Towards a classification of Fermionic Rational CFTs

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Based on: J. Bae, ZD, K. Lee, S. Lee, M. Sarkis, arXiv: 2010.12392/2108.01647

ZD, K. Lee, S. Lee, L. Li, arXiv: 2210.06805

Joint HEP-TH seminar

KIAS

Overview

- Holomorphic modular bootstrap
- Extension to fermionic theories
- Integrality
- Summary

Motivation

- Describe critical phenomena of phase transitions

- Describe 2D world-sheet of string theory

- Deep mathematical structure

CFT Basics

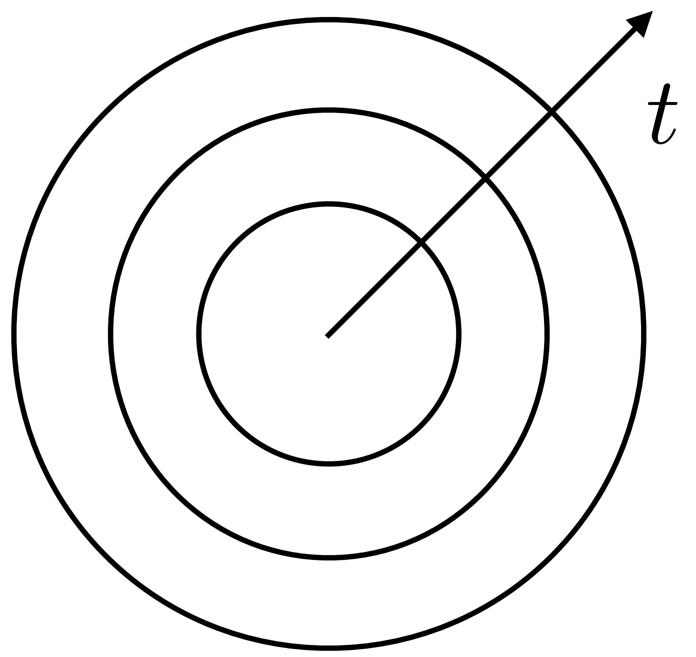
- 2D conformal field theory contains two copies of Virasoro algebras as its symmetry algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \quad (m \in \mathbb{Z})$$

- Conformal primaries are highest-weight states:

$$L_0|h\rangle = h|h\rangle, \quad L_{n>0}|h\rangle = 0$$ Conformal descendants: $$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle$$

- Radial quantization

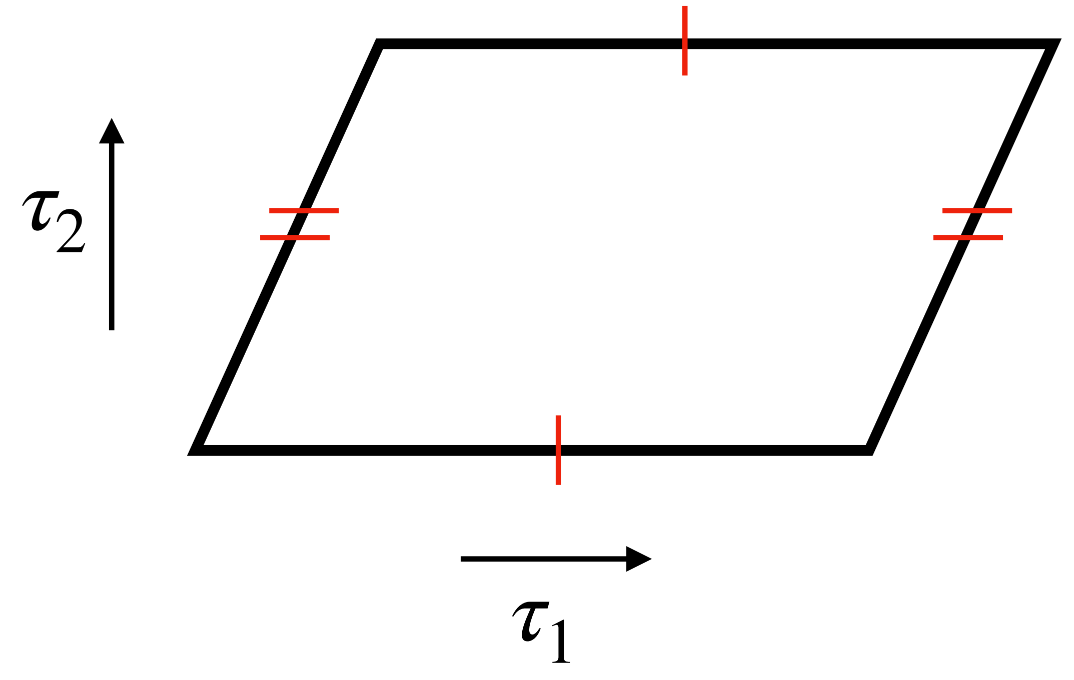


$$\mathcal{H}_{S^1} = \bigoplus_{h,\bar{h}} \overbrace{d_{h,\bar{h}}}^{\text{Degeneracy}} V_h \otimes V_{\bar{h}}$$

Highest-weight Representation

CFT Basics

- Torus Partition Function



$$\begin{aligned} Z(\tau,\bar\tau) &= \mathrm{Tr}_{\mathcal{H}_{S^1}}\left[\exp\left(2\pi i\tau_1 P - 2\pi\tau_2 H\right)\right] \\ &= \mathrm{Tr}_{\mathcal{H}_{S^1}}\left[q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right] \\ &= \sum_{h,\bar h} d_{h,\bar h}\, \underbrace{\mathrm{Tr}_{V_h}\left[q^{L_0-\frac{c}{24}}\right]}_{\chi_h(\tau)}\, \underbrace{\mathrm{Tr}_{V_{\bar h}}\left[\bar{q}^{\bar{L}_0-\frac{c}{24}}\right]}_{\chi_{\bar h}(\bar\tau)} \end{aligned}$$

$$\begin{aligned} q &= e^{2\pi i(\tau_1+i\tau_2)} \\ &= e^{2\pi i\tau} \end{aligned}$$

Character: counting states in rep.

