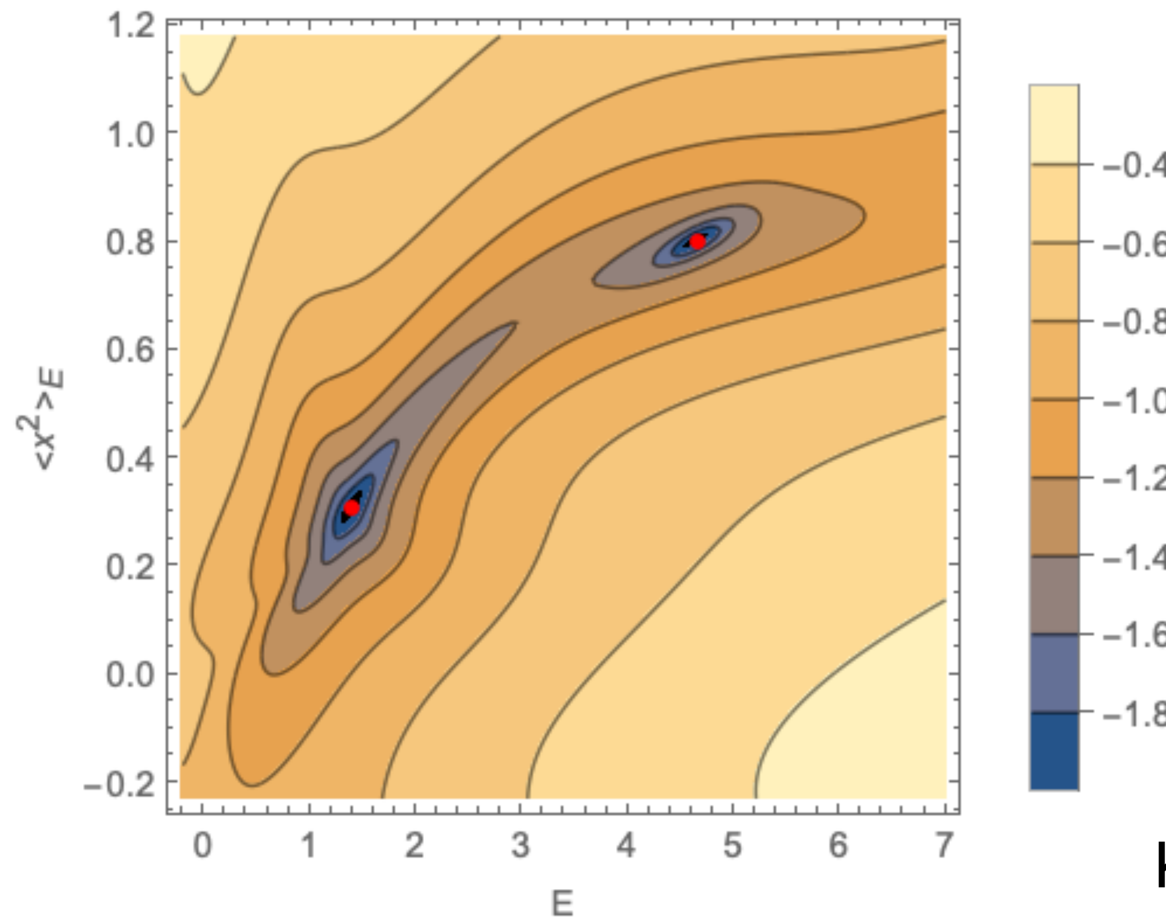
Quartic theory with η minimization

$$|\psi^{(K)}_{\text{null}}\rangle = \sum_{m=0}^{K} \sum_{n=0}^{K-m} a_{mn}\, x^m (ip)^n |\psi_E\rangle$$

$$\langle\psi^{(L)}_{\text{test}}| = \sum_{m=0}^{L} \sum_{n=0}^{L-m} b_{mn}\, \langle\psi_E| x^m (ip)^n$$

L=K+2





K=2

High precision results

$\Delta E_n^{(K)}$	$K=1$	$K=2$	$K=3$	$K=4$
$n=0$	-1×10^{-3}	-2×10^{-3}	-4×10^{-10}	-7×10^{-12}
$n=1$		3×10^{-3}	-3×10^{-5}	2×10^{-11}
$n=2$			5×10^{-6}	6×10^{-7}
$n=3$				1×10^{-7}

$E_0 = 1.39235164153029...$

$\Delta\langle x^2\rangle_n^{(K)}$	$K=1$	$K=2$	$K=3$	$K=4$
$n=0$	-1×10^{-2}	-1×10^{-4}	2×10^{-9}	1×10^{-11}
$n=1$		1×10^{-3}	-1×10^{-6}	1×10^{-11}
$n=2$			-3×10^{-6}	6×10^{-8}
$n=3$				2×10^{-8}

$\Delta\langle 0\|x^m\|n\rangle^{(M)}$	$m=1$	$m=2$	$m=3$	$m=4$
$n=1, M=1$	4×10^{-6}		1×10^{-5}	
$n=2, M=2$		3×10^{-7}		1×10^{-6}
$n=1, M=3$	1×10^{-11}		2×10^{-11}	

Beyond Hermitian

- Hamiltonian eigenstates satisfy consistency relations

$$\langle \psi_E | \mathcal{O} H | \psi_E \rangle = E \langle \psi_E | \mathcal{O} | \psi_E \rangle = \langle \psi_E | H \mathcal{O} | \psi_E \rangle$$

- Inner product

1. Hermitian Hamiltonian

$$\langle \psi_1 | \psi_2 \rangle^{\mathcal{H}} = \int \mathrm{d}x [\psi_1(x)]^* \psi_2(x)$$

2. Non-Hermitian Hamiltonian (PT symmetric)

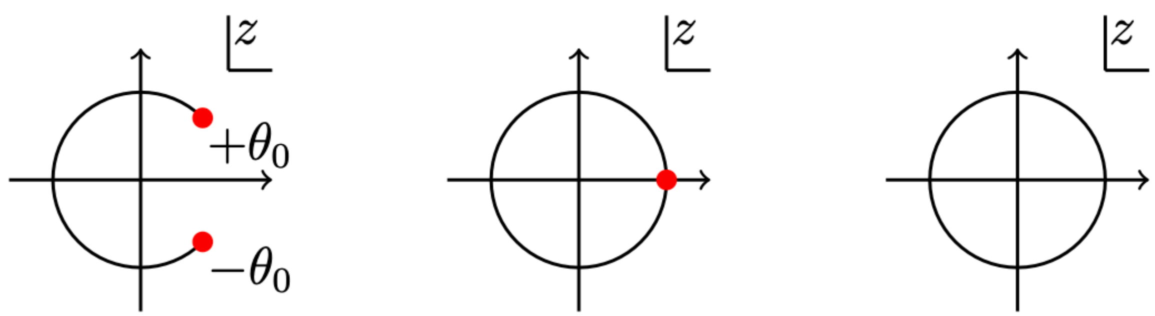
$$\langle \psi_1 | \psi_2 \rangle^{\mathcal{PT}} = C \int \mathrm{d}x [\psi_1(-x)]^* \psi_2(x)$$

