$\mathcal{W}_\infty$ and integrability

Tomáš Procházka

Institute of the Physics of the Czech Academy of Sciences

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$\mathcal{W}_\infty$:

- (spherical) degenerate double affine Hecke algebra (Cherednik)
- cohomological Hall algebra (Kontsevich & Soibelman)
- Yangian of $\widehat{\mathfrak{gl}(1)}$
- quantum toroidal algebra, Ding-Iohara-Miki algebra (q-deformed)

Plan:

- review of $\mathcal{W}$-algebras and $\mathcal{W}_\infty$
- Yangian description
- R-matrix
- ILW Bethe ansatz equations

W algebras - motivation

$\mathcal{W}$-algebas: extensions of the Virasoro algebra (2d CFT) by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV/KP) $\rightsquigarrow$ $\mathcal{W}$ is quant. KP
- (old) matrix models
- instanton partition functions and AGT
- holographic dual description of 3d higher spin theories
- quantum Hall effect
- topological strings
- higher spin square (Gaberdiel, Gopakumar)
- $\mathcal{N} = 4$ SYM at junction of three codimension 1 defects (Gaiotto, Rapčák)
- geometric representation theory (equivariant cohomology of various moduli spaces)

Zamolodchikov $\mathcal{W}_3$ algebra

$\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

To close the algebra we need to find the OPE of W with itself consistent with associativity (Jacobi, crossing symmetry...).

The result:

$$
\begin{aligned}
W(z)W(w) \quad \sim \quad & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
& + \frac{1}{(z-w)^2}\left(\frac{32}{5c+22}\Lambda(w) + \frac{3}{10}\partial^2 T(w)\right) \\
& + \frac{1}{z-w}\left(\frac{16}{5c+22}\partial\Lambda(w) + \frac{1}{15}\partial^3 T(w)\right) + reg.
\end{aligned}
$$

Λ is a quasiprimary 'composite' (spin 4) field,

$$
\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T(z).
$$

The algebra is non-linear, not a Lie algebra in the usual sense (linearity should not be expected for spins ≥ 3 or non-abelian ≥ 2).

Constructions

Since even this simplest extension of Virasoro algebra looks quite complicated and the result of imposing Jacobi identities is rather non-trivial, the question is whether one can find other ways of systematically producing $\mathcal{W}$-algebras, say starting from affine Lie algebras. Well-known procedures:

- Hamiltonian reduction (Drinfeld-Sokolov reduction) - impose certain Hamiltonian constraints in affine Lie algebra, either classically or using BRST on quantum level
- Goddard-Kent-Olive coset construction

$$\frac{\mathfrak{su}(n)_k \times \mathfrak{su}(n)_l}{\mathfrak{su}(n)_{k+l}} \simeq \frac{\mathfrak{u}(k+l)_n}{\mathfrak{u}(k)_n \times \mathfrak{u}(l)_n}$$

  (for $k=1$ or $l=1$), generalizes the Casimir construction $\mathfrak{su}(n)_k/\mathfrak{su}(n)$
- free field representations (Miura transformation) - more later

Interpolating algebras

- let us focus on algebras $\mathcal{W}_N$ generated by spins $2, 3, \ldots, N$
- problem: no natural embedding $\mathcal{W}_N \subset \mathcal{W}_{N+1}$
  [reason - $(\mathfrak{su}(2) \subset \mathfrak{su}(N)) \not\subset (\mathfrak{su}(2) \subset \mathfrak{su}(N+1))$]
- solution: an interpolating algebra $\mathcal{W}_\infty$ that unifies $\mathcal{W}_N$
- construction similar to $\mathfrak{hs}(\lambda)$ interpolating between $\mathfrak{sl}(N)$
- consider an algebra of $N \times N$ matrices and embed $\mathfrak{su}(2)$ in it via N-dimensional irreducible representation. As a vector space, it decomposes into representations of spin $0, 1, \ldots, N-1$. We can choose a basis of $N \times N$ matrices $\{T^l_m\}$ such that the matrix multiplication is

$$T^{l_1}_{m_1} \star T^{l_2}_{m_2} = \sum_{|l_1-l_2| \leq l \leq l_1+l_2} C^{l_1 l_2 l}_{m_1 m_2}(N) \, T^l_{m_1+m_2}$$