- Modular invariance

SL$(2,\mathbb{Z})$ symmetry group generated by T and S

$$\mathcal{T} : Z(\tau,\bar\tau) \to Z(\tau+1,\bar\tau+1) = \mathrm{Tr}[e^{2\pi i J} q^{L_0-\frac{c}{24}}\bar{q}^{\bar{L}_0-\frac{c}{24}}]$$

States must have integer spin

$$\mathcal{S} : Z(\tau,\bar\tau) \to Z(-\frac{1}{\tau},-\frac{1}{\bar\tau})$$

Impose constraints on the spectrum

Rational CFT

- Rational CFT has only finitely many conformal primaries $\Longrightarrow$ c and all h rational!

[Anderson, Moore 1988] [Vafa 1988]

$$Z(\tau,\bar\tau) = \sum_{i,j=0}^{n-1} \chi_i \, d_{ij} \, \bar\chi_j$$

Lattice CFT, WZW model, etc

$$= \underbrace{\left(\chi_0, \quad \cdots, \quad \chi_{n-1}\right)}_{\downarrow\ \text{Classify?}} \cdot \mathcal{M} \cdot \begin{pmatrix} \bar\chi_0 \\ \vdots \\ \bar\chi_{n-1} \end{pmatrix}$$

- (Holomorphic) characters form a weight zero vector-valued modular form for SL$(2,\mathbb{Z})$

$$\chi_i(\tau+1) = \sum_{j=0}^{n-1} T_{ij}\chi_j(\tau), \quad \chi_i(-\frac{1}{\tau}) = \sum_{j=0}^{n-1} S_{ij}\chi_j(\tau)$$

Modularity

- Recall MF of weight k

Weight

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^k f(\tau), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \qquad \text{e.g.} \qquad E_k(\tau)=\sum_{\substack{m,n\in\mathbb{Z}\\ m,n\neq 0}} \frac{1}{(m\tau+n)^k}$$

$\tau \in \mathbb{H}$

Graded ring of holomorphic modular forms $M_k\big(\mathrm{SL}(2,\mathbb{Z})\big)$ freely generated by E_4 and E_6

Modular Forms also have $q=\exp(2\pi i\tau)$-expansion

- Characters span solutions to n-th order Modular Linear Differential Equation (MLDE)

[Anderson, Moore 1988]

Related to null vectors [Gaberdiel, Keller 0804.0489]

Modular form of weight $2(n-a)$

$$\left[D_\tau^n+\sum_{a=0}^{n-1}\phi_a(\tau)D_\tau^a\right]\chi_i(\tau)=0$$

Serre Derivative $\quad D_\tau=\frac{1}{2\pi i}\partial_\tau-\frac{k}{12}E_2: \quad M_k\to M_{k+2}$

MLDE

- Wronskian method

$$D_\tau^n \chi + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a \chi = 0$$

$$\det \begin{pmatrix} \chi & \chi_0 & \cdots & \chi_{n-1} \\ D\chi & D\chi_0 & \cdots & D\chi_{n-1} \\ \vdots & \vdots & & \vdots \\ D^k\chi & D^k\chi_0 & \cdots & D^k\chi_{n-1} \\ \vdots & \vdots & & \vdots \\ D^n\chi & D^n\chi_0 & \cdots & D^n\chi_{n-1} \end{pmatrix} = 0 \quad \Longrightarrow \quad W_k = \det \begin{pmatrix} \chi_0 & \cdots & \chi_{n-1} \\ D_\tau\chi_0 & \cdots & D_\tau\chi_{n-1} \\ \vdots & & \vdots \\ D_\tau^{k-1}\chi_0 & \cdots & D_\tau^{k-1}\chi_{n-1} \\ D_\tau^{k+1}\chi_0 & \cdots & D_\tau^{k+1}\chi_{n-1} \\ \vdots & & \vdots \\ D_\tau^n\chi_0 & \cdots & D_\tau^n\chi_{n-1} \end{pmatrix}, \quad \phi_k(\tau) = (-1)^{n-k}\frac{W_k}{W_n}$$

Meromorphic

- Physical constraint: characters have no poles in upper half-plane

- Poles of $\phi_a(\tau)$ all come from zeros of W_n

Holomorphic modular bootstrap

[Mathur, Mukhi, Sen 1988]

- Classify RCFT by n (number of χ_i) and l (order of zeros for W_n)

① Physical constraints on q-expansion

Integrality; Positivity; Unique Vacuum; Unitarity…

vacuum

$$\begin{cases} \chi_0 = q^{-c/24}(1+\sum a_i q^i), & \underline{a_i \in \mathbb{Z}^{\geq 0}} \\ \chi_i = q^{h_i - c/24}(1+\sum a_i q^i), & \underline{a_i \in \mathbb{Q}^{\geq 0}} \end{cases}$$

Bounded denominator

② Identify or disprove the solutions

- 2nd order holomorphic $(l=0)$ MLDE [Mathur, Mukhi, Sen 1988]

$M_k(SL(2,\mathbb{Z}))$ generated by E_4 and E_6

$$\left[D_\tau^2 + \mu E_4\right] f(\tau) = 0$$

Finite unitary family

$$\mu = -\frac{c(c+4)}{576}, \quad c = \left\{\frac{2}{5}, 1, 2, \frac{14}{5}, 4, \frac{26}{5}, 6, 7, \frac{38}{5}, 8\right\}$$

Mathur-Mukhi-Sen family

- MMS (Cvitanović-Deligne) Series

$$\left\{\frac{2}{5},1,2,\frac{14}{5},4,\frac{26}{5},6,7,(\frac{38}{5}),8\right\}$$
$$\Downarrow$$
$$\{\mathrm{LY},\underbrace{\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{g}_2,\mathfrak{d}_4,\mathfrak{f}_4,\mathfrak{e}_6,\mathfrak{e}_7,(\mathfrak{e}_{7\frac{1}{2}}),\mathfrak{e}_8}\}$$

