Heisenberg Spin Chains And Supersymmetric Gauge Theories

Wei Gu

Max Planck Institute for Mathematik

Presentation at Tsinghua University

Based on Wei Gu arXiv : 2212.11288

Motivation

In this talk, I will tell you that the spin chain can emerge from a two-dimensional supersymmetric gauge theory.
But why do we need this fact?

- It is well-investigated that the Gromov-Witten theory can be regarded as a topological sector of two-dimensional supersymmetric gauge theory (GLSMs).

Motivation

In this talk, I will tell you that the spin chain can emerge from a two-dimensional supersymmetric gauge theory.
But why do we need this fact?

- It is well-investigated that the Gromov-Witten theory can be regarded as a topological sector of two-dimensional supersymmetric gauge theory (GLSMs).
- Mathematicians used the techniques of integrable systems to prove the theorems of the Gromov-Witten theory.

Motivation

In this talk, I will tell you that the spin chain can emerge from a two-dimensional supersymmetric gauge theory.
But why do we need this fact?

- It is well-investigated that the Gromov-Witten theory can be regarded as a topological sector of two-dimensional supersymmetric gauge theory (GLSMs).
- Mathematicians used the techniques of integrable systems to prove the theorems of the Gromov-Witten theory.
- Physicists found several similarities between the supersymmetric gauge theory and the integrable system. For example, vacuum equations of gauge theory are the same as the Bethe-ansatz equations of the correspondence integrable system.

Motivation

In this talk, I will tell you that the spin chain can emerge from a two-dimensional supersymmetric gauge theory.
But why do we need this fact?

- It is well-investigated that the Gromov-Witten theory can be regarded as a topological sector of two-dimensional supersymmetric gauge theory (GLSMs).
- Mathematicians used the techniques of integrable systems to prove the theorems of the Gromov-Witten theory.
- Physicists found several similarities between the supersymmetric gauge theory and the integrable system. For example, vacuum equations of gauge theory are the same as the Bethe-ansatz equations of the correspondence integrable system.
- Possible others ...

One of my goals in this talk is to convince you that the third point is a fact, then it explains why the three "different subjects" are inherently related.

Spin Chain

Many of you may be already familiar with the spin chain. It has rich important physics, but the setup is simple.

Spin Chain

Many of you may be already familiar with the spin chain. It has rich important physics, but the setup is simple.
It has N-sites, each site has a vector space $\mathcal{V}$ and the associated Hamiltonian as

$$H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} \right) + \dots$$

Figure: Spin chain.

Spin Chain

Examples. Consider a spin-1/2 system: $\mathcal{V} = \{+, -\}$.
- XX model, $H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} \right)$.

Spin Chain

Examples. Consider a spin-1/2 system: $\mathcal{V} = \{+, -\}$.

- XX model, $H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} \right)$.
- XXX model, $H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} + 2 S_{z,i} S_{z,i+1} \right)$.

Spin Chain

Examples. Consider a spin-1/2 system: $\mathcal{V} = \{+, -\}$.

- XX model, $H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} \right)$.
- XXX model, $H = \sum_{i=1}^{N} \left(S_{+,i} S_{-,i-1} + S_{-,i} S_{+,i-1} + 2 S_{z,i} S_{z,i+1} \right)$.

...

Spin Chain

Examples. Consider a spin-1/2 system: $\mathcal{V} = \{+, -\}$.
• XX model, $H = \sum_{i=1}^{N} \left(S_{+,i}S_{-,i-1} + S_{-,i}S_{+,i-1}\right)$.
• XXX model, $H = \sum_{i=1}^{N} \left(S_{+,i}S_{-,i-1} + S_{-,i}S_{+,i-1} + 2S_{z,i}S_{z,i+1}\right)$.
...
The k spin-up configuration corresponds to the configuration of k-magnons.
We will see that these can be mapped to the supersymmetric gauge theory. Magnons map to the vacuum states of supersymmetric gauge theory, and the interaction $S_{+,i}S_{-,i-1}$ and $S_{-,i}S_{+,i-1}$ map to the fundamental domainwalls $\cdots$

A brief review of 2d $\mathcal{N}$=(2,2) GLSMs

Some relevant (and new) points:

- Setup.

  • I only focus on $U(k)$ gauge group in this talk. Super gauge field V, its field strength Σ with F_{01} to be one component field.

  • A twisted superpotential $\tilde{W} = -t \cdot tr\Sigma$, $t = r - i\theta$, where r is FI-parameter, θ is the usual theta angle in a gauge theory.

$$\Sigma = \sigma + i\theta^{+}\bar{\lambda}_{+} - i\bar{\theta}^{-}\lambda_{-} + \ldots, \quad \bar{\Sigma} = \bar{\sigma} - i\bar{\theta}^{+}\lambda_{+} + i\theta^{-}\bar{\lambda}_{-} + \ldots$$

  • This talk only considers fundamental matter fields Φ_i. For N matters, the global symmetry is $SU(N)$. One can break this global symmetry by turning on the so-called the twisted masses $m_i = \langle\overline{\Sigma}\rangle$, where $\overline{\Sigma}$ is the superfield strength of the background gauge field $\overline{V}$ for $SU(N)$ global symmetry.