- C are structure constants - *rational functions* of N

- we can therefore define the $\star$ product for any value of $\lambda \sim N$, not only positive integers (using the same rational functions)
- since the associativity conditions are algebraic, they are satisfied for any $\lambda \in \mathbb{C}$
- restricting $\lambda \to N$ (and throwing away spin $N+$) $\rightsquigarrow$ $End(N)$
- for $\lambda \to \infty$ we reduce to multiplication on S^2 and Poisson bracket, so the algebra is a quantization of S^2 (fuzzy sphere)
- the whole construction can be described compactly by

$$\mathfrak{hs}(\lambda) \leftrightarrow \frac{\mathcal{U}(\mathfrak{so}(3))}{\langle X^2 + Y^2 + Z^2 - (\lambda^2 - 1)\rangle}$$

- notice that this is exactly the geometric way of representing quantum S^2

$\mathcal{W}_\infty$ algebra

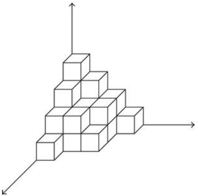
- $\mathcal{W}_\infty$: interpolating algebra for W_N series; spins $2, 3, \ldots$
- Gaberdiel-Gopakumar: solving associativity conditions for this field content $\rightsquigarrow$ two-parameter family
- parameters: central charge c and rank parameter λ
- choosing $\lambda = N \to$ truncation of $\mathcal{W}_\infty$ to $\mathcal{W}_N = \mathcal{W}[\mathfrak{sl}(N)]$, i.e. $\mathcal{W}_\infty$ is interpolating algebra for the whole $\mathcal{W}_N$ series
- adding spin 1 field, we have $\mathcal{W}_{1+\infty} \rightsquigarrow$ many simplifications
- surprise: $\mathcal{W}_{1+\infty}$ contains all $\mathcal{W}[\mathfrak{g}]$ for all simple $\mathfrak{g}$ (except possibly $\mathfrak{f}_4$?)
- **triality** symmetry of the algebra (Gaberdiel & Gopakumar) $\mathcal{W}_\infty[c, \lambda_1] \simeq \mathcal{W}_\infty[c, \lambda_2] \simeq \mathcal{W}_\infty[c, \lambda_3]$

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \qquad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$

- MacMahon function as vacuum character of the algebra (enumerating all the local fields in the algebra)

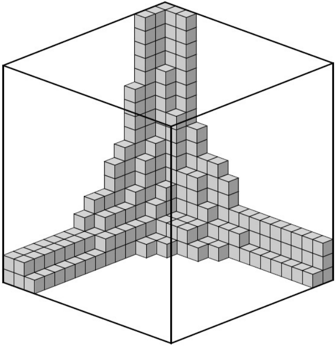
$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$

- The same generating function is well-known to count the plane partitions (3d Young diagrams)



- triality acts by permuting the coordinate axes
- restriction to $\mathcal{W}_N$ corresponds to max N boxes in one of the directions

- this can be generalized to degenerate primaries (not only vacuum rep) by allowing 2d Young diagram asymptotics
- counting exactly as in topological vertex $\rightsquigarrow$ topological vertex can be interpreted as being a character of degenerate $\mathcal{W}_{1+\infty}$ representations



- box counting generalizes also to minimal models (Ising...)

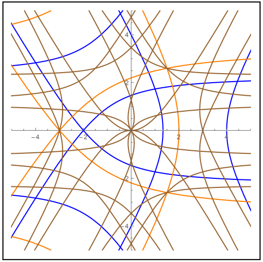
Truncations

- for $\lambda_3 = N$ we reduce to $\mathcal{W}_N$ (i.e. $\lambda_3 = 2$ is Virasoro)
- more general truncations: Gaiotto-Rapčak $Y_{N_1 N_2 N_3}$

$$\frac{N_1}{\lambda_1} + \frac{N_2}{\lambda_2} + \frac{N_3}{\lambda_3} = 1$$

has singular vector at level $(N_1+1)(N_2+1)(N_3+1)$

- $Y_{k,k+1,0}$ gives k-th unitary minimal models of $\mathcal{W}_N$
- intersections: minimal models (Ising: $Y_{120} \cap Y_{002}$)



Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$\begin{aligned}
0 &= [e_{j+3}, e_k] - 3[e_{j+2}, e_{k+1}] + 3[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad + \sigma_2 [e_{j+1}, e_k] - \sigma_2 [e_j, e_{k+1}] - \sigma_3 \{e_j, e_k\} \\
0 &= [f_{j+3}, f_k] - 3[f_{j+2}, f_{k+1}] + 3[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\
&\quad + \sigma_2 [f_{j+1}, f_k] - \sigma_2 [f_j, f_{k+1}] + \sigma_3 \{f_j, f_k\} \\
0 &= [\psi_{j+3}, e_k] - 3[\psi_{j+2}, e_{k+1}] + 3[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\
&\quad + \sigma_2 [\psi_{j+1}, e_k] - \sigma_2 [\psi_j, e_{k+1}] - \sigma_3 \{\psi_j, e_k\} \\
0 &= [\psi_{j+3}, f_k] - 3[\psi_{j+2}, f_{k+1}] + 3[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\
&\quad + \sigma_2 [\psi_{j+1}, f_k] - \sigma_2 [\psi_j, f_{k+1}] + \sigma_3 \{\psi_j, f_k\} \\
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k]
\end{aligned}$$

'initial/boundary conditions'

$$[\psi_0, e_j] = 0, \qquad [\psi_1, e_j] = 0, \qquad [\psi_2, e_j] = 2e_j,$$
$$[\psi_0, f_j] = 0, \qquad [\psi_1, f_j] = 0, \qquad [\psi_2, f_j] = -2f_j$$

and finally the Serre relations

$$0 = \mathrm{Sym}_{(j_1,j_2,j_3)} \left[e_{j_1}, [e_{j_2}, e_{j_3+1}]\right], \quad 0 = \mathrm{Sym}_{(j_1,j_2,j_3)} \left[f_{j_1}, [f_{j_2}, f_{j_3+1}]\right].$$

Parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{C}$ constrained by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and

$$\begin{aligned} \sigma_2 &= \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3 \\ \sigma_3 &= \epsilon_1\epsilon_2\epsilon_3. \end{aligned}$$

We have both commutators and anticommutators in defining quadratic relations (but no $\mathbb{Z}_2$ grading) - for $\sigma_3 \neq 0$ not a Lie (super)-algebra.

Introducing generating functions (currents)

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

the first set of formulas above (almost!) simplify to

$$e(u)e(v) = \varphi(u-v)e(v)e(u), \quad f(u)f(v) = \varphi(v-u)f(v)f(u),$$
$$\psi(u)e(v) = \varphi(u-v)e(v)\psi(u), \quad \psi(u)f(v) = \varphi(v-u)f(v)\psi(u)$$

with rational structure function (scattering phase in BAE)

$$\boxed{\varphi(u) = \frac{(u+\epsilon_1)(u+\epsilon_2)(u+\epsilon_3)}{(u-\epsilon_1)(u-\epsilon_2)(u-\epsilon_3)}}$$

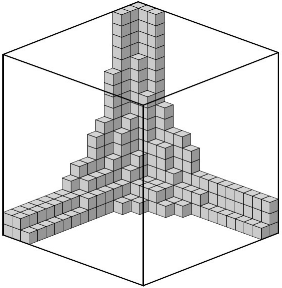
The representation theory of the algebra is much simpler in this Yangian formulation and it is controlled by this function

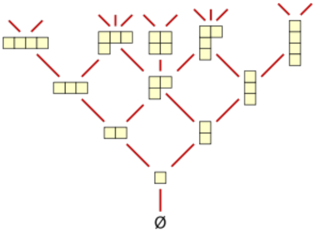
$\psi(u), e(u)$ and $f(u)$ in representations act like

$$\psi(u)\,|\Lambda\rangle = \psi_0(u) \prod_{\square \in \Lambda} \varphi(u-\epsilon_\square)\,|\Lambda\rangle$$

$$e(u)\,|\Lambda\rangle = \sum_{\square \in \Lambda^+} \frac{E(\Lambda \to \Lambda + \square)}{u-\epsilon_\square}\,|\Lambda + \square\rangle$$

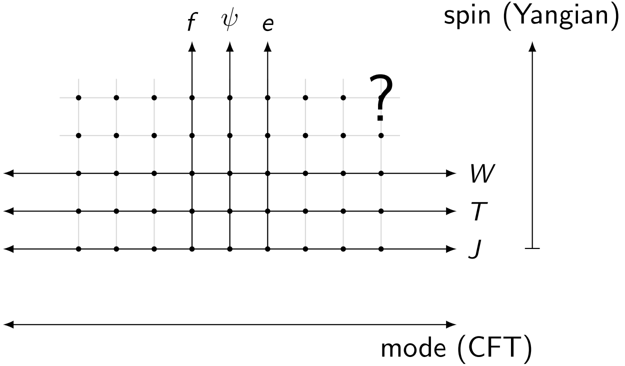
where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions,...) and where $\epsilon_\square = \sum_j \epsilon_j x_j(\square)$ is the weighted geometric position of the box.