Level 1 WZW model

$$\dim L(\theta) = \dim \mathfrak{g} = \frac{2(h^\vee+1)(5h^\vee-6)}{h^\vee+6}$$

$$\dim L(2\theta) = 5(h^\vee)^2\frac{(2h^\vee+3)(5h^\vee-6)}{(h^\vee+12)(h^\vee+6)} \quad \cdots$$

- Appears also
  - At the boundary of 2d numerical modular bootstrap [Collier, Lin, Yin 1608.06241][Bae, Lee, Song 1708.08815]
  - Gauge algebra (except $\mathfrak{a}_1, \mathfrak{g}_2$) of minimal 6d $\mathcal{N}=(1{,}0)$ SCFTs [Morrsion, Tyalor 1201.1943]
  - Instanton moduli space of simply laced $\mathfrak{g}$ = Higgs branch of 4d rank 1 $\mathcal{N}=2$ SCFT

[Beem et al. 1312.5344] $\Downarrow$ [Beem, Rastelli 1707.07679]

(Modified) Schur index = Vacuum character of WZW model at level $-\frac{h^\vee}{6}-1$ = non-unitary solution of 2nd order MLDE

Generalize to fermionic theories!

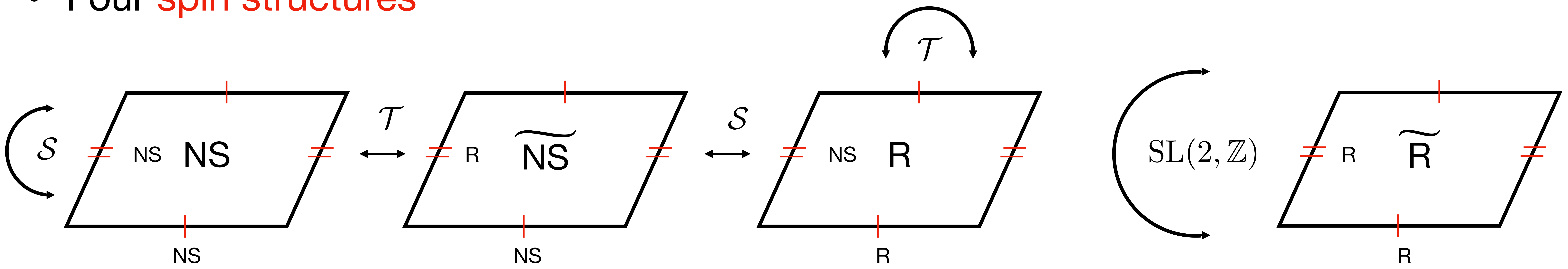
[J. Bae, ZD, K. Lee, S. Lee, M. Sarkis, 2010.12392]

Are there analogs of MMS family?

Torus Partition Function Revisited

- Four spin structures

NS = Anti-Periodic   R = Periodic



- Four partition functions

$$Z_{\text{NS}}(\tau,\bar\tau) = \text{Tr}_{\mathcal{H}_{\text{NS}}}\left[q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0 - c/24}\right], \quad Z_{\widetilde{\text{NS}}}(\tau,\bar\tau) = \text{Tr}_{\mathcal{H}_{\text{NS}}}\left[(-1)^F q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0 - c/24}\right]$$

$$Z_{\text{R}}(\tau,\bar\tau) = \text{Tr}_{\mathcal{H}_{\text{R}}}\left[q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0 - c/24}\right], \quad Z_{\widetilde{\text{R}}}(\tau,\bar\tau) = \text{Tr}_{\mathcal{H}_{\text{R}}}\left[(-1)^F q^{L_0-\frac{c}{24}}\,\bar q^{\bar L_0 - c/24}\right]$$

Witten index if ∃ SUSY

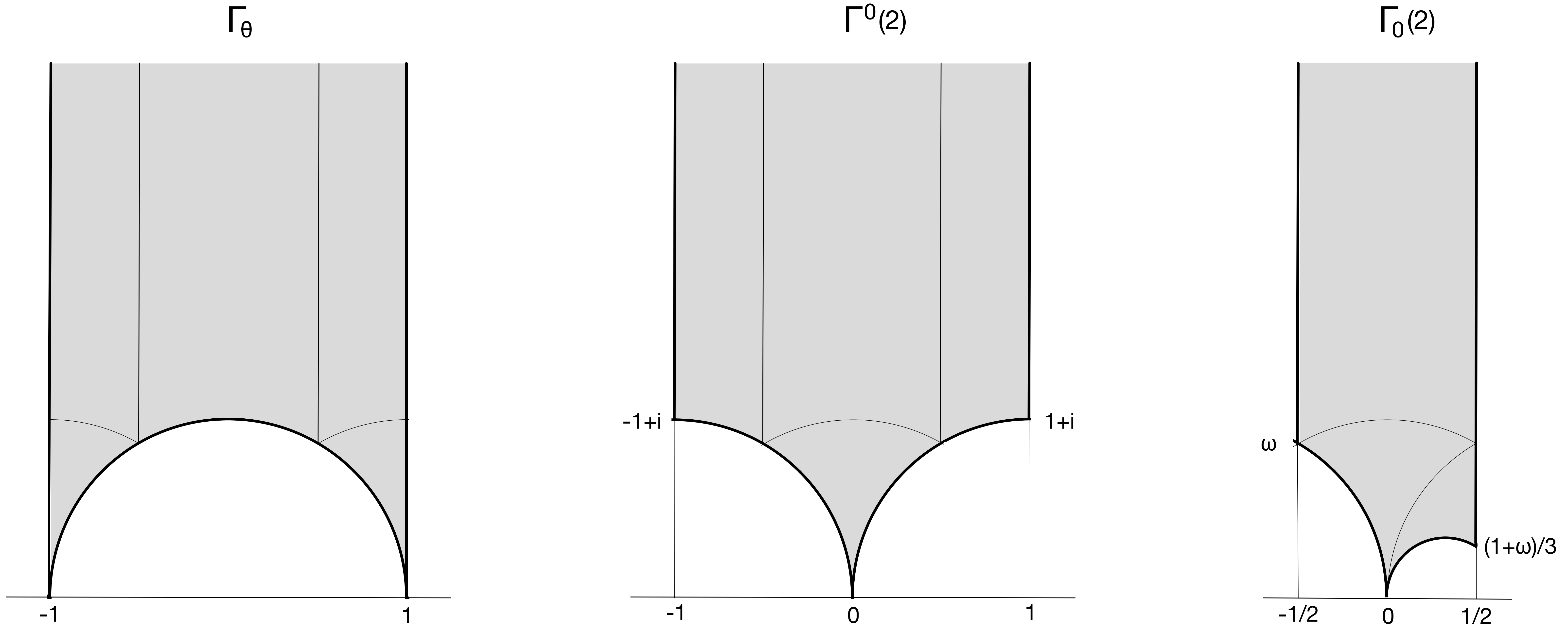
$$h_i \Longrightarrow h_i^{\text{NS}},\ h_i^{\text{R}}$$