A brief review of 2d $\mathcal{N}$=(2,2) GLSMs

Some relevant (and new) points:

- Setup.
  • I only focus on $U(k)$ gauge group in this talk. Super gauge field V, its field strength Σ with F_{01} to be one component field.
  • A twisted superpotential $\tilde{W} = -t \cdot tr\Sigma$, $t = r - i\theta$, where r is FI-parameter, θ is the usual theta angle in a gauge theory.

$$\Sigma = \sigma + i\theta^{+}\bar{\lambda}_{+} - i\bar{\theta}^{-}\lambda_{-} + \ldots, \quad \bar{\Sigma} = \bar{\sigma} - i\bar{\theta}^{+}\lambda_{+} + i\theta^{-}\bar{\lambda}_{-} + \ldots$$

  • This talk only considers fundamental matter fields Φ_i. For N matters, the global symmetry is $SU(N)$. One can break this global symmetry by turning on the so-called the twisted masses $m_i = \langle\overline{\Sigma}\rangle$, where $\overline{\Sigma}$ is the superfield strength of the background gauge field $\overline{V}$ for $SU(N)$ global symmetry.
- Phases $\simeq$ ground states in a semi-classical region. Symmetries are important.
  • Lorentz symmetry: $U(t; \phi_i, F_{01})$. Perturbative RG-flow $r = \sum_i \rho_i \log \frac{\mu}{\Lambda}$.
  • SUSY: $U = 0$.
  • One-form symmetries: F_{01} was missed in the previous studies. It is important! See my work '21 on the emergent dynamical decomposition.
  • If the semi-classical analysis is not trustful, then we consider the effective theory by integrating out matters:

$$\widetilde{W}_{eff} = -(t + i(k-1)\pi)\sum_{a=1}^{k}\Sigma_a - \sum_{i,a}\rho_i^a\Sigma_a\left(\log\left(\rho_i^a\Sigma_a\right) - 1\right).$$

Gr(k;N)

Gauged linear sigma model for Gr(k;N) is well-studied. See Witten '93, Gu-Sharpe '18. It is a $U(k)$ gauge theory with N fundamental fields.

- When $r \gg 0$, the semi-classical vacuum configuration is the geometry: Gr(k;N).
- When $r \ll 0$, the vacuum structure is described by a twisted effective superpotential

$$\widetilde{W}_{eff} = -(t + i(k-1)\pi) \sum_{a=1}^{k} \Sigma_a - \sum_a N\Sigma_a \left(\log\left(\Sigma_a\right) - 1\right).$$

  The vacuum equations

$$e^{\frac{\partial \widetilde{W}_{eff}}{\partial \sigma_a}} = 1$$

  give

$$(\sigma_a)^N = \widetilde{q}, \quad \widetilde{q} = e^{-t-i\pi(k-1)}, \qquad \mathrm{a} \in \{1, \cdots, k\}, \quad \sigma_a \neq \sigma_b.$$

We will see the above equations are also the Bethe-ansatz equations of the N-sites XX-spin chain.

Exact Results

In Gu-Sharpe '18, an exact quantum theory of 2d supersymmetric gauge theory can be described by the Landau-Ginzburg model. It is target space $\left(\mathbb{C}^{\star}\right)^{kN} \times (\mathbb{C})^{k^2} / S_k$ with a superpotential

$$W = \sum_a \Sigma_a \left(\sum_{ib} \rho^a_{ib} Y_{ib} + \sum_{\mu\nu} \alpha^a_{\mu\nu} Z_{\mu\nu} - t \right) + \sum_{ia} \exp(-Y_{ia}) + \sum_{\mu \neq \nu} X_{\mu\nu},$$

where $\rho^a_{ib} = \delta^a_b$, $\alpha^a_{\mu\nu} = -\delta^a_\mu + \delta^a_\nu$.
The Weyl orbifold maps weights to weights

$$Y_i \mapsto Y_j \qquad \sum_a \Sigma_a \rho^a_i \mapsto \sum_a \Sigma_a \rho^a_i.$$

and roots to roots

$$X_\mu \mapsto X_\nu \qquad \sum_a \Sigma_a \alpha^a_\mu \mapsto \sum_a \Sigma_a \alpha^a_\nu.$$

.

Vacua

By computing

$$e^{dW} = 1,$$

gives

$$\exp\left(-y_{ia}\right) = \sigma_a, \qquad x_{\mu\nu} = -\sigma_\mu + \sigma_\nu,$$

and

$$(\sigma_a)^N = \widetilde{q}, \qquad \mathrm{a} \in \{1, \cdots, k\}, \quad \sigma_a \neq \sigma_b.$$

But it contains more degrees of freedom than the one in the twisted effective superpotential $\widetilde{W}_{eff}$, which will play an important role in our story.

BPS Spectrum

The reason we mention the exact result is that it includes the BPS spectrum in the dynamics. The BPS domainwall that fluctuates from one vacuum to another one will be useful in our story.

BPS Spectrum

The reason we mention the exact result is that it includes the BPS spectrum in the dynamics. The BPS domainwall that fluctuates from one vacuum to another one will be useful in our story.

For projective space, or a $U(1)$ gauge theory, it was observed first by Witten '79 that the fundamental domainwall, which connects two adjacent vacua, has a gauge charge 1. A candidate of this domainwall is the fundamental field ϕ. The mass of this domainwall is proportional to the dynamical scale $\Lambda^N = q$:

$$Z \approx \Lambda$$

BPS Spectrum

The reason we mention the exact result is that it includes the BPS spectrum in the dynamics. The BPS domainwall that fluctuates from one vacuum to another one will be useful in our story.