Two different descriptions of the algebra:

- usual CFT point of view with local fields $J(z), T(z), W(z), \ldots$ with increasingly complicated OPE as we go to higher spins
- Yangian point of view (Arbesfeld-Schiffmann-Tsymbaliuk) where all the spins are included in the generating functions $\psi(u), e(u)$ and $f(u)$ but accessing higher mode numbers is difficult



How to connect these two descriptions?

- the parameters can be identified as $\lambda_1 = -\psi_0 \epsilon_2 \epsilon_3$
- the generators with low spin and mode can be identified

$$\psi_2 = 2L_0, \quad e_0 = J_{-1}, \quad f_0 = -J_{+1}$$

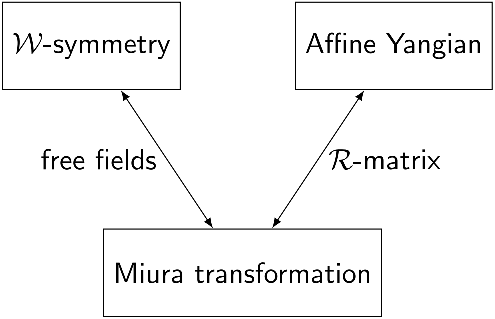
$$\psi_3 = (W_3 + \ldots)_0 + \sigma_3 \sum_{m>0} (3m-1) J_{-m} J_m$$

(cut & join operator, Benjamin-Ono hierarchy,
not a zero mode of a local field, but need Hilbert transform)

- this is sufficient to find the map in principle, but what is the more conceptual way to understand the map?
- Negut: closed form of the map $\mathcal{W} \to \mathcal{Y}$ using shuffle algebra

Miura transformation and $\mathcal{R}$-matrix

very powerful (free field rep, coproduct, integrability)

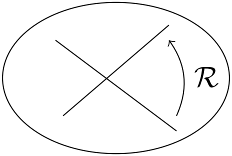


- consider the following factorization of N-th order differential operator

$$(\partial + \partial\phi_1(z)) \cdots (\partial + \partial\phi_N(z)) = \sum_{j=0}^{N} U_j(z)\partial^{N-j}$$

with N commuting free fields $\partial\phi_j(z)\partial\phi_k(w) \sim \delta_{jk}(z-w)^{-2}$
- OPEs of U_j generate $\mathcal{W}_N$ and furthermore are quadratic
- $\mathcal{W}_N \leftrightarrow$ quantization of N-th order differential operators
- the embedding of $\mathcal{W}_N$ in the bosonic Fock space depends on the way we order the fields
- Maulik-Okounkov: $\mathcal{R}$-matrix as transformation (intertwiner) between two embeddings, $\mathcal{R} : \mathcal{F}^{\otimes 2} \to \mathcal{F}^{\otimes 2}$

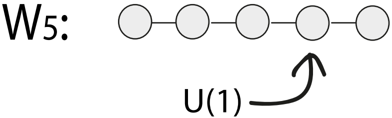
$$(\partial + \partial\phi_1)(\partial + \partial\phi_2) = \mathcal{R}^{-1}(\partial + \partial\phi_2)(\partial + \partial\phi_1)\mathcal{R}$$



- $\mathcal{R}$ defined in this way satisfies the Yang-Baxter equation (two ways of reordering $321 \to 123$)

$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) = \\ = \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2)$$

- the spectral parameter u - the global $U(1)$ charge
- $\mathcal{R}$-matrix satisfying YBE $\rightsquigarrow$ apply the algebraic Bethe ansatz



- spin chain of length $N \rightsquigarrow \mathcal{W}_N$ algebra
- this explains the Yangian structure, the Yangian levels corresponding to copies of $\hat{\mathfrak{u}}(1)$