Symmetry

- Level-two congruence subgroups of SL$(2,\mathbb{Z})$

$$\underbrace{\Gamma_\theta = \langle T^2, S\rangle}_{\mathrm{NS}} \quad \overset{\mathcal{T}}{\longleftrightarrow} \quad \underbrace{\Gamma^0(2) = \langle T^2, STS^{-1}\rangle}_{\widetilde{\mathrm{NS}}} \quad \overset{\mathcal{S}}{\longleftrightarrow} \quad \underbrace{\Gamma_0(2) = \langle T, ST^{-2}S^{-1}\rangle}_{\mathrm{R}}$$

- Fundamental domain



Fermionic MLDE

- MLDE for NS, $\widetilde{\text{NS}}$ and R

Focus on $\longrightarrow$

$$\left[D_\tau^n + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a\right] f_i^{\text{NS}}(\tau) = 0, \quad f_i^{\text{NS}} = q^{h_i^{\text{NS}} - \frac{c}{24}} \Big(\sum_{i=0}^{\infty} a_i q^{i/2}\Big)$$

$\mathcal{T}$

$$\left[D_\tau^n + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a\right] f_i^{\widetilde{\text{NS}}}(\tau) = 0, \quad f_i^{\widetilde{\text{NS}}} = q^{h_i^{\text{NS}} - \frac{c}{24}} \Big(\sum_{i=0}^{\infty} (-1)^i a_i q^{i/2}\Big)$$

$\mathcal{S}$

$$\left[D_\tau^n + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a\right] f_i^{\text{R}}(\tau) = 0, \quad f_i^{\text{R}} = q^{h_i^{\text{R}} - \frac{c}{24}} \Big(\sum_{i=0}^{\infty} b_i q^{i}\Big)$$

- Ring of holomorphic modular forms

$$M_{2k}(\Gamma_\theta) = \text{Sym}\langle -\vartheta_2^4, \vartheta_4^4 \rangle$$

$$M_{2k}(\Gamma^0(2)) = \text{Sym}\langle \vartheta_2^4, \vartheta_3^4 \rangle$$

ϑ_i Jacobi theta functions, weight 1/2

$$M_{2k}(\Gamma_0(2)) = \text{Sym}\langle \vartheta_3^4, \vartheta_4^4 \rangle$$

$(k \geq 1)$

Classification of fermionic RCFTs

- Holomorphic first order ($n = 1,\ l = 0$) MLDE for Γ_θ

$$M_k(\Gamma_\theta) = \mathrm{Sym}\langle -\vartheta_2^4, \vartheta_4^4\rangle \qquad \Big[D_\tau + \mu\left(-\vartheta_2^4(\tau) + \vartheta_4^4(\tau)\right)\Big] f^{\mathrm{NS}} = 0, \qquad \mu = \frac{c}{12}$$

$$\Downarrow \qquad \text{Change of variables}$$

$$\Big[\frac{d}{d\lambda} + \frac{c}{12}\frac{1-2\lambda}{\lambda(1-\lambda)}\Big] f^{\mathrm{NS}} = 0, \qquad \lambda = \frac{\vartheta_2^4}{\vartheta_3^4}$$

- All Solutions

$$f_0^{\mathrm{NS}}(\lambda) = \left[\frac{16}{\lambda(1-\lambda)}\right]^{\frac{c}{12}}$$

To have non-negative coefficients $c = \frac{k}{2}, \quad k \in \mathbb{N}$

- Tensor products of $k = 1$ theory. What is it? Majorana-Weyl fermion!

Ising Model

- Unitary minimal model $(m+1, m)$ with $c = 1 - \dfrac{6}{m(m+1)}$

- $m = 3$ is Ising model: $h = 0, \dfrac{1}{2}$ and $\dfrac{1}{16}$

- $$\chi_0(\tau) = \frac{1}{2}\left[\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} + \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right] = q^{-\frac{1}{24}\cdot\frac{1}{2}}(1 + q^2 + q^3 + 2q^4 + \cdots)$$
  $$\chi_\epsilon(\tau) = \frac{1}{2}\left[\sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} - \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}}\right] = q^{\frac{1}{2}-\frac{1}{24}\cdot\frac{1}{2}}(1 + q + q^2 + q^3 + \cdots)$$
  $$\chi_\sigma(\tau) = \frac{1}{\sqrt{2}}\sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}} = q^{\frac{1}{16}-\frac{1}{24}\cdot\frac{1}{2}}(1 + q + q^2 + 2q^3 + \cdots)$$