For projective space, or a $U(1)$ gauge theory, it was observed first by Witten '79 that the fundamental domainwall, which connects two adjacent vacua, has a gauge charge 1. A candidate of this domainwall is the fundamental field ϕ. The mass of this domainwall is proportional to the dynamical scale $\Lambda^N = q$:

$$Z \approx \Lambda$$

It has a finite mass, so, unlike the bosonic theory, 2d $\mathcal{N} = (2,2)$ massive theory is not confined. More generally, we have

$$Z_{\ell_1 \ell_2} = W\,|_{\ell_2} - W\,|_{\ell_1}, \qquad \ell \text{ labels the vacua.}$$

See Hannay-Hori '97 and Hori-Vafa '20 for abelian theories. See also Dorey '98 about the connection to the BPS-spectrum in Seiberg-Witten theory. For nonabelian theories, see Gu '22. If turning on the twisted mass, the central charge expression will be modified.

An intermediate scale

In quantum field theory, the physics depends on scale. Now if we consider a scale

$$\Lambda \ll e(\mu),$$

where e is the gauge coupling. The mass of the perturbative spectrum is proportional to the gauge coupling e. At this scale, the physics degrees of freedom are domainwalls. We will show that this is the theory for spin chains.

State Space For Vacua

Now, we come back the vacuum equations

$$(\sigma_a)^N = \widetilde{q}.$$

So the vacua can be understood by picking k sites out of N-values on the Coulomb field σ.

State Space For Vacua

Now, we come back the vacuum equations

$$(\sigma_a)^N = \widetilde{q}.$$

So the vacua can be understood by picking k sites out of N-values on the Coulomb field σ.

.

How about the Hilbert space representation? See Witten '82 or Hori.et.al mirror symmetry book '2000, we have the ground states

$$\prod_{a=1}^{k} \delta^2(\overline{\sigma}_a) \prod_{a=1}^{k} \bar{\lambda}_{+,a} \mid 0\rangle.$$

State Space For Vacua

Now, we come back the vacuum equations

$$(\sigma_a)^N = \widetilde{q}.$$

So the vacua can be understood by picking k sites out of N-values on the Coulomb field σ.

.

How about the Hilbert space representation? See Witten '82 or Hori.et.al mirror symmetry book '2000, we have the ground states

$$\prod_{a=1}^{k} \delta^2(\overline{\sigma}_a) \prod_{a=1}^{k} \bar{\lambda}_{+,a} \mid 0\rangle.$$

From the structure, we can enlarge the fermionic degrees of freedom as

$$\overline{\lambda}_{\pm,j} := \int d^2\sigma \delta^2(\sigma_a - \overline{\sigma}_a)\overline{\lambda}_{\pm}, \qquad \lambda_{\pm,j} := \int d^2\sigma \delta^2(\sigma_a - \overline{\sigma}_a)\lambda_{\pm},$$

and their canonical commutation relations

$$\{\overline{\lambda}_{\pm,j}, \lambda_{\pm,i}\} = \delta_{ji}.$$

State Space

If we replace the formal state

$$|0\rangle \mapsto \prod_{i=1}^{N-k} \bar{\lambda}_{-,i}|\Omega\rangle$$

Now we can represent the state-space by a formal notation, for example, for k=2

$$|-, \cdots, +, -, \cdots, -, +, -, \cdots, -\rangle.$$

This notation is not necessary for quantum field theory, however, it would be useful for presenting the data of the spin chain.

Spin Operators

One can construct local spin operators on each site as

$$S_{+,i} = \bar{\lambda}_{+,i}\lambda_{-,i}, \quad S_{-,i} = \bar{\lambda}_{-,i}\lambda_{+,i}, \quad S_{z,i} = \frac{1}{2}\left(\bar{\lambda}_{+,i}\lambda_{+,i} - \bar{\lambda}_{-,i}\lambda_{-,i}\right).$$

It is readily to show that

$$[S_+, S_-] = 2S_z.$$

This explains why we need $\mathcal{N} = (2,2)$ supersymmetry for getting a spin chain. For more general global symmetry in gauge theory, it would have one more index for defining the local spin operators, see Gu '22.

Domainwalls In The Spin Chain

The domainwall can be regarded as a map such as

$$|-, \cdots, -_{i-1}, +_i, -_{i+1} \cdots\rangle \mapsto |-, \cdots, -_{i-2}, +_{i-1}, -_i \cdots\rangle$$

$$\phi :\mapsto \mathcal{D}_i = S_{-,i} S_{+,i-1}.$$

One can similarly define the anti-domainwall. For more general domainwalls, a similar construction applies.

Constraints From Domainwalls

In our case, we have the global symmetry $SU(N)$ in our target space. The domainwalls are charged under the center group $\mathbb{Z}_N$ of the flavor group $SU(N)$. So the dynamic domainwalls suggested that all of the physical state space should be neutral under the center group $\mathbb{Z}_N$. The expectation value of σ field is also charged under $\mathbb{Z}_N$ group. So, for example, consider the k=1 case, the physical state space at the intermediate scale can be represented as

$$\sum_{i=1}^{N} \sigma^i \mid \cdots +_i \cdots \rangle.$$

This is actually the state for one "magnon" in the spin chain if we replace σ with e^{ip}, where p is the momentum of the magnon.