Non-Hermitian PT theory

- Yang-Lee edge singularity

Yang-Lee, Kortman-Griffiths, Fisher, Cardy, ...

$$z = e^{-\beta h} = e^{i\theta}$$



$T > T_c$ $T = T_c$ $T < T_c$

distribution of the Yang-Lee zeros for the Ising partition function

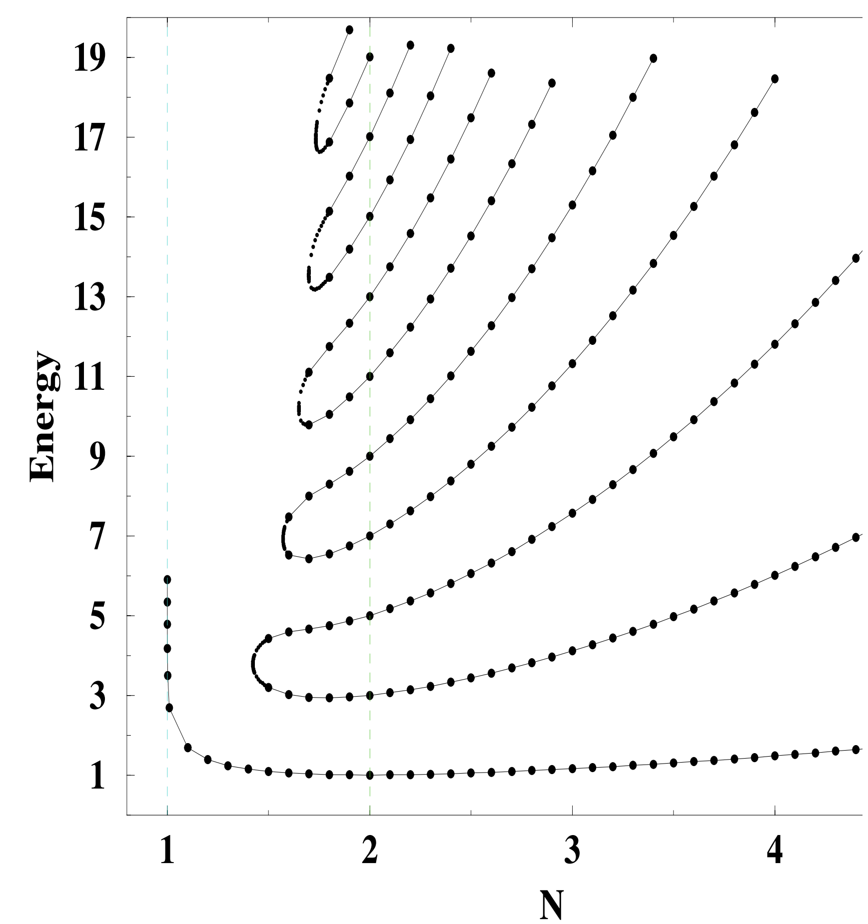
$$\mathcal{L}_{YL} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi + (h - ih_c)\varphi + i\gamma\varphi^3 + \ldots$$

- PT symmetry

$$H = p^2 - (ix)^N$$

real and bounded spectrum

Bender-Boettcher, 1998



Non-Hermitian cubic theory

- Hamiltonian

$$H = p^2 + ix^3$$

- Results

$E_0 = 1.156267071988...$

$\Delta E_n^{(K)}$	$K=1$	$K=2$	$K=3$
$n=0$	4×10^{-4}	-8×10^{-7}	1×10^{-11}
$n=1$		2×10^{-3}	-3×10^{-9}
$n=2$			-1×10^{-4}

$\langle x\rangle_0 = -0.590072533091i$

$\Delta\langle x\rangle_n^{(K)}$	$K=1$	$K=2$	$K=3$
$n=0$	$-3\times 10^{-2}i$	$2\times 10^{-6}i$	$-2\times 10^{-11}i$
$n=1$		$-8\times 10^{-4}i$	$1\times 10^{-8}i$
$n=2$			$2\times 10^{-6}i$

The null bootstrap
2. Lagrangian

Dyson-Schwinger

- Path integral

$$Z[J] = \int \mathcal{D}\phi \, e^{-S[\phi] + \int d^D x J(x)\phi(x)}$$

- Green's function

$$G_n(x_1, \dots, x_n) \equiv \langle T\{\phi(x_1) \dots \phi(x_n)\}\rangle$$

- Quantum equation of motion

$$\langle \delta S[\phi]/\delta\phi(x)\rangle = \langle J(x)\rangle$$

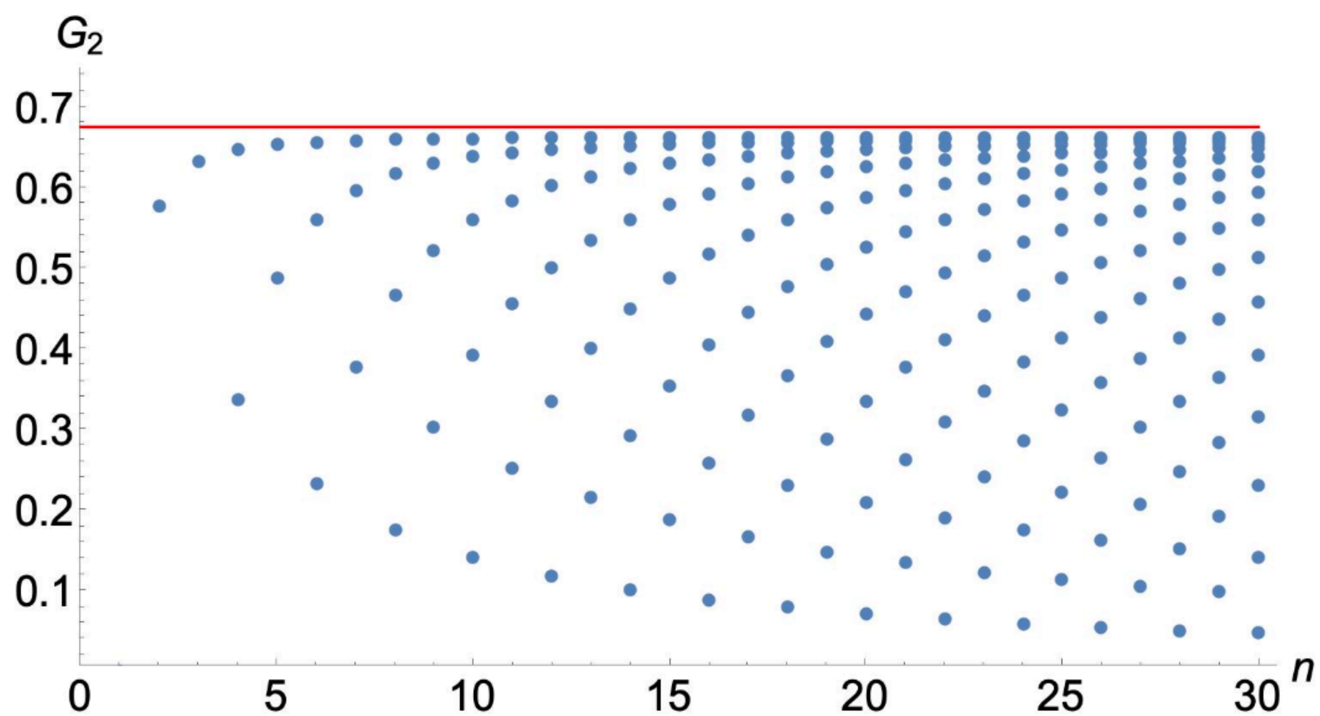
- Dyson-Schwinger equations: take J derivatives and then J=0

example $$\langle \phi(x_1) \, \delta S[\phi]/\delta\phi(x_2)\rangle = \delta(x_1 - x_2)$$