They satisfy a third order holomorphic bosonic MLDE

Rank 1 solution

- $k = 1\,(c = \frac{1}{2})$ solution for fermionic MLDE

$$(h = \frac{1}{2})$$

$$f_0^{\mathrm{NS}} = \left[\frac{16}{\lambda(1-\lambda)}\right]^{\frac{1}{24}} = \sqrt{\frac{\vartheta_3(\tau)}{\eta(\tau)}} = \chi_0(\tau) + \chi_\epsilon(\tau)$$

$$\mathcal{T} \Updownarrow$$

$$f_0^{\widetilde{\mathrm{NS}}} = \sqrt{\frac{\vartheta_4(\tau)}{\eta(\tau)}} = \chi_0(\tau) - \chi_\epsilon(\tau) \quad \Longrightarrow \quad \text{MW fermion}$$

$$\mathcal{S} \Updownarrow$$

$$f_0^{\mathrm{R}} = \sqrt{\frac{\vartheta_2(\tau)}{\eta(\tau)}} = \sqrt{2}\chi_\sigma(\tau)$$

- Partition function:

$$Z = |\chi_0|^2 + |\chi_\epsilon|^2 + |\chi_\sigma|^2$$
$$= \frac{1}{2}\Big(\underbrace{|\chi_0+\chi_\epsilon|^2}_{Z_{\mathrm{NS}}} + \underbrace{|\chi_0-\chi_\epsilon|^2}_{Z_{\widetilde{\mathrm{NS}}}} + \underbrace{|\sqrt{2}\chi_\sigma|^2}_{Z_{\mathrm{R}}} + \underbrace{0}_{Z_{\widetilde{\mathrm{R}}}}\Big)$$

due to fermonic zero mode

Fermionization

Ising model $\Longleftrightarrow$ Majorana fermion

Non-anomalous $\mathbb{Z}_2$ symmetry g — Gauging $\mathbb{Z}_2$ — $(-1)^F$symmetry

- More precisely

e.g. [Karch, Tong, Turner 1902.05550]

Couple to spin TQFT and gauge $\mathbb{Z}_2$ background field

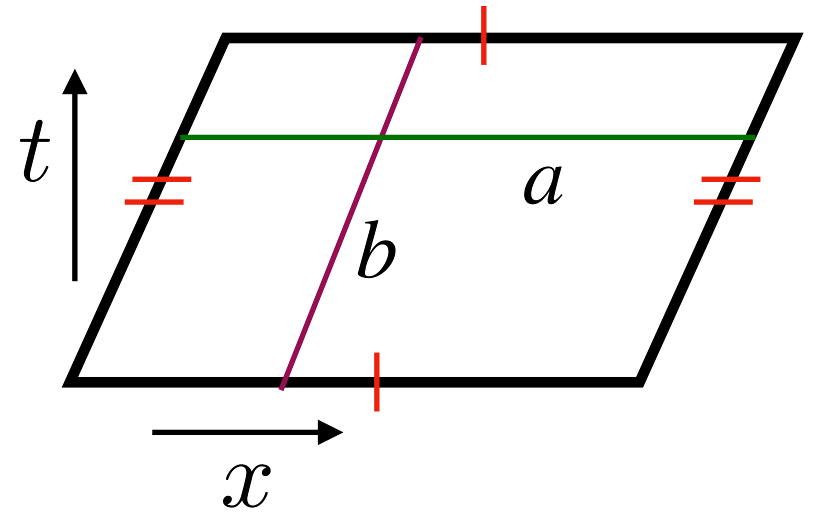
$$Z_{\mathscr{F}}(\rho) = \frac{1}{2^g} \sum_{t \in H^1(\Sigma_g, \mathbb{Z}_2)} Z_{\mathscr{B}}[t+\rho] \ \text{exp} \left[i\pi \left(\text{Arf}[t+\rho] + \text{Arf}[\rho] + \int t \cup S \right) \right]$$

sum over all spin structures t (GSO projection)

$$Z_{\mathscr{B}}(T) = \frac{1}{2^g} \sum_{s \in H^1(\Sigma_g, \mathbb{Z}_2)} Z_{\mathscr{F}}[s+\rho] \ \text{exp} \left[i\pi \left(\text{Arf}[T+\rho] + \text{Arf}[\rho] + \int s \cup T \right) \right]$$

Denote $Z_{(a,b)}$ the partition function with $\mathbb{Z}_2$ symmetry line a, b in spatial and time direction

$$\mathrm{Tr}_{V_{h_b}} [a \, q^{L_0 - \frac{c}{24}}]$$



$$Z_{\mathcal{F}}^{\mathrm{NS}} = \frac{1}{2} \left(Z_{(1,1)} + Z_{(g,1)} + Z_{(1,g)} - Z_{(g,g)} \right) ,$$

$$Z_{\mathcal{F}}^{\widetilde{\mathrm{NS}}} = \frac{1}{2} \left(Z_{(1,1)} + Z_{(g,1)} - Z_{(1,g)} + Z_{(g,g)} \right) ,$$

$$Z_{\mathcal{F}}^{\mathrm{R}} = \frac{1}{2} \left(Z_{(1,1)} - Z_{(g,1)} + Z_{(1,g)} + Z_{(g,g)} \right) ,$$

$$Z_{\mathcal{F}}^{\widetilde{\mathrm{R}}} = \frac{1}{2} \left(Z_{(1,1)} - Z_{(g,1)} - Z_{(1,g)} - Z_{(g,g)} \right) .$$

$$Z_{\mathcal{B}} = Z_{(1,1)} = Z_{\mathcal{F}}^{\mathrm{NS}} + Z_{\mathcal{F}}^{\widetilde{\mathrm{NS}}} + Z_{\mathcal{F}}^{\mathrm{R}} + Z_{\mathcal{F}}^{\widetilde{\mathrm{R}}}$$

Two-Character Case

- Holomorphic fermionic MLDE $\left[D_\tau^2 + \mu_1\left(-\vartheta_2^4(\tau) + \vartheta_4^4(\tau)\right)D_\tau + \mu_2\theta_3^8(\tau) + \mu_3 E_4(\tau)\right] f^{\rm NS}(\tau) = 0$

c and $h^{\rm NS}$ can only fix two of μ_i

Vacuum inside NS sector

Extra constraint $\mu_3 = 0$ or $a_1 = 0$ in $f_0^{\rm NS} = q^{-\frac{c}{24}}(1 + a_1 q^{\frac{1}{2}} + \cdots)$