Finite Symmetries: CPT

CPT-symmetries in quantum field theory can also descend to the spin chain. They act as the following:

$$
\begin{aligned}
\mathcal{P} S_{\pm,i} \mathcal{P}^{-1} &= S_{\mp,i} & \qquad \mathcal{P} S_{z,i} \mathcal{P}^{-1} &= -S_{z,i}, \\
\mathcal{T} S_{\pm,i} \mathcal{T}^{-1} &= -S_{\pm,i} & \qquad \mathcal{T} S_{z,i} \mathcal{T}^{-1} &= S_{z,i}, \\
\mathcal{C} S_{\pm,i} \mathcal{C}^{-1} &= -S_{\mp,i} & \qquad \mathcal{C} S_{z,i} \mathcal{C}^{-1} &= -S_{z,i}.
\end{aligned}
$$

These symmetries act on the domain walls as

Finite Symmetries: CPT

CPT-symmetries in quantum field theory can also descend to the spin chain. They act as the following:

$$\mathcal{P}S_{\pm,i}\mathcal{P}^{-1} = S_{\mp,i} \qquad \mathcal{P}S_{z,i}\mathcal{P}^{-1} = -S_{z,i},$$
$$\mathcal{T}S_{\pm,i}\mathcal{T}^{-1} = -S_{\pm,i} \qquad \mathcal{T}S_{z,i}\mathcal{T}^{-1} = S_{z,i},$$
$$\mathcal{C}S_{\pm,i}\mathcal{C}^{-1} = -S_{\mp,i} \qquad \mathcal{C}S_{z,i}\mathcal{C}^{-1} = -S_{z,i}.$$

These symmetries act on the domain walls as

$$\mathcal{P}\mathcal{D}_i\mathcal{P}^{-1} = \bar{\mathcal{D}}_i \qquad \mathcal{P}\mathcal{D}_N\mathcal{P}^{-1} = \left(\frac{\widetilde{q}}{|\widetilde{q}|}\right)^2 \bar{\mathcal{D}}_N,$$
$$\mathcal{T}\mathcal{D}_i\mathcal{T}^{-1} = \mathcal{D}_i \qquad \mathcal{T}\mathcal{D}_N\mathcal{T}^{-1} = \left(\frac{\widetilde{q}}{|\widetilde{q}|}\right)^{-2} \mathcal{D}_N,$$
$$\mathcal{C}\mathcal{D}_i\mathcal{C}^{-1} = \bar{\mathcal{D}}_i \qquad \mathcal{C}\mathcal{D}_N\mathcal{C}^{-1} = \bar{\mathcal{D}}_N,$$

where we have used the fact that $\mathcal{T}i\mathcal{T}^{-1} = -i$ for $i^2+1=0$. From the above, one can observe that $\mathcal{P}$ and $\mathcal{T}$ may be violated individually unless $\widetilde{q} = \pm 1$. However, we will see that the scattering factor could also break the $\mathcal{T}$-symmetry if it is not a pure phase factor.

Open v.s Closed Spin Chain

The general vacuum equations look like the following:

$$\left(e^{ip_a}\right)^N = \widetilde{q} \prod_{b \neq a} S_{ab}.$$

Open v.s Closed Spin Chain

The general vacuum equations look like the following:

$$\left(e^{ip_a}\right)^N = \widetilde{q} \prod_{b \neq a} S_{ab}.$$

• When the scattering factor, S_{ab}, is a pure phase: Besides a usually closed spin chain with $\widetilde{q} = 1$ defined in the literature, we claim that the anti-periodic spin chain with $\widetilde{q} = -1$ is also a *closed* one. On the other hand, the open spin chain in this situation has $\widetilde{q} \neq \pm 1$.

Open v.s Closed Spin Chain

The general vacuum equations look like the following:

$$\left(e^{ip_a}\right)^N = \widetilde{q} \prod_{b \neq a} S_{ab}.$$

• When the scattering factor, S_{ab}, is a pure phase: Besides a usually closed spin chain with $\widetilde{q} = 1$ defined in the literature, we claim that the anti-periodic spin chain with $\widetilde{q} = -1$ is also a *closed* one. On the other hand, the open spin chain in this situation has $\widetilde{q} \neq \pm 1$.
• When the scattering factor is not a pure phase: It is always an open spin chain for any $\widetilde{q}$.

Hamiltonian

For a closed spin chain, it is natural to propose the Hamiltonian as

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i + f(\mathcal{D}_i, \bar{\mathcal{D}}_i),$$

where the factor $f(\mathcal{D}_i, \bar{\mathcal{D}}_i)$ can be fixed if we require that the state spaces are eigenstates of this Hamiltonian. We have assumed the fact that all other matter representations are a subspace of the tensor product of fundamental matters. This is not true for general, and for a general case, one may need to take into account the so-called higher symmetries. We will not focus on this in this talk.

Hamiltonian

For a closed spin chain, it is natural to propose the Hamiltonian as

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i + f(\mathcal{D}_i, \bar{\mathcal{D}}_i),$$

where the factor $f(\mathcal{D}_i, \bar{\mathcal{D}}_i)$ can be fixed if we require that the state spaces are eigenstates of this Hamiltonian. We have assumed the fact that all other matter representations are a subspace of the tensor product of fundamental matters. This is not true for general, and for a general case, one may need to take into account the so-called higher symmetries. We will not focus on this in this talk.