- consider an *auxiliary* Fock space $\mathcal{F}_A$ and a *quantum* space $\mathcal{F}_Q \equiv \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_N$
- we associate to this the monodromy matrix $\mathcal{T}_{AQ} : \mathcal{F}_A \otimes \mathcal{F}_Q \to \mathcal{F}_A \otimes \mathcal{F}_Q$ defined as

$$\mathcal{T}_{AQ} = \mathcal{R}_{A1}\mathcal{R}_{A2} \cdots \mathcal{R}_{AN}$$

- if the individual $\mathcal{R}$-matrices satisfy the YBE, $\mathcal{T}$ will also satisfy YBE with respect to two auxiliary spaces A and B

$$\mathcal{R}_{AB}\mathcal{T}_A\mathcal{T}_B = \mathcal{T}_B\mathcal{T}_A\mathcal{R}_{AB}$$

- the algebra of matrix elements of $\mathcal{T}$ satisfying this equation is the Yangian - RTT presentation

- in our situation the Fock-Fock $\mathcal{R}$-matrix controlling the commutation relations of Yangian has rows and columns labeled by Young diagrams ($\rightsquigarrow$ infinite number of generators)
- we can however restrict to simple matrix elements, i.e. upper left corner of the $\mathcal{R}$-matrix

$$\mathcal{H} = \langle 0|_A \mathcal{T} |0\rangle_A , \quad \mathcal{E} = \langle 0|_A \mathcal{T} |1\rangle_A , \quad \mathcal{F} = \langle 1|_A \mathcal{T} |0\rangle_A$$

- the YBE now implies relations between these operators like

$$0 = [\mathcal{H}(u), \mathcal{H}(v)]$$

(infinite set of commuting Hamiltonians) or

$$(u - v + h_3)\mathcal{H}(u)\mathcal{E}(v) = (u - v)\mathcal{E}(v)\mathcal{H}(u) + h_3\mathcal{H}(v)\mathcal{E}(u)$$

(ladder operators)

- these generating functions can be related to AST Yangian

$$\psi(u) = \frac{u+\sigma_3\psi_0}{u}\frac{\mathcal{H}(u+\epsilon_1)\mathcal{H}(u+\epsilon_2)}{\mathcal{H}(u)\mathcal{H}(u+\epsilon_1+\epsilon_2)}$$

$$e(u) = \epsilon_3^{-1}\mathcal{H}(u)^{-1}\mathcal{E}(u)$$

$$f(u) = -\epsilon_3^{-1}\mathcal{F}(u)\mathcal{H}(u)^{-1}$$

- using $\mathcal{R}$-matrix one can find these Yangian generators systematically following the algorithm of QISM (the only input is the Miura transformation)

How does the $\mathcal{R}$-matrix look like?

- a fermionic form obtained by A. Smirnov by studying the large N limit of $\mathfrak{gl}(N)$ $\mathcal{R}$-matrix - complicated
- a bosonic formula is not known so far, but results of Nazarov-Sklyanin can be interpreted as matrix elements of $\mathcal{R}$-matrix in mixed representation where the two bosonic Fock spaces are associated to different asymptotic directions

$$\mathcal{R}(u) = \mathbb{1} - \frac{1}{u}\sum_{j>0} a_{-j}a_j + \frac{1}{2!u(u+1)}\sum_{j,k>0}(a_{-j}a_{-k}+a_{-j-k})(a_j a_k + a_{j+k})$$
$$- \frac{1}{3!u(u+1)(u+2)}\sum_{j,k,l>0}(a_{-j}a_{-k}a_{-l} + a_{-j-k}a_{-l} + a_{-j-l}a_{-k}$$
$$+ a_{-k-l}a_{-j} + 2a_{-j-k-l})(\ldots) + \ldots$$

with $a_j \equiv a_j^{(1)} - a_j^{(2)}$.

KdV and KP - classical limits of Virasoro and $\mathcal{W}_\infty$

Consider a one-dimensional Schrödinger operator

$$L^2 \equiv \partial^2 + u(x).$$

where $u(x)$ is (minus) the potential. Typically one is interested in the *spectrum* of this differential operator. Let us instead have a look at *isospectral deformations* of this potential. Clearly the spatial translations, i.e. the deformations of the type

$$\partial_{t_1} u = \partial_x u$$

leave the spectrum of L^2 invariant.
But perhaps surprisingly there are more complicated deformations of u which have the same property, actually infinitely many of them.