"BPS" condition

Number of free MW fermions

- Strategy

① Physical constraints on q-expansion $\begin{cases} f_0^{\rm NS} = q^{-c/24}(1 + \sum a_i q^{\frac{i}{2}}), & \underline{a_i \in \mathbb{Z}^{\geq 0}} \\ f_1^{\rm NS} = q^{h^{\rm NS} - c/24}(1 + \sum a_i q^{\frac{i}{2}}), & \underline{a_i \in \mathbb{Q}^{\geq 0}} \end{cases}$

Put bound on the denominators of $c, h^{\rm NS}$ and numerically search

Bounded denominator

② Identify or disprove the solutions

Check the fusion algebra

Check the Ramond sector [Benjamin, Lin 2005.02394]

Result

type	property	central charge c
BPS, **I**	$h^{\mathrm{R}}_{-} = \frac{c}{24}, a_1 = 0$	$1, \frac{9}{4}, 6, \frac{39}{4}, 11, 12$
BPS, **II**	$h^{\mathrm{R}}_{-} = \frac{c}{24}, a_1 \neq 0$	$\frac{3}{4}, \frac{3}{2}, 3, 6, 9, \frac{21}{2}, \frac{45}{4}, 12$
non-BPS, **I**	$h^{\mathrm{R}}_{-} > \frac{c}{24}, h^{\mathrm{NS}} \neq \frac{1}{2}$	$\frac{7}{10}, \frac{133}{10}, \frac{91}{5}, \frac{102}{5}, 21, \frac{85}{4}, 22, \frac{114}{5}$
non-BPS, **II**	$h^{\mathrm{R}}_{-} > \frac{c}{24}, h^{\mathrm{NS}} = \frac{1}{2}$	$\frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}$
non-BPS, **III**	one-parameter family	16
non-BPS, **IV**	single-character	$\frac{17}{2}, 9, \frac{19}{2}, 10, \cdots, \frac{47}{2}$

[J. Bae, ZD, K. Lee, S. Lee, M. Sarkis, 2010.12392]

Result

c	1	$\frac{9}{4}$	6	$\frac{39}{4}$	11	12
h^{NS}	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	$[1]$
h^{R}	$\frac{1}{24}, \frac{3}{8}$	$\frac{3}{32}, \frac{15}{32}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{13}{32}, \frac{33}{32}$	$\frac{11}{24}, \frac{9}{8}$	$\frac{1}{2}, [1]$

- A_k type $\mathcal{N}=2$ superconformal minimal model

$$c(k) = \frac{3k}{k+2}\,, \qquad (k = 1, 2, 3, \ldots)$$

- For $k = 1\,(c = 1)$, $\quad (h^{\mathrm{NS}}, \underset{\downarrow\, U(1)_R}{Q}) = (0,0), (\frac{1}{6}, \pm\frac{1}{3})$

$$\begin{cases} f_0^{\mathrm{NS}} = \chi^{\mathrm{NS}}_{(0,0)} = q^{-\frac{1}{24}}\left(1 + q + 2q^{\frac{3}{2}} + 2q^2 + 2q^{\frac{5}{2}} + \cdots\right) \\ f_1^{\mathrm{NS}} = \chi^{\mathrm{NS}}_{(\frac{1}{6}, \pm\frac{1}{3})} = q^{\frac{1}{6}-\frac{1}{24}}\left(1 + q^{\frac{1}{2}} + q + q^{\frac{3}{2}} + 2q^2 + \cdots\right) \end{cases}$$

Result

c	1	$\frac{9}{4}$	6	$\frac{39}{4}$	11	12
h^{NS}	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	$[1]$
h^{R}	$\frac{1}{24}, \frac{3}{8}$	$\frac{3}{32}, \frac{15}{32}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{13}{32}, \frac{33}{32}$	$\frac{11}{24}, \frac{9}{8}$	$\frac{1}{2}, [1]$

- Fermionization of $\mathfrak{a}_1$ level six WZW model (7 primaries, $\mathbb{Z}_2$ symmetry from Verlinde line)

At the level of partition function:

$$
\begin{aligned}
Z &= |\chi_0|^2 + |\chi_1|^2 + |\chi_2|^2 + |\chi_3|^2 + |\chi_4|^2 + |\chi_5|^2 + |\chi_6|^2 \\
&= \frac{1}{2}\Big(\underbrace{|\chi_0+\chi_6|^2 + |\chi_2+\chi_4|^2}_{Z_{\mathrm{NS}}} + \underbrace{|\chi_0-\chi_6|^2 + |\chi_2-\chi_4|^2}_{Z_{\widetilde{\mathrm{NS}}}} + \underbrace{|\chi_1+\chi_5|^2 + |\sqrt{2}\chi_3|^2}_{Z_{\mathrm{R}}} + \underbrace{|\chi_1-\chi_5|^2}_{Z_{\widetilde{\mathrm{R}}} \;=\; 4} \Big)
\end{aligned}
$$

Macdonald Identity (The yellow book Ex.14.9)

Result

c	1	$\frac{9}{4}$	6	$\frac{39}{4}$	11	12
h^{NS}	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	$[1]$
h^{R}	$\frac{1}{24}, \frac{3}{8}$	$\frac{3}{32}, \frac{15}{32}$	$\frac{1}{4}, \frac{3}{4}$	$\frac{13}{32}, \frac{33}{32}$	$\frac{11}{24}, \frac{9}{8}$	$\frac{1}{2}, [1]$

- Fermionization of six copies of $\mathfrak{a}_1$ level one WZW model

  Conformal weights: (lots of degeneracies) $h = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \frac{5}{4}, \frac{3}{2}$ $\quad (\chi_{\frac{1}{4}} - \chi_{\frac{5}{4}} = 2)$

  $\mathcal{N} = 4$ supersymmetry? [Harvey, Moore 2003.13700]

- One-character theory. The second solution does not make sense as a character.