For an open spin chain, we take the (anti-)holomorphic part of the above.

A Summary

Let us summarize what we have. Before Nekrasov and Shatashvili about the correspondence between the integrable system and the gauge theory:

$\mathrm{Y}(p_a)$	$\leftrightarrow$	$\widetilde{W}_{\mathrm{eff}}(\sigma_a)$
p_a	$\leftrightarrow$	σ_a
k-particle sector	$\leftrightarrow$	gauge group $U(k)$
N-sites	$\leftrightarrow$	flavor group $SU(N)$
twisted boundary	$\leftrightarrow$	$t = r - i\theta$
(in-)homogeneities	$\leftrightarrow$	twisted masses

Spin chain is not manifest in their studies.

A Summary

Let us summarize what we have. Before Nekrasov and Shatashvili about the correspondence between the integrable system and the gauge theory:

$\mathrm{Y}(p_a)$	$\leftrightarrow$	$\widetilde{W}_{\mathrm{eff}}(\sigma_a)$
p_a	$\leftrightarrow$	σ_a
k-particle sector	$\leftrightarrow$	gauge group $U(k)$
N-sites	$\leftrightarrow$	flavor group $SU(N)$
twisted boundary	$\leftrightarrow$	$t = r - i\theta$
(in-)homogeneities	$\leftrightarrow$	twisted masses

Spin chain is not manifest in their studies.
Based on the vacuum structure and the dynamics of the domainwalls. We have shown that the spin chain can emerge from the $\mathcal{N} = (2, 2)$ supersymmetric gauge theory in two steps.

A Summary

Let us summarize what we have. Before Nekrasov and Shatashvili about the correspondence between the integrable system and the gauge theory:

$\Upsilon(p_a)$	$\leftrightarrow$	$\widetilde{W}_{\text{eff}}(\sigma_a)$
p_a	$\leftrightarrow$	σ_a
k-particle sector	$\leftrightarrow$	gauge group $U(k)$
N-sites	$\leftrightarrow$	flavor group $SU(N)$
twisted boundary	$\leftrightarrow$	$t = r - i\theta$
(in-)homogeneities	$\leftrightarrow$	twisted masses

Spin chain is not manifest in their studies.
Based on the vacuum structure and the dynamics of the domainwalls. We have shown that the spin chain can emerge from the $\mathcal{N} = (2,2)$ supersymmetric gauge theory in two steps.

- Write the ground states as the Hilbert space formula, and (isomorphic-)map them to the magnon's configuration in the spin chain.
- Kinetic and dynamics of magnons map to the kinetic and dynamics of domainwalls.

So the above similarities found by Nekrasov and Shatashvili between the integrable systems and the gauge theories are just a consequence of our framework.

Examples: Gr(k;N)

Now we focus on the simplest example: gauged linear sigma model for Gr(k;N). The vacuum equations are

$$(\sigma_a)^N = \Lambda^N.$$

If we use the variable $\widetilde{\sigma}_a = \sigma_a \Lambda^{-1}$, the vacuum equations reduce to

$$(\widetilde{\sigma}_a)^N = 1.$$

These equations are Bethe-equations of the XX model. And we expect that the spin chain is a closed one.

Examples: Gr(k;N)

Now we focus on the simplest example: gauged linear sigma model for Gr(k;N). The vacuum equations are

$$(\sigma_a)^N = \Lambda^N.$$

If we use the variable $\widetilde{\sigma}_a = \sigma_a \Lambda^{-1}$, the vacuum equations reduce to

$$(\widetilde{\sigma}_a)^N = 1.$$

These equations are Bethe-equations of the XX model. And we expect that the spin chain is a closed one.

The Hamiltonian is simple:

$$H = \sum_i \mathcal{D}_i + \sum_i \bar{\mathcal{D}}_i.$$

The eigenstates are k-magnons

$$\mid k\rangle := A_k \sum_{1 \le j_1 < \cdots < j_k \le N} \prod_{a=1}^{k} \sum_{\mathcal{W} \in S_k} \left(\widehat{\sigma}_{\mathcal{W}(a)}\right)^{j_a} |j_1, \cdots, j_k\rangle_k.$$

The eigenvalue is

$$H \cdot \mid k\rangle := (e_1(\sigma) + e_1(\bar{\sigma})) \mid k\rangle$$

One might have noticed that the holomorphic part of the eigenvalue $e_1(\sigma)$ is the cohomology generator of $H^{1,1}(Gr(k;N))$. One may expect that all of the generators can be constructed as the eigenvalues of the corresponding operators. It was first done by mathematicians (Postnikov '05, C. Korff and C. Stroppe '09) that one can construct a general elementary function:

One might have noticed that the holomorphic part of the eigenvalue $e_1(\sigma)$ is the cohomology generator of $H^{1,1}(Gr(k;N))$. One may expect that all of the generators can be constructed as the eigenvalues of the corresponding operators. It was first done by mathematicians (Postnikov '05, C. Korff and C. Stroppe '09) that one can construct a general elementary function:

$$e_r(\mathcal{D}).$$

The eigenvalue of this is $e_r(\sigma)$. One can further construct the Schur operators $S_\lambda(\mathcal{D})$ that the eigenvalues are

$$S_\lambda(\sigma).$$

Quantum Cohomology

One might have noticed that the holomorphic part of the eigenvalue $e_1(\sigma)$ is the cohomology generator of $H^{1,1}(Gr(k;N))$. One may expect that all of the generators can be constructed as the eigenvalues of the corresponding operators. It was first done by mathematicians (Postnikov '05, C. Korff and C. Stroppe '09) that one can construct a general elementary function:

$$e_r(\mathcal{D}).$$

The eigenvalue of this is $e_r(\sigma)$. One can further construct the Schur operators $S_\lambda(\mathcal{D})$ that the eigenvalues are

$$S_\lambda(\sigma).$$

In this context, the quantum cohomology of Gr(k;N) can be reconstructed from the XX model.