Consider the evolution of u given by KdV equation,

$$4\partial_{t_3} u = 6u\partial_x u + \partial_x^3 u.$$

This is the famous *Korteweg-de Vries equation* (Boussinesq 1877, KdV 1895).

One way to see that this does not change the spectrum is to write the KdV equation in the Lax form

$$\partial_{t_3} L^2 = \left[\partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u', L^2\right].$$

This Lax form guarantees that the evolution of L^2 in the time direction t_3 preserves its spectrum (quantum mechanics).
There is in fact an infinite number of commuting flows analogous to this one (labeled by odd integers).
These flows deform the potential while preserving the spectrum.

To construct these flows, we apply a formalism of Sato school - the pseudodifferential operators. Consider formal expressions of the type

$$\partial^N + \sum_{j=1}^{\infty} f_j(x)\partial^{N-j}$$

(a monic pseudodifferential operator of Nth order) with composition by Leibniz rule

$$\partial^n f = f\partial^n + \sum_{j=1}^{\infty} \binom{n}{j} f^{(j)}\partial^{n-j}.$$

Introduction of pseudodifferential operators (instead of working with just differential operators) has many advantages. In particular, for every such operator there exists a formal ath power for any $a \in \mathbb{Q}$ (in particular every such operator has a formal inverse).

In our example of Schrödinger operator we have

$$L = \left(\partial^2 + u\right)^{1/2} = \partial + \frac{u}{2}\partial^{-1} - \frac{u'}{4}\partial^{-2} + \frac{u'' - u^2}{8}\partial^{-3} + \\ + \frac{6uu' - u'''}{16}\partial^{-4} + \frac{2u^3 - 11u'^2 - 14uu'' + u^{(4)}}{32}\partial^{-5} + \ldots$$

and

$$L^3 = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u' + \mathcal{O}\left(\partial^{-1}\right)$$

The differential part of this operator is exactly the operator that we saw before. It generates the evolution of L^2 in the direction of KdV time t_3,

$$\partial_{t_3} L^2 = \left[(L^3)_+, L^2\right].$$

This formula generalizes to all odd integer powers of L and also explains why are the even time evolutions trivial.

So far the equations of KdV hierarchy were written in the Lax form. One can also introduce a Poisson bracket and turn the whole system to a Hamiltonian integrable system. One of the possible Poisson brackets is the *Gelfand-Dickey bracket*

$$\{u(x), u(y)\} = -\delta'''(x-y) - 4u(x)\delta'(x-y) - 2u'(x)\delta(x-y)$$

(*second Poisson structure* of KdV hierarchy). In terms of Fourier modes this gives the classical Virasoro algebra. Although it looks complicated, one can get it easily using Miura transformation: Let us assume that the Schrödinger operator L^2 factorizes,

$$\partial^2 + u = (\partial - j)(\partial + j)$$

(in other words that $u = j' - j^2$). Imposing the canonical bracket

$$\{j(x), j(y)\} = \delta'(x-y),$$

(Heisenberg algebra for Fourier modes of j) it is easy to check that this induces the Gelfand-Dickey bracket for u.

Conserved quantities

The commuting flows are generated by Hamiltonians which can be written compactly as

$$I_j = \int \mathrm{res}_\partial (L^j)$$

where res_∂ is the formal *residue* of pseudodifferential operator, i.e. coefficient of ∂^{-1}. It has an important property that $\mathrm{res}_\partial [A, B]$ is a total x-derivative so in particular $\int \mathrm{res}_\partial$ with suitable boundary conditions has properties of a trace. We can assemble these conserved charges into a generating function

$$D(\lambda) = \exp \int \mathrm{res}_\partial \log \left(1 - \frac{L}{\lambda}\right) = \exp \left[- \int \sum_{j=1}^{\infty} \mathrm{res}_\partial \left(\frac{L^j}{j \lambda^j} \right) \right]$$

which has properties of the *spectral determinant* of the associated Schrödinger operator.