What is it? Why we list it here?

$C = 12$ Theory

- Solution $\quad f_0^{\mathrm{NS}} = K(\tau) = \dfrac{16}{\lambda(1-\lambda)} - 24, \quad \lambda = \dfrac{\theta_2^4}{\theta_3^4}$

  Describe $\mathbb{Z}_2$ orbifold of eight free bosons + eight free fermions

  $\exists\ \mathcal{N} = 1$ super vertex operator algebra

  [Duncan 0502267]

- Very large symmetry group known as Conway group Co_0 (order ~ 8×10^8)

  Start from 24 free fermions: to generate the required supercurrent, $W = \sum_{\alpha=1}^{4096} \epsilon^\alpha W_\alpha$

  A choice of ϵ^α breaks $O(24)$ symmetry to Co_0

- Conjectured to be holographically dual to the pure $\mathcal{N} = 1$ supergravity in AdS$_3$ [Witten 0706.3359]

Bilinear Relation

c	1	$\frac{9}{4}$	6	$\frac{39}{4}$	11	12
$\mathcal{M}$	2	1	15	1	2	$[0]$

Primary degeneracy

$$f_0^{\text{NS},(c)} f_0^{\text{NS},(12-c)} + \mathcal{M} f_1^{\text{NS},(c)} f_1^{\text{NS},(12-c)} = f_0^{\text{NS},(12)} = K(\tau)$$

- Decompose the one-character theory into two sub-theories

Generalized coset construction [Gaberdiel, Hampapura, Mukhi 1602.01022] Extended to 4-point functions [Muhki, Poddar 2011.09487]

Related to Hecke image of characters [Harvey, Wu 1804.06860][ZD, Lee, Sun 2206.07478]

Conway Moonshine

- Bilinear relation:

  Decompose Conway CFT into theories with subgroup symmetry

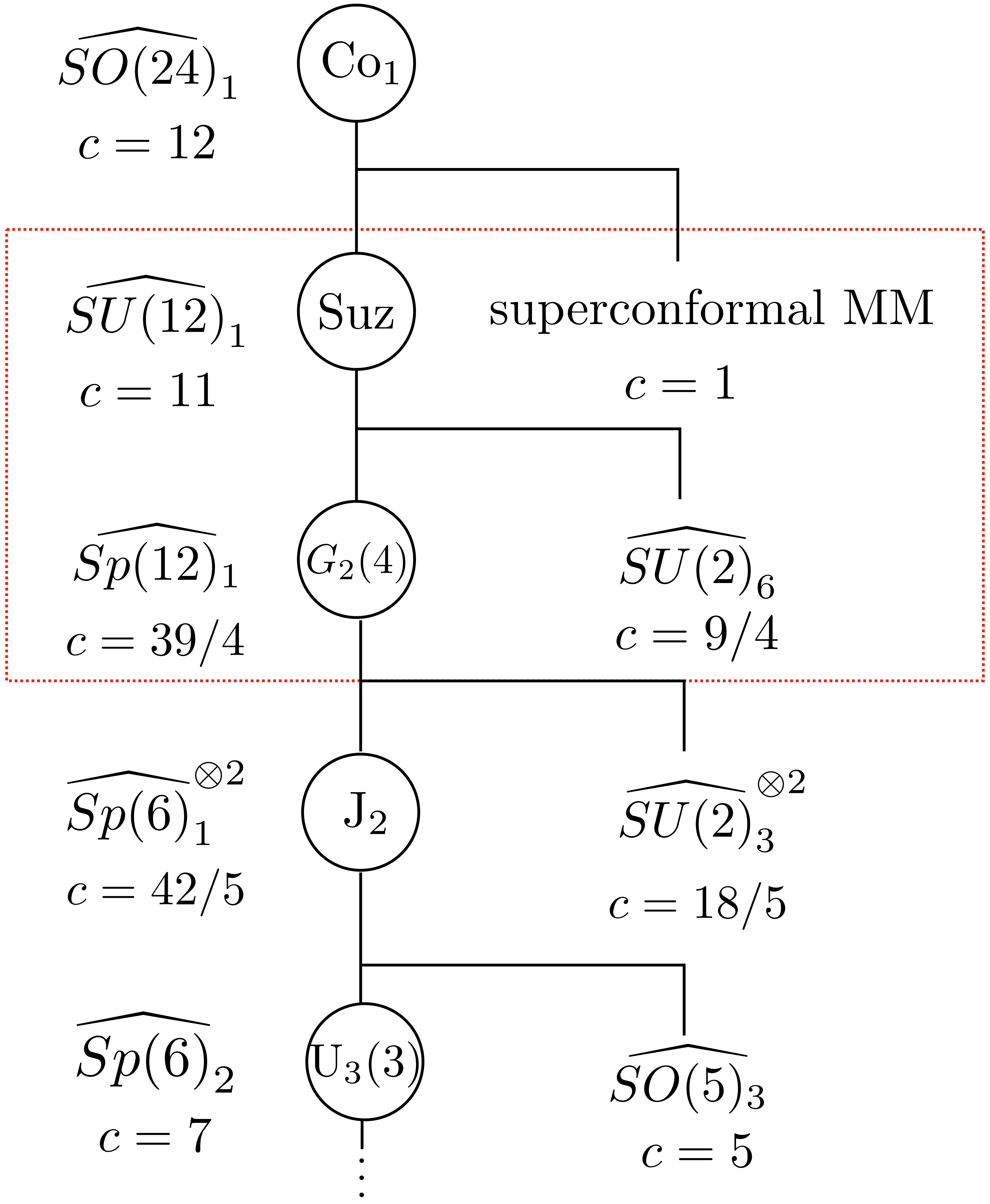
  $$f_0^{\mathrm{NS},(c)} f_0^{\mathrm{NS},(12-c)} + \mathcal{M} f_1^{\mathrm{NS},(c)} f_1^{\mathrm{NS},(12-c)} = f_0^{\mathrm{NS},(12)} = K(\tau)$$