One might have noticed that the holomorphic part of the eigenvalue $e_1(\sigma)$ is the cohomology generator of $H^{1,1}(Gr(k;N))$. One may expect that all of the generators can be constructed as the eigenvalues of the corresponding operators. It was first done by mathematicians (Postnikov '05, C. Korff and C. Stroppe '09) that one can construct a general elementary function:

$$e_r(\mathcal{D}).$$

The eigenvalue of this is $e_r(\sigma)$. One can further construct the Schur operators $S_\lambda\left(\mathcal{D}\right)$ that the eigenvalues are

$$S_\lambda\left(\sigma\right).$$

In this context, the quantum cohomology of Gr(k;N) can be reconstructed from the XX model.

Open XX model

If we start from the vacuum equations

$$(\sigma_a)^N = \Lambda^N .$$

The emergent spin chain is an open one.

Open XX model

If we start from the vacuum equations

$$(\sigma_a)^N = \Lambda^N.$$

The emergent spin chain is an open one.
All the discussions are similar to the closed one by only focusing on the (anti-)holomorphic part.

K-theoretic of XX model

One can lift gauged linear sigma models to the 3d CS-matter theories on space-time $\mathbb{R}^2 \times S^1$. Compared to the 2d theory, one has one more parameter to label the vacua and others: the so-called Chern-Simons levels. We only focus on the gauge Chern-Simons level $k_{U(k)}$. We split it into two factors: $k_{U(1)}$ and $k_{SU(k)}$.

K-theoretic of XX model

One can lift gauged linear sigma models to the 3d CS-matter theories on space-time $\mathbb{R}^2 \times S^1$. Compared to the 2d theory, one has one more parameter to label the vacua and others: the so-called Chern-Simons levels. We only focus on the gauge Chern-Simons level $k_{U(k)}$. We split it into two factors: $k_{U(1)}$ and $k_{SU(k)}$.

• If we choose (Kapustin-Willet '2013)

$$k_{U(1)} = k_{SU(k)} = -\frac{N}{2},$$

the vacuum equations are

$$(1 - x_a)^N = \widetilde{q}.$$

So the integrable system is almost identical to one for two-dimensional gauge theory with the only change being to replace σ by $1 - x$.

K-theoretic XX model

- If we choose (Jockers et.al, Ueda and Yoshida '2019, Gu et.al '20 and '22)

$$k_{U(1)} = -\frac{N}{2}, \qquad k_{SU(k)} = k - \frac{N}{2},$$

the vacuum equations are

$$(1 - x_a)^N = \widetilde{q}\,\frac{(x_a)^k}{\prod_{b=1}^{k} x_b}.$$

K-theoretic XX model

- If we choose (Jockers et.al, Ueda and Yoshida '2019, Gu et.al '20 and '22)

$$k_{U(1)} = -\frac{N}{2}, \qquad k_{SU(k)} = k - \frac{N}{2},$$

the vacuum equations are

$$(1 - x_a)^N = \widetilde{q} \frac{(x_a)^k}{\prod_{b=1}^k x_b}.$$

There is "interaction factor" on the right hand side

$$S_{ab} = -\frac{x_a}{x_b}.$$

This factor means the magnons interacts each other. Since this factor of this case is not a pure phase, the spin chain of this case is always an open one regardless of what the value of $\widetilde{q}$ is. This is because if we regard

$$1 - x_a := e^{ip_a}$$

The momentum p_a is not real.

State Space

One can propose the interacting state space as follows:

$$A_k \sum_{1\le j_1<\cdots<j_k\le N} \sum_{\mathcal{W}\in S_k} S_{\mathcal{W}(a)} e^{\sum_{a=1}^{k} ip_{\mathcal{W}(a)} j_a} |j_1 < \cdots < j_k\rangle_k ,$$

where A_k is an overall normalization factor, and $\sum_{\mathcal{W}\in S_k} S_{\mathcal{W}}$ is a multiplication of scattering factors S_{ab}. One can show that the proposed state space is consistent with the symmetries.

State Space

One can propose the interacting state space as follows:

$$A_k \sum_{1 \le j_1 < \cdots < j_k \le N} \sum_{\mathcal{W} \in S_k} S_{\mathcal{W}(a)} e^{\sum_{a=1}^{k} i p_{\mathcal{W}(a)} j_a} |j_1 < \cdots < j_k\rangle_k ,$$

where A_k is an overall normalization factor, and $\sum_{\mathcal{W} \in S_k} S_{\mathcal{W}}$ is a multiplication of scattering factors S_{ab}. One can show that the proposed state space is consistent with the symmetries. The fundamental Hamiltonian of this system is a complex one, which was first proposed in a slightly different context by Gorbounov and Korff '2014:

$$h = \sum_{i}^{N} \mathcal{D}_i - \sum_{|i_1 - i_2| \mod N > 1}^{N} \mathcal{D}_{i_1} \mathcal{D}_{i_2} + \sum_{|i_a - i_b| \mod N > 1}^{N} \mathcal{D}_{i_1} \mathcal{D}_{i_2} \mathcal{D}_{i_3} + \cdots .$$

The number of sites is N, so only finitely many terms act non-trivially, and the series therefore terminates.