Quartic theory

- Usual approach
  1. A finite set of DS equations (underdetermined)
  2. Set high-point connected Green's functions to zero
  3. Solve the finite system

- However, this does gives the correct answer! $\mathcal{L}(\phi) = \frac{1}{4}\phi^4$



D=0, quartic theory

$$G_4 = -3G_2^2 + 1$$
$$G_6 = -12G_2G_4 - 6G_2^3$$

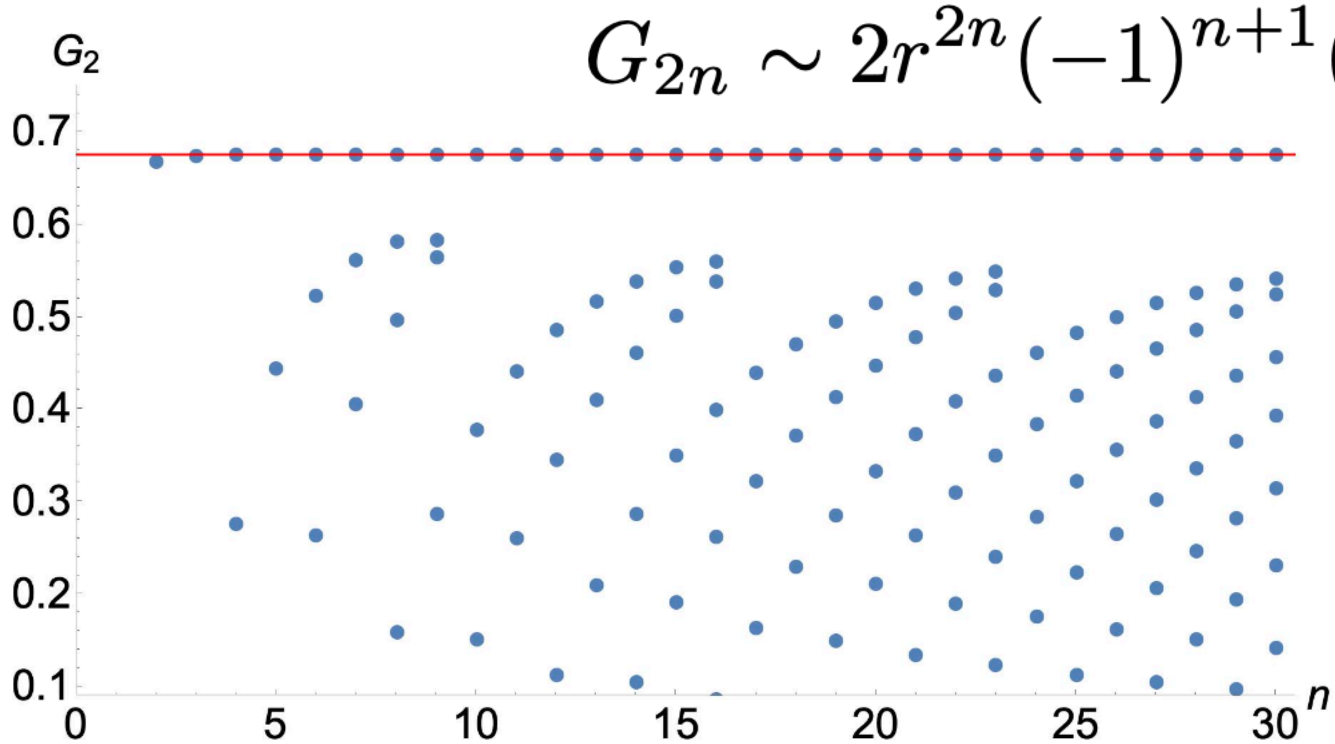
Bender-Karapoulitidis-Klevansky, 2022

Quartic theory

- Usual approach
  1. A finite set of DS equations (underdetermined)
  2. Set high-point connected Green's function to zero
  3. Solve the finite system

- Solution: use asymptotic behaviour at large n (# of points)