Example

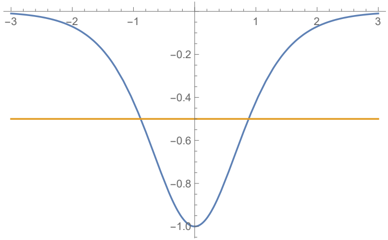
The simplest example is single-soliton solution of KdV equation

$$u(x,t,\ldots) = \frac{2p^2}{\cosh^2(px + p^3 t + p^5 t_5 + \ldots)}$$

which is in quantum mechanics known as Pöschl-Teller potential (for $p = 1$). It is reflectionless and supports exactly one bound state.

The spectral determinant is

$$D(\lambda) = \frac{\lambda - p}{\lambda + p}$$



For a multisoliton solutions of KdV equation $D(\lambda)$ is multiplicative.

Generalizations

Instead of second order differential operator one can consider an arbitrary N-th order differential operator $L^N = \partial^N + \ldots$. Everything discussed previously has straightforward generalization and we find the N-KdV hierarchy. For $N = 3$ one finds the Boussinesq equation.

If one instead starts directly with a pseudodifferential operator

$$L = \partial + u_2 \partial^{-1} + u_3 \partial^{-2} + \ldots,$$

one finds a system of an infinitely many non-linear partial differential equations for an infinitely many unknown functions in infinitely many times. This is the *Kadomtsev-Petviashvili hierarchy* named after KP equation which is

$$3\partial_{t_2}^2 u = \partial_{t_1} \left(4\partial_{t_3} u - \partial_{t_1}^3 u - 12 u \partial_{t_1} u \right).$$

This is a generalization of KdV equation with additional transverse direction t_2.

Conserved quantities

Quantization of the Poisson bracket for KdV leads to Virasoro algebra, N-KdV to $\mathcal{W}_N$ algebra and KP hierarchy to $\mathcal{W}_\infty$.
What is the fate of the integrals of motion of the classical theory?
In the case of Virasoro algebra one can easily write down a set of commuting operators which are zero modes of quantized local currents, i.e.

$$I_1 = \int T(x)dx = L_0 - \frac{c}{24}$$

$$I_3 = \int (TT)(x)dx = L_0^2 + 2\sum_{m=1}^{\infty} L_{-m}L_m - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}$$

$$I_5 = \sum_{m_1+m_2+m_3=0} :L_{m_1}L_{m_2}L_{m_3}: + \sum_{m=1}^{\infty}\left(\frac{c+11}{6}m^2 - 1 - \frac{c}{4}\right)L_{-m}L_m$$

$$+ \frac{3}{2}\sum_{m=1}^{\infty} L_{1-2m}L_{-1+2m} - \frac{c+4}{8}L_0^2 + \frac{(c+2)(3c+20)}{576}L_0 + const.$$

What is the spectrum of this infinite set of commuting operators?

Since L_0 is one of the integrals of motion, for diagonalization we can restrict to eigenspaces of L_0, i.e. the problem reduces to diagonalization of finite matrices. But it is hard to diagonalize these explicitly. The only known (to me) example is Virasoro at $c = -2$ (BLZ 1996).

In an important series of papers Bazhanov, Lukyanov, Zamolodchikov (1994, ...) studied this problem introducing quantum monodromy matrix and Baxter's Q operator and writing the thermodynamic Bethe ansatz / Destri-de Vega equations for these. Later, in the light of ODE/IM correspondence (Dorey, Tateo), BLZ (2003) came up with much simpler Bethe ansatz solution (considered also by Fioravanti (2004)):

For a given central charge c, conformal dimension Δ and the Virasoro level M let us consider the differential operator

$$-\partial_z^2 + \frac{\ell(\ell+1)}{z^2} + \frac{1-\sum_{j=1}^{M}\gamma_j}{z} + \sum_{j=1}^{M}\left(\frac{2}{(z-z_j)^2} + \frac{\gamma_j}{z-z_j}\right) + \lambda z^{h^2-2}$$

where h is related to c and ℓ to Δ. For $h^2 \in \mathbb{N}$ we have a regular singularity at $z=0$. At $z=\infty$ we have an irregular singularity. Requiring trivial monodromy of solutions of ODE around points z_j (aparent singularities) determines γ_j and imposes M equations for *M Bethe roots z_j*

$$\sum_{k\neq j} \frac{z_j(h^4 z_j^2 - (h^2-2)(2h^2+1)z_j z_k + (h^2-1)(h^2-2)z_k^2)}{(z_j-z_k)^3} = (1-h^2)z_j - h^4\Delta.$$

These are the *Bethe ansatz equations* of BLZ that diagonalize the local integrals of motion of quantum KdV.