Gr(2;N)

Let us check whether the ground state wave functions proposed in our paper for this case are indeed the eigenstates of the Hamiltonian. To shorten the notation, we denote the ground state as $|\,\omega\rangle_2 = \sum_{1\le j_1<j_2\le N} a(j_1,j_2)|j_1 < j_2\rangle_2$, where

$$a(j_1,j_2) = A\left(e^{i(p_1 j_1 + p_2 j_2)} + S_{21} e^{i(p_2 j_1 + p_1 j_2)}\right).$$

The boundary condition is $a(j_1, j_2 + N) = q \cdot a(j_2, j_1)$, which gives

$$e^{i(p_1 j_1 + p_2 j_2)} e^{ip_2 N} + S_{21} e^{ip_1 N} e^{i(p_2 j_1 + p_1 j_2)} = q\left(e^{i(p_1 j_2 + p_2 j_1)} + S_{21} e^{i(p_2 j_2 + p_1 j_1)}\right).$$

The vacuum equations can be read from the above as

$$e^{ip_1 N} = qS_{12}, \qquad e^{ip_2 N} = qS_{21}.$$

So it is consistent.

Gr(2;N)

To compute $H \mid \omega\rangle_2$, special care is needed when two overturned spins are sitting next to each other. We find

$$\begin{aligned} H \mid \omega\rangle_2 &= \sum_{1\leq j_1<j_2\leq N} (a(j_1+1, j_2) + a(j_1, j_2+1) - a(j_1+1, j_2+1)) \mid j_1 < j_2\rangle_2 \\ &\quad - \sum_{1\leq j\leq N} (a(j+1, j+1) - a(j+1, j+2)) \mid j < j+1\rangle_2. \end{aligned}$$

In order to obey the eigenstate condition, the contact terms in the last line of the above equation should be vanishing:

$$a(j+1, j+1) - a(j+1, j+2) = 0.$$

If we test the coefficient as $a(j_1, j_2) = Ce^{i(p_1 j_1 + p_2 j_2)} + De^{i(p_1 j_2 + p_2 j_1)}$, the vanishing contact terms all give

$$\frac{C}{D} = -\frac{x_1}{x_2}.$$

This is certainly consistent with our scattering factor S_{12} in the vacuum equations. The procedure applies to a general Grassmannian.

Geometric basis

The Hamiltonian actually has a geometrical meaning. See Buch and Mihalcea '08. The eigenvalue of this Hamiltonian is the first Schubert class of the $Gr(k;N)$:

$$H \mid \omega\rangle_k = \mathcal{O}_\Box \mid \omega\rangle_k$$

where

$$\mathcal{O}_\Box := 1 - \prod_{a=1}^{k} x_a.$$

Geometric basis

The Hamiltonian actually has a geometrical meaning. See Buch and Mihalcea '08. The eigenvalue of this Hamiltonian is the first Schubert class of the $Gr(k; N)$:

$$H \mid \omega\rangle_k = \mathcal{O}_\square \mid \omega\rangle_k$$

where

$$\mathcal{O}_\square := 1 - \prod_{a=1}^{k} x_a.$$

For example, if k=2 the factor in the above equation is

$$a(j_1+1, j_2) + a(j_1, j_2+1) - a(j_1+1, j_2+1) = (z_1 + z_2 - z_1 z_2) a(j_1, j_2)$$

where the coefficient $(z_1 + z_2 - z_1 z_2) = 1 - x_1 x_2$ (where $z = 1 - x$) is indeed the first Schubert class of $Gr(2, N)$. Thus, one may naturally expect that higher Schubert classes are the eigenvalues of the higher Hamiltonian as well. Quantum K-theory of $Gr(k; N)$ from the integrable model has been investigated by Gorbounov and Korff '2014, although their construction was based on a five-vertex model rather than a spin chain.

Seiberg(-like) Duality is P-symmetry

In gauge theories, there is a so-called Seiberg(-like) duality between two gauge theories:

$$Gr(k;N) \cong Gr(N-k;N).$$

The gauged linear sigma models for them are different: one is $U(k)$ gauge group and the other one is $U(N-k)$. However, the two gauge theories share the same vacuum structures and BPS spectrum. This relation is nontrivial in the sense that it is not a symmetry of a single gauge theory.

Seiberg(-like) Duality is P-symmetry

In gauge theories, there is a so-called Seiberg(-like) duality between two gauge theories:

$$Gr(k; N) \cong Gr(N-k; N).$$

The gauged linear sigma models for them are different: one is $U(k)$ gauge group and the other one is $U(N-k)$. However, the two gauge theories share the same vacuum structures and BPS spectrum. This relation is nontrivial in the sense that it is not a symmetry of a single gauge theory.
But from the spin chain, it is a P-symmetry. To see this, we notice that the spin up/down maps to the spin down/up under the P symmetry.

$$P : |+\rangle \mapsto |-\rangle.$$

This says that the k-magnons map to the N-k magnons. This is exactly the Seiberg(-like) duality between two gauge theories.