$$G_{2n} \sim 2r^{2n}(-1)^{n+1}(2n-1)! \quad (n \to \infty)$$



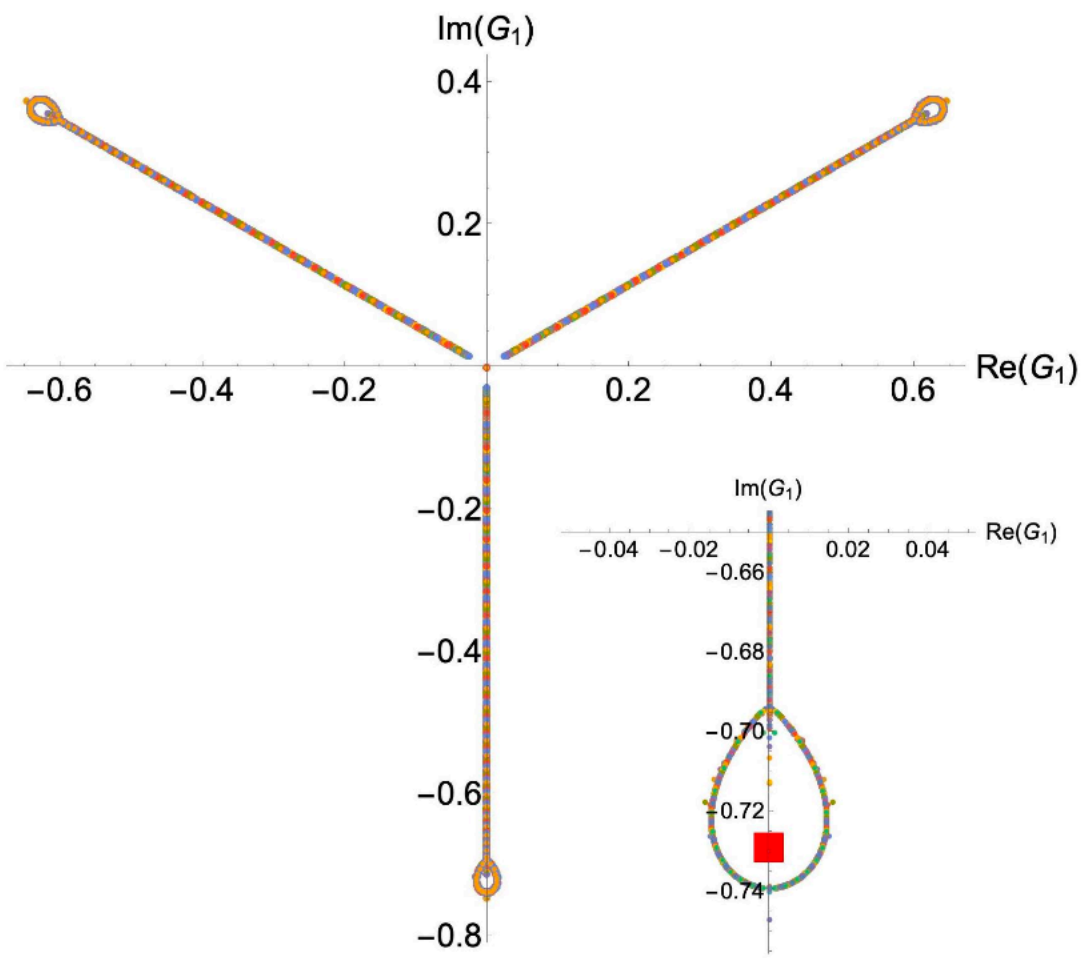
D=0, quartic theory

Bender-Karapoulitidis-Klevansky, 2022

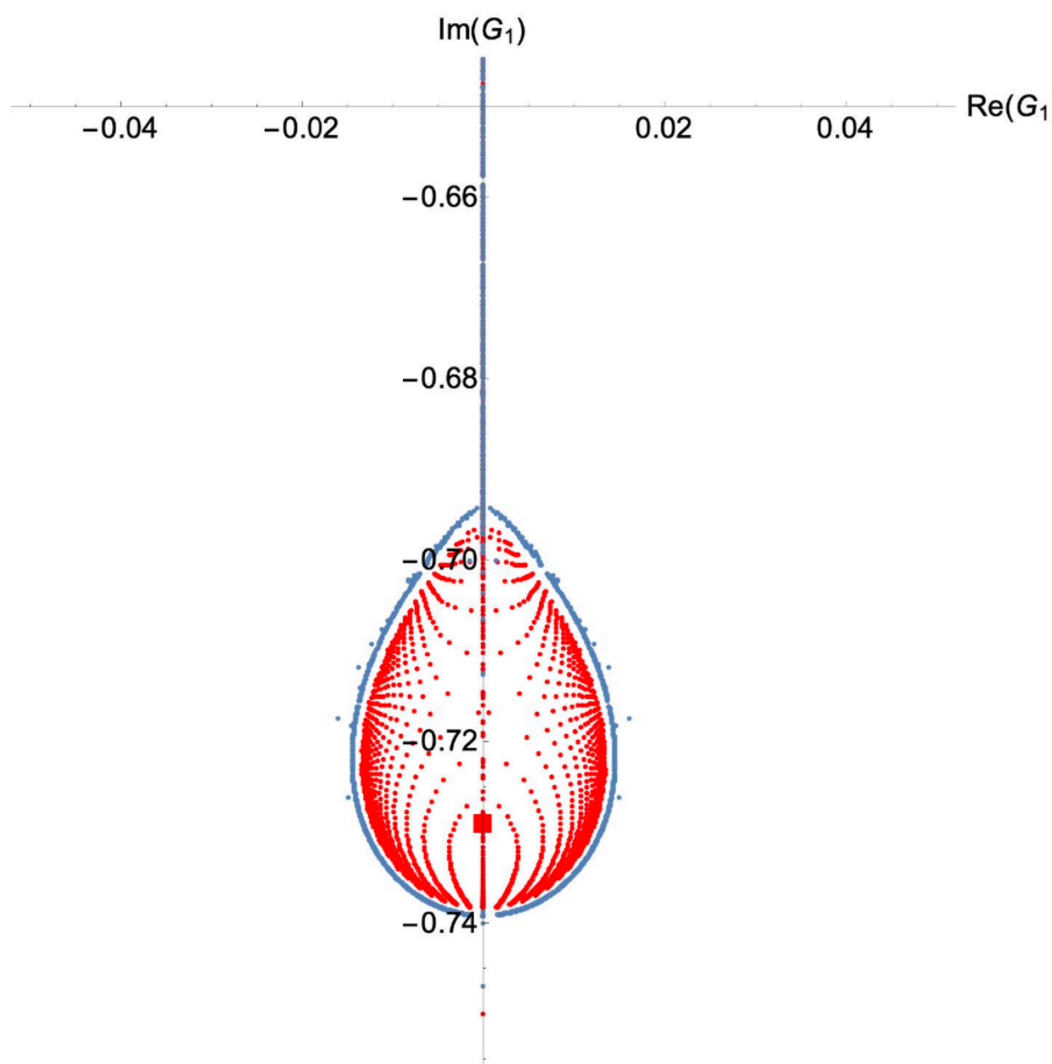
Cubic theory

$$\mathcal{L} = \frac{1}{3} i\phi^3$$

Bender-Karapoulitidis-Klevansky, 2022

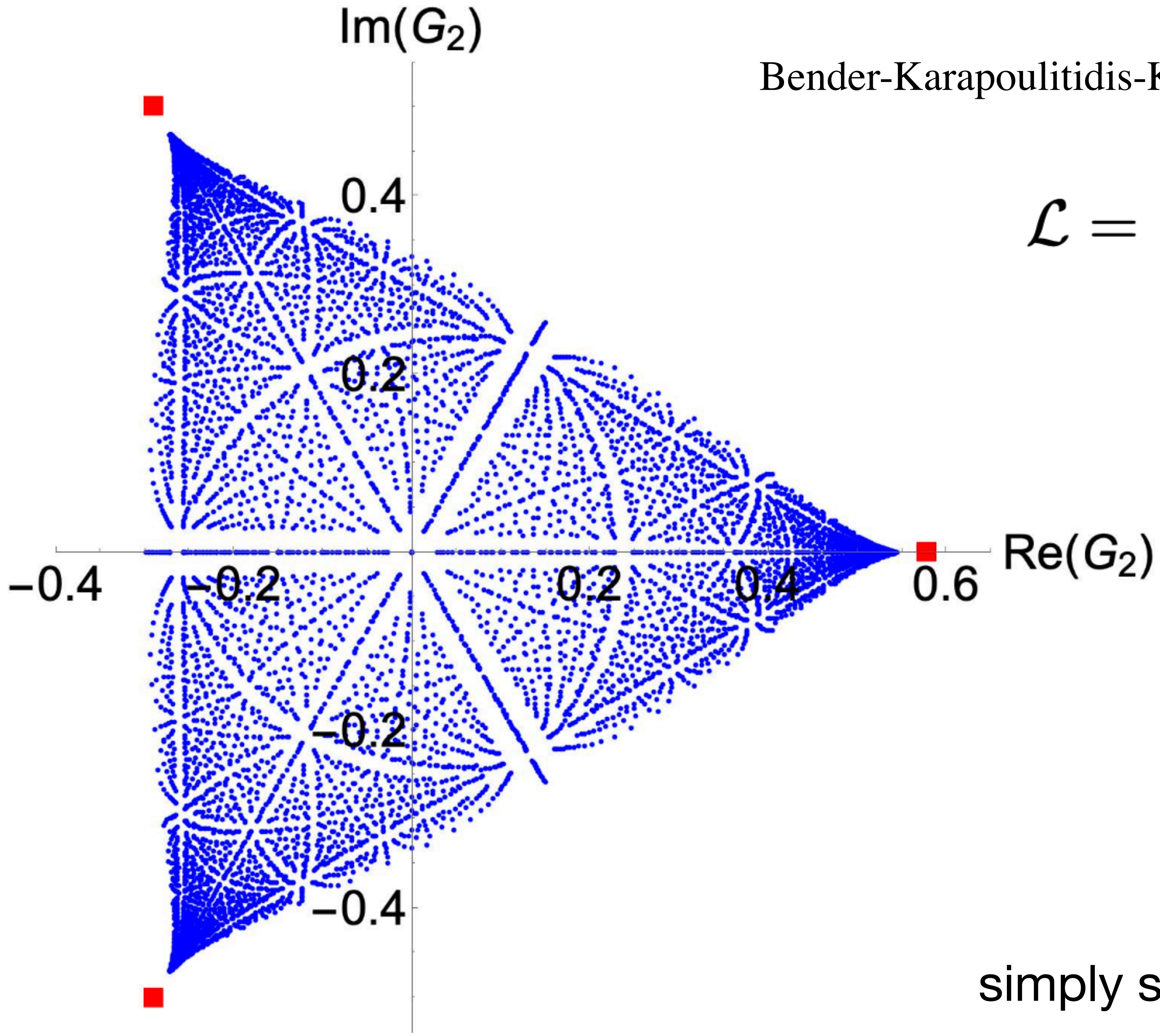


simply set to zero



use large-n asymptotic behaviour

Sextic



Bender-Karapoulitidis-Klevansky, 2022

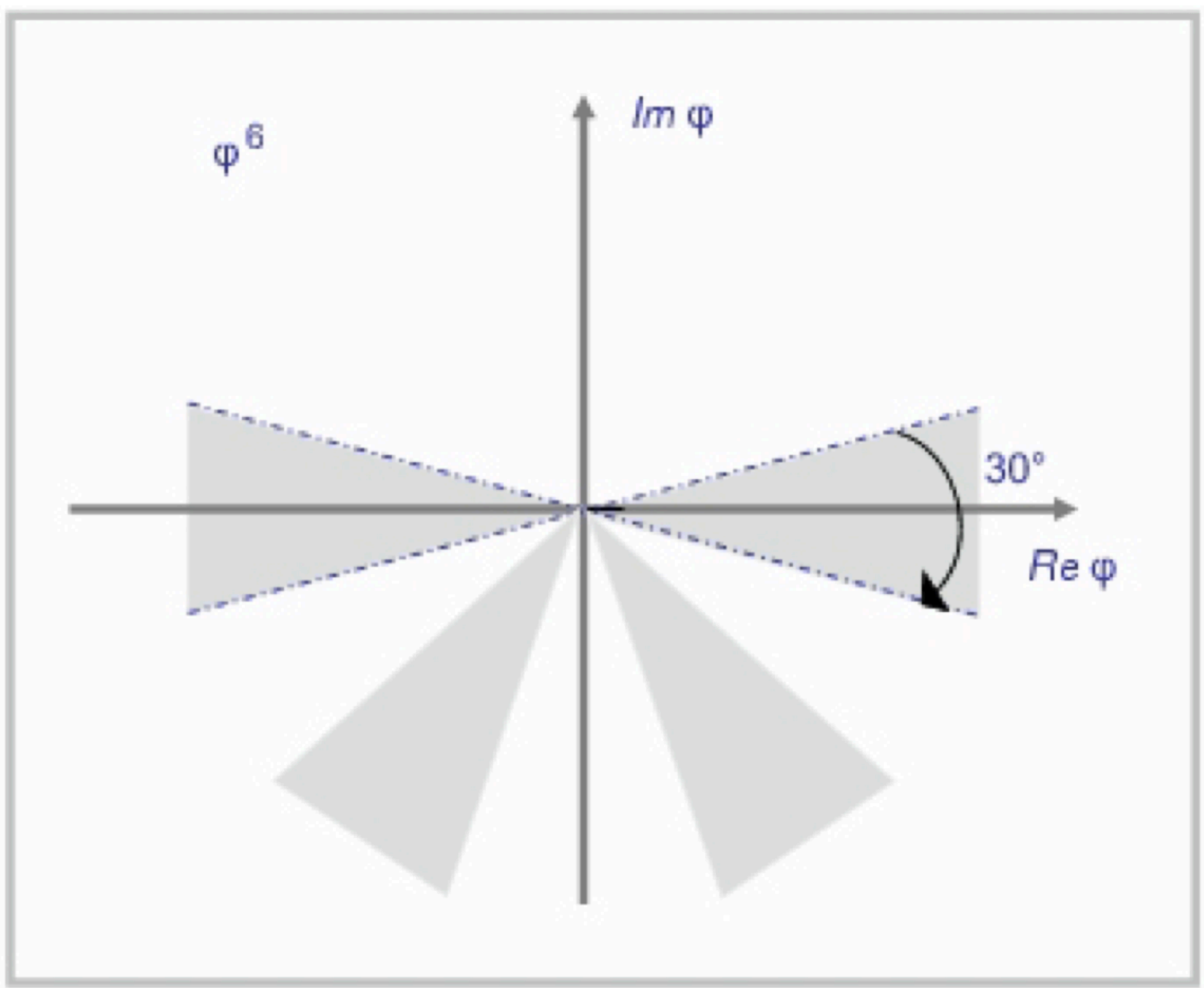
$$\mathcal{L} = \frac{1}{6}\phi^6$$

simply set to zero

Stokes sectors

- Different integration paths give different results

$$\mathcal{L} = \frac{1}{6}\phi^6$$



1. $(0^\circ\ ,\ 180^\circ)$
2. $(-60^\circ\ ,\ -120^\circ)$
3. $(60^\circ\ ,\ 120^\circ)$

Bender-Klevansky, 2010

Null state condition

- DS equations is not sensitive to the choice of Stokes sectors

- Add a quantization condition
  1. boundary condition or asymptotic behaviour
  2. unitarity/positivity constraints (Hermitian solution)
  3. null state condition

  -> determined system

Quartic theory

- Lagrangian $$\mathcal{L} = \frac{1}{2}(\dot{\phi})^2 + \frac{1}{2}\phi^2 + \frac{1}{2}\phi^4$$
- DS equations

$$\left(-\partial_{\tau_1}^2 + 1\right) G_n(\tau_1, \tau_2, \ldots) + 2G_{n+2}(\tau_1, \tau_1, \tau_1, \tau_2, \ldots)$$
$$= \sum_{i=2}^{n} \delta(\tau_1 - \tau_i)\, G_{n-2}(\tau_2, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots)$$