- $\mathcal{N} = 1$ SVOA with symmetry Suzuki chain subgroups are identified in [Johnson-Freyd 1908.11012]

  Dual theories naturally appear from the classification

- Beyond two characters starting from J_2 group

  [J. Bae, ZD, K. Lee, S. Lee, M. Sarkis, 2108.01647]



Comparison with MTC classification

- Modular Tensor Category (MTC) contains objects with topological spin, ∃ fusion, braiding operations…
- Similarity: modularity plays an important role
- Difference:

Holomorphic bootstrap		MTC Classification
Explicit form of characters	$\xrightarrow{\text{Reduce}}$	Modular data (T, S)
Cannot distinguish degeneracies	$\xrightarrow{\text{Increase rank}}$	No degeneracy
New constraints? Refinement?	$\xleftarrow{\text{Input}}$	More structures

- Fermionic theories: Spin/Super/Fermionic MTC

[Bruillard et al, 1705.05293]
[Bruillard, Plavnik, Rowell, Zhang, 1909.09843]

$\downarrow$

First non-trivial example: $\mathfrak{a}_1$ level six WZW model

Rank 3

- Problem: number of parameters grows too fast

For holomorphic case, $c, h_1^{\mathrm{NS}}, h_2^{\mathrm{NS}}$ determine three out of five; impose both "BPS" condition and absence of free fermions.

Class	$(c, h_1^{\mathrm{NS}}, h_2^{\mathrm{NS}}, h_0^{\mathrm{R}}, h_1^{\mathrm{R}}, h_2^{\mathrm{R}})$
I	$\left(\frac{3m}{2}, \frac{1}{2}, \frac{m}{8}, \frac{m}{16}, \frac{1}{16}(2m+4+\lvert m-4\rvert), \frac{1}{16}(2m+4-\lvert m-4\rvert)\right)$
I	$\left(12, \frac{1}{2}, 1, \frac{1}{2}, \frac{3}{2}, 1\right)$, $\left(18, \frac{1}{2}, \frac{3}{2}, \frac{3}{4}, \frac{9}{4}, \frac{5}{4}\right)$, $\left(24, \frac{1}{2}, 2, 1, 3, \frac{3}{2}\right)$
II	$(2, \frac{1}{6}, \frac{1}{3}, \frac{1}{12}, \frac{3}{4}, \frac{5}{12})$, $(\frac{18}{5}, \frac{3}{10}, \frac{2}{5}, \frac{3}{20}, \frac{3}{4}, \frac{11}{20})$, $(\frac{9}{2}, \frac{1}{4}, \frac{1}{2}, \frac{3}{16}, \frac{15}{16}, \frac{9}{16})$, $(5, \frac{1}{3}, \frac{1}{2}, \frac{5}{24}, \frac{7}{8}, \frac{5}{8})$, $(7, \frac{1}{2}, \frac{2}{3}, \frac{7}{24}, \frac{7}{8}, \frac{5}{8})$, $(\frac{15}{2}, \frac{1}{2}, \frac{3}{4}, \frac{5}{16}, \frac{15}{16}, \frac{9}{16})$, $(\frac{42}{5}, \frac{3}{5}, \frac{7}{10}, \frac{7}{20}, \frac{19}{20}, \frac{3}{4})$, $(10, \frac{2}{3}, \frac{5}{6}, \frac{5}{12}, \frac{13}{12}, \frac{3}{4})$
III	$(\frac{39}{2}, \frac{3}{4}, \frac{3}{2}, \frac{13}{16}, \frac{33}{16}, \frac{23}{16})$, $(\frac{66}{5}, \frac{4}{5}, \frac{11}{10}, \frac{11}{20}, \frac{27}{20}, \frac{3}{4})$, $(22, \frac{5}{6}, \frac{5}{3}, \frac{11}{12}, \frac{9}{4}, \frac{19}{12})$
IV	$(1, \frac{1}{6}, \frac{1}{2}, \frac{1}{24}, \frac{3}{8}, \frac{1}{8})$, $(\frac{3}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{5}{16}, \frac{3}{16})$, $(3, \frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \frac{11}{24}, \frac{3}{8})$, $(9, \frac{1}{2}, \frac{2}{3}, \frac{3}{8}, \frac{9}{8}, \frac{25}{24})$, $(\frac{21}{2}, \frac{1}{2}, \frac{3}{4}, \frac{7}{16}, \frac{21}{16}, \frac{19}{16})$, $(11, \frac{1}{2}, \frac{5}{6}, \frac{11}{24}, \frac{11}{8}, \frac{9}{8})$

[J. Bae, ZD, K. Lee, S. Lee, M. Sarkis, 2108.01647]

- Improvement: make use of integrality!

Already appeared in MTC literature [W. Eholzer 9408160], heavily used e.g. in [Ng, Rowell, Wang, Wen, 2203.14829]

First appeared in holomorphic bootstrap [Kaidi, Lin, Parra-Martinez 2107.13557]

Integrality for SL$(2,\mathbb{Z})$

- Unbounded Denominator Conjecture (congruence property in MTC):

If coefficients of a vector-valued modular form (characters in particular) are all integral, each component is invariant under a fixed principal congruence subgroup $\Gamma(N)$

Smallest N possible

proved by [Calegari, Dimitrov, Tang, 2109.09040]

- $\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ d & e \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), N \in \mathbb{N}, \quad \begin{pmatrix} a & b \\ d & e \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\}$

- Action of $\mathrm{SL}(2,\mathbb{Z})$ is reduced to: $\mathrm{SL}(2,\mathbb{Z})/\Gamma(N) = \mathrm{SL}(2,\mathbb{Z}_N)$

Finite group

- No assumptions on positivity; $-\mathrm{id}$ acts trivially, so actually $\mathrm{PSL}(2,\mathbb{