For example, for the first non-trivial integral of motion I_3 we have

$$I_3 = (\Delta+M)^2 - \frac{c+2}{12}(\Delta+M) + \frac{c(5c+22)}{2880} + 4(h^{-4} - h^{-2}) \sum_{j=1}^{M} z_j.$$

This in principle reduces the problem of finding the spectrum of I_j to solving a system of Bethe ansatz equations. In practice, for numerical evaluations the diagonalization of matrices is much easier. On the other hand, having a system of Bethe ansatz equations can be simpler for theoretical considerations like studying the thermodynamic limit etc.

Generalizations: (Feigin, Frenkel, Hernandez; Masoero, Raimondo, Valeri) for other Lie algebras

How about $\mathcal{W}_\infty$ and its additional symmetries (triality...)?

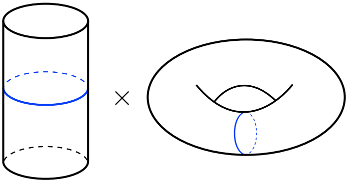
ILW Bethe ansatz

- in our construction of commuting Yangian invariants using algebraic Bethe ansatz (from a spin chain) we constructed commuting Hamiltonians by taking vacuum-to-vacuum matrix element

$$\mathcal{H}_0(u) = \langle 0|_A \mathcal{T}(u) |0\rangle_A$$

  which leads to non-local Hamiltonians from CFT point of view
- from the usual BAE point of view we should instead take the trace of the auxiliary Hilbert space, but this is now infinite
- we can instead regularize the trace using the energy

$$\mathcal{H}_q(u) = \mathrm{Tr}_A \, q^{L_{A,0}} \mathcal{T}(u)$$



- this leads for every q to a different infinite family of commuting Hamiltonians, Hamiltonians of *intermediate long wave equation*
- at $q \to 0$ these reduce to Yangian Hamiltonians, in $q \to 1$ limit to local Hamiltonians from CFT point of view (the higher spin generalizations of BLZ)
- these Hamiltonians can be diagonalized by Bethe ansatz equations (Litvinov, Nekrasov, Shatashvili, Bonelli, Sciarappa, Tanzini, Vasko)

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon_3}{u_j + a_l} \prod_{k \neq j} \frac{(u_j - u_k + \epsilon_1)(u_j - u_k + \epsilon_2)(u_j - u_k + \epsilon_3)}{(u_j - u_k - \epsilon_1)(u_j - u_k - \epsilon_2)(u_j - u_k - \epsilon_3)}$$

- these equations are the same as in the simplest Heisenberg XXX $SU(2)$ spin chain, except for the fact that the interaction between Bethe roots is now a degree 3 rational function instead of degree 1

- the limit $q \to 1$ is rather singular (actually any q a root of unity), but in this limit the Heisenberg subalgebra and $\mathcal{W}_\infty$ decouple in rather interesting way
- in particular, the Bethe roots associated to $\mathcal{W}_\infty$ remain finite in the $q \to 1$ limit while those associated to Heisenberg subalgebra diverge in rather interesting way (work in progress)
- once we solve Bethe ansatz equations, the spectrum of $\mathcal{H}_q(u)$ can be written as

$$\frac{\mathcal{H}_q(u)}{\mathcal{H}_{q=0}(u)} \to \frac{1}{\sum_\lambda q^{|\lambda|}} \sum_\lambda q^{|\lambda|} \prod_{\square \in \lambda} \psi_\Lambda(u - \epsilon_\square + \epsilon_3)$$

where

$$\psi_\Lambda(u) = A(u) \prod_j \varphi(u - x_j)$$

which is a conjectural Yangian version of formula by Feigin-Jimbo-Miwa-Mukhin (TP & Akimi Watanabe)

Questions

Many questions

- how are the ILW and BLZ Bethe ansatz equations related? understanding this could shed light on mysterious fiber-base duality / Miki automorphism
- how can the ILW generating function be regularized to extract interesting information in $q \to 1$ limit?
- another set of Bethe ansatz equations based on affine Gaudin model
- quantum spectral curves, mirror symmetry in topological string
- elliptic Calogero model (TBA equations of Nekrasov-Shatashvili)

Thank you!