A Unification of Gauge Theories Via Spin Chain

In the spin chain, we have spin operators: $S_\pm$. They have no field theoretic correspondence since they change the number of magnons which can not happen in a field theoretic process.

A Unification of Gauge Theories Via Spin Chain

In the spin chain, we have spin operators: $S_{\pm}$. They have no field theoretic correspondence since they change the number of magnons which can not happen in a field theoretic process. If we map the k-magnons of the spin chain to the U(k) gauge theory, then from the point of view of the spin chain. Different gauge theories can be connected via the spin operators. Maybe something deep behind this...

The Yang-Baxter Equation

In the Yang-Baxter equation, the R-matrix is the fundamental variable used to describe the dynamics. It is defined by a map

$$R_{ab} : \mathcal{V}_a \otimes \mathcal{V}_b \mapsto \mathcal{V}_a \otimes \mathcal{V}_b.$$

The Yang-Baxter Equation

In the Yang-Baxter equation, the R-matrix is the fundamental variable used to describe the dynamics. It is defined by a map

$$R_{ab} : \mathcal{V}_a \otimes \mathcal{V}_b \mapsto \mathcal{V}_a \otimes \mathcal{V}_b.$$

For the cases in this talk, the basis chosen for $\mathcal{V}$ is

$$|+,+\rangle, \quad |-,+\rangle, \quad |+,-\rangle, \quad |-,-\rangle.$$

Since the basis is transformed under the R-matrix, we expect that the off-diagonal entries of the R-matrix correspond to the (anti)-domain walls in our context.

The Yang-Baxter Equation

In the Yang-Baxter equation, the R-matrix is the fundamental variable used to describe the dynamics. It is defined by a map

$$R_{ab} : \mathcal{V}_a \otimes \mathcal{V}_b \mapsto \mathcal{V}_a \otimes \mathcal{V}_b.$$

For the cases in this talk, the basis chosen for $\mathcal{V}$ is

$$|+,+\rangle, \quad |-,+\rangle, \quad |+,-\rangle, \quad |-,-\rangle.$$

Since the basis is transformed under the R-matrix, we expect that the off-diagonal entries of the R-matrix correspond to the (anti)-domain walls in our context.
Example: For the Grassmannian, we have the R-matrix

$$\begin{pmatrix} a & & & \\ & b & c & \\ & \widetilde{c} & \widetilde{b} & \\ & & & \widetilde{a} \end{pmatrix}.$$

So two options for a generic $\widetilde{q}$: c or $\widetilde{c}$ is vanishing. They are the so-called five-vertex models.

R-matrix

For XXX or XXZ model, the Hamiltonian is a hermitian operator, which says:

$$a = \widetilde{a}, \qquad b = \widetilde{b}, \qquad c = \widetilde{c}.$$

R-matrix

For XXX or XXZ model, the Hamiltonian is a hermitian operator, which says:

$$a = \widetilde{a}, \qquad b = \widetilde{b}, \qquad c = \widetilde{c}.$$

For XYZ model, we also have domainwall configurations:

$$|-,-\rangle \mapsto |+,+\rangle, \quad \text{or} \quad |+,+\rangle \mapsto |-,-\rangle.$$

The R-matrix for this model should be enlarged as

$$\begin{pmatrix} a & & & d \\ & b & c & \\ & c & b & \\ d & & & a \end{pmatrix}.$$

One can derive the Yang-Baxter equation from the dynamic constraint of domainwalls. See Gu '22.

A comment on the four-dimensional CS theory

In 4d CS theory the space-time is $\Sigma \times C$, and the integrable systems depend on the C:
- if $C = \mathbb{C}$, it gives XXX model.

A comment on the four-dimensional CS theory

In 4d CS theory the space-time is $\Sigma \times C$, and the integrable systems depend on the C:
- if $C = \mathbb{C}$, it gives XXX model.
- if $C = \mathbb{C}^\star$, it leads to XXZ model.

A comment on the four-dimensional CS theory

In 4d CS theory the space-time is $\Sigma \times C$, and the integrable systems depend on the C:
- if $C = \mathbb{C}$, it gives XXX model.
- if $C = \mathbb{C}^\star$, it leads to XXZ model.
- if $C = \mathrm{Torus}$, it corresponds to XYZ model.

A comment on the four-dimensional CS theory

In 4d CS theory the space-time is $\Sigma \times C$, and the integrable systems depend on the C:
- if $C = \mathbb{C}$, it gives XXX model.
- if $C = \mathbb{C}^\star$, it leads to XXZ model.
- if $C = \mathrm{Torus}$, it corresponds to XYZ model.

Recall that the spin operator, $S_z = \sum_i S_{z,i}$, is actually half of the axial-R symmetry operator

$$S_z = \frac{1}{2} F_A$$

If we regard the GLSM as an effective theory of 4d $\mathcal{N}$=1 gauge theory on $\Sigma \times C$, and the operator F_A depends on the sharp of C, which is exactly what the 4d-CS theory suggested. There should be more deep connections...

Conclusion

- We showed how a spin chain emerges from the two-dimensional supersymmetric gauge theory. **Domainwalls play a crucial role**.
- We discussed several examples.
- We talked about the Yang-Baxter equations with new insight, and commented on the four-dimensional CS theory.

Thanks!