- Independent parameters in the equal-time limit

$$F_n = \partial_{\tau_2}^n G_2(\tau_1, \tau_2)\Big|_{\tau_1 \to \tau_2 + 0^+} = \left\langle \phi(\tau)\, \frac{d^n \phi(\tau)}{d\tau^n} \right\rangle$$

Quartic theory

- The composite operators are

$$\langle\phi\dot{\phi}\rangle = \frac{1}{2}\,, \quad \langle(\dot{\phi})^2\rangle = -F_2\,, \quad \langle\phi^4\rangle = -\frac{F_0}{2} + \frac{F_2}{2}\,,$$

$$\langle\phi^3\dot{\phi}\rangle = \frac{3F_0}{2}\,, \quad \langle\phi^2(\dot{\phi})^2\rangle = \frac{1}{2} + \frac{F_2}{6} - \frac{F_4}{6}$$

- Null state condition $\langle \text{test}^{(L)} | \text{null}^{(K)} \rangle = \langle \mathcal{O}^{(L)}_{\text{test}} \mathcal{O}^{(K)}_{\text{null}} \rangle = 0$

$$\mathcal{O}^{(K)}_{\text{null}} = \sum_{m=0}^{K} a_m \frac{d^m\phi(\tau)}{d\tau^m}\,, \quad \mathcal{O}^{(L)}_{\text{test}} = \sum_{m=0}^{L} b_m \frac{d^m\phi(\tau)}{d\tau^m}$$

L=2K

Quartic theory

- For K=1

$$\{1, \partial_{\tau_1}, \partial^2_{\tau_1}\}\langle\phi(\tau_1)\mathcal{O}^{(K)}_{\text{null}}(\tau_2)\rangle|_{\tau_1\to\tau_2} = 0$$

$$\downarrow$$

$$\left\{\frac{a_1}{2a_0} + F_0\,,\ \frac{1}{2} + \frac{a_1}{a_0}F_2\,,\ \frac{a_1}{2a_0} + \frac{3a_1}{a_0}F_0 + F_2\right\} = 0$$

This implies $F_0 = \langle\phi^2\rangle$ is a root of $24x^3 + 4x^2 - 1$

real root at x = 0.2991; exact value $\langle\phi^2\rangle = 0.30581365$

Quartic theory

- The null state condition for unequal time $\mathcal{O}_{\text{null}}^{(K)} = \sum_{m=0}^{K} a_m \frac{d^m \phi(\tau)}{d\tau^m}$

$$(a_0 + a_1 \partial_{\tau_2})\, G_2^{(K=1)}(\tau_1, \tau_2) = 0$$

The solution is

$$G_2^{(K=1)}(\tau_1, \tau_2) = c_1 e^{\frac{a_0}{a_1}|\tau_1 - \tau_2|}$$

with

$$-a_0/a_1 \; = \; 1.6717\ldots.$$

- Exact energy gap $E_{\text{gap}} \; = \; E_1 \, - \, E_0 \; = \; 1.62823$

Quartic theory

- Roots of the "null polynomial" $\sum_{m=0}^{K} a_m \, x^m$

  encode the energies of the intermediate states $E_m - E_0$

$$G_2^{(K)}(\tau_1, \tau_2) = \sum_{m=1}^{K} c_m \, e^{-\Delta E_m |\tau_1 - \tau_2|}$$

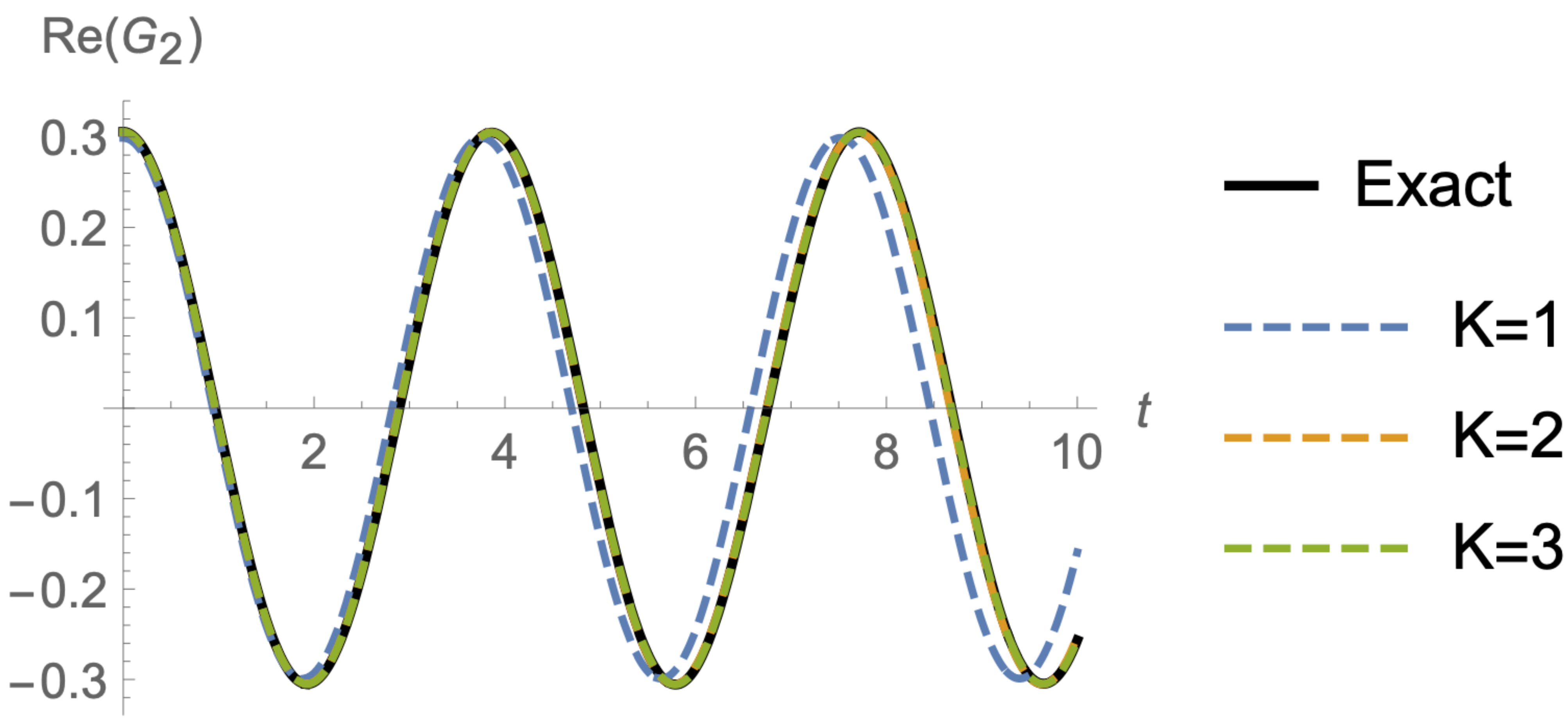
  the coefficients are associated with $\langle n|\phi|0\rangle$

$$\mathcal{O}_{\text{null}}^{(K)} = \sum_{m=0}^{K} a_m \frac{d^m \phi(\tau)}{d\tau^m}$$

- For a bounded-from-below spectrum,
  all roots should are positive.
  This selects a unique solution to the polynomial system!

Quartic theory

- Reconstruct the 2-point function at real time



Quartic theory

- For K=6

  ΔE = {1.628230589 . . ., 5.882239..., 10.9536..., 16.661..., 23.3..., 32.5...}

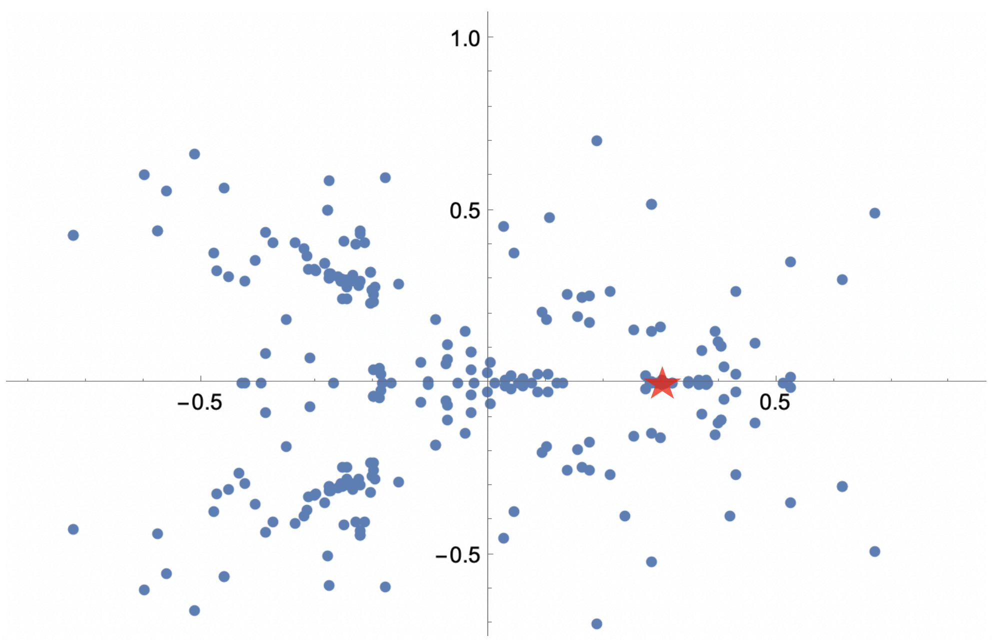
  Exact = {1.628230531..., 5.882226..., 10.9525..., 16.624..., 22.8..., 29.4... }

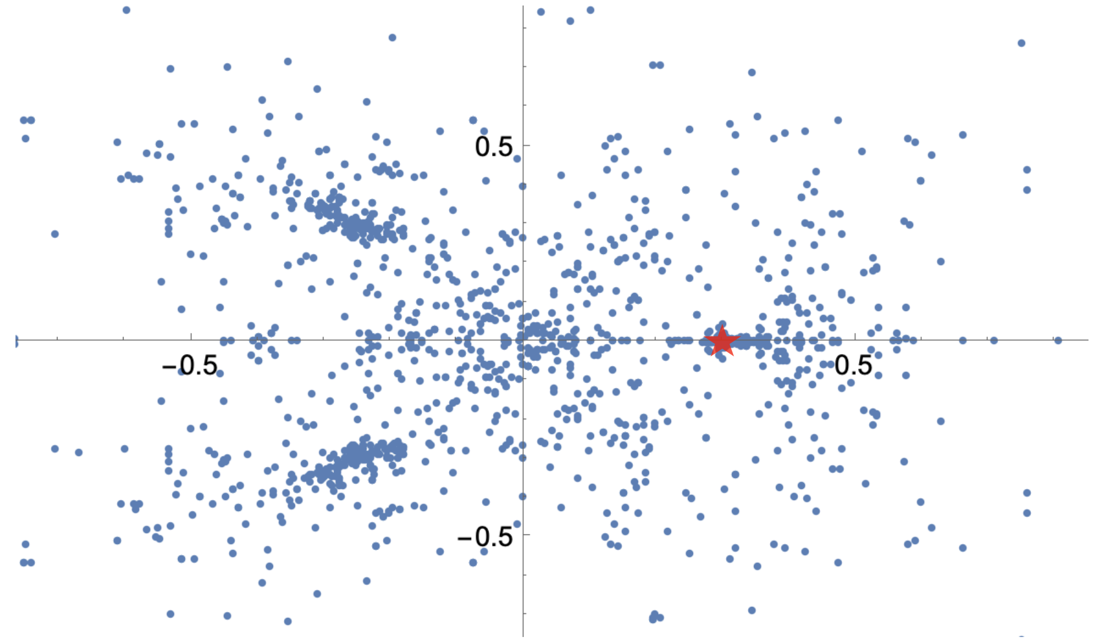
$$c_1^{1/2} = 0.5525659561\ldots \text{ and } c_2^{1/2} = 0.021994704\ldots$$

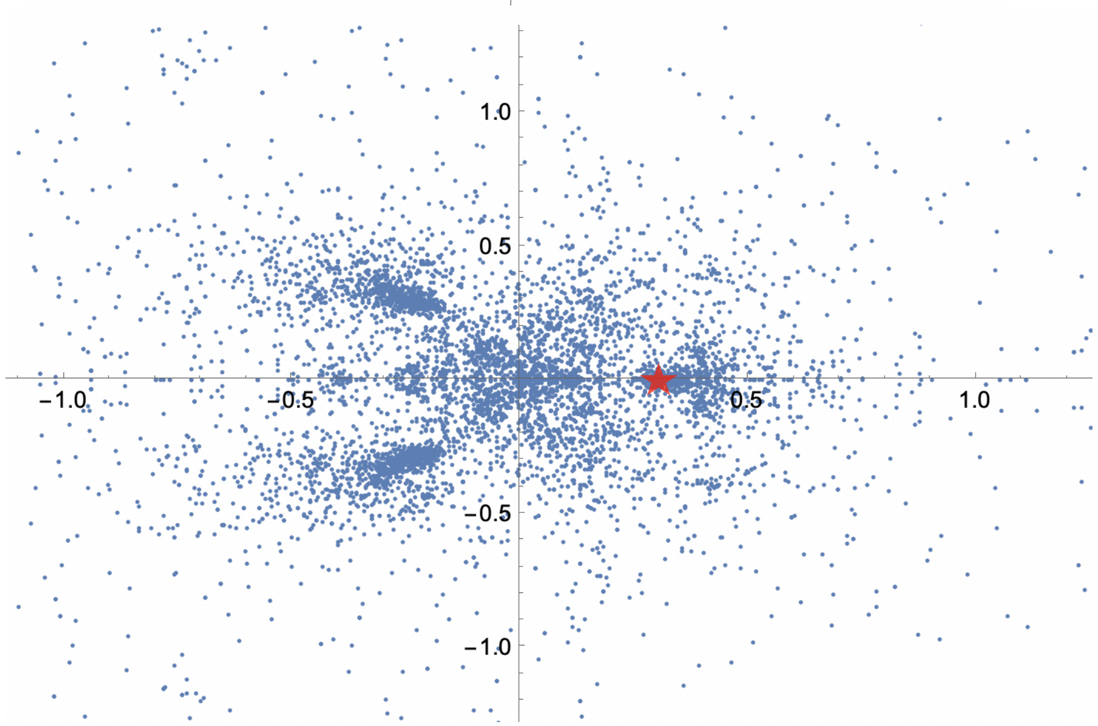
Exact: $\langle 1|\phi|0\rangle = 0.5525659593\ldots$

$$\langle 3|\phi|0\rangle = 0.021994761\ldots$$

Root accumulation
K=4,5,6







$$\langle \phi^2 \rangle = 0.30581365$$

Non-Hermitian cubic theory

- Lagrangian

$$\mathcal{L} = \frac{1}{2}(\dot{\phi})^2 + \frac{i}{2}\phi^3$$

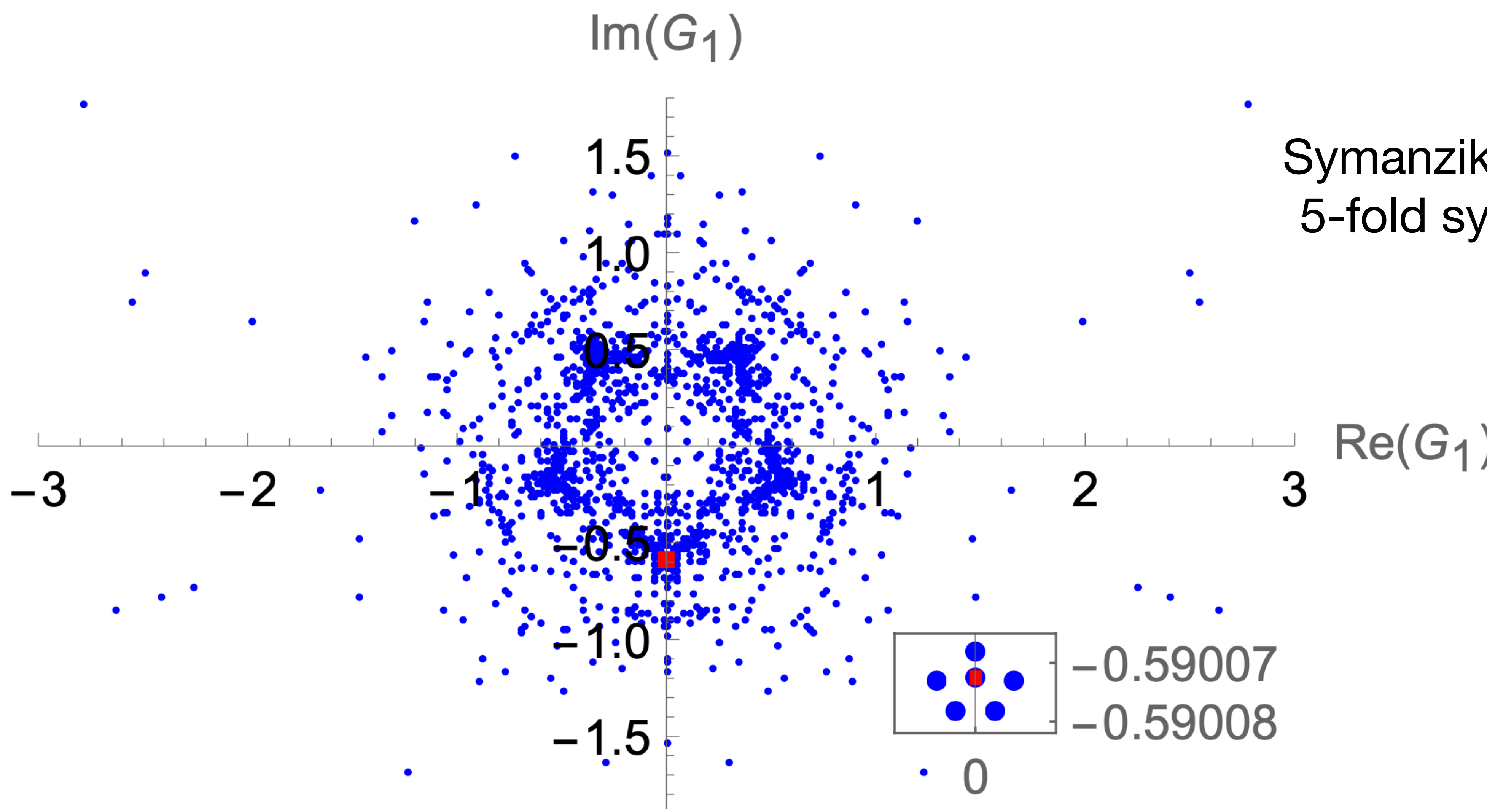
- DS equations

$$-\partial_{\tau_1}^2 G_n(\tau_1, \tau_2, \ldots) + \frac{3i}{2} G_{n+1}(\tau_1, \tau_1, \tau_2, \ldots)$$
$$= \sum_{i=2}^{n} \delta(\tau_1 - \tau_i)\, G_{n-2}(\tau_2, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots)\,.$$

- For K=6, $\langle\phi\rangle$ = –0.590072522 . . . i (exact value –0.590072533 . . . i)

  null polynomial -> intermediate spectrum
  bounded-from-below spectrum -> unique solution with positive roots

Root accumulation at K=6



Symanzik rotation
5-fold symmetry

FIG. 2. The $K = 6$ solutions for the 1D non-Hermitian $i\phi^3$ theory. The red square indicates the exact value at $G_1 = -0.5900725\ldots i$. We find 123 roots of distance less than 10^{-1} from this exact value, while $\{44, 24, 12, 6\}$ of them are of distance less than $\{10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}\}$. Inset: The 6 solutions are obtained by iteratively discarding the most distant root from the average.

Outlook

- Null state condition as a quantization condition

  Is there any connection to the resurgent WKB method (exact quantization condition)?

- Towards more degrees of freedom

  quantum many-body systems, higher dimensions, matrix models, ...
  spin chains, QED3, QCD and hadron physics

Back to CFT

- Non-Hermitian CFT
  (multi-critical) Yang-Lee edge singularity

- Complex CFT
  weakly first-order transition in statistical and condensed matter physics (deconfined quantum criticality)
  gauge theory (walking)

- Beyond relativistic CFTs

  Galilean $c \to \infty$, Carrollian $c \to 0$,
  anisotropic scaling, ...

Thank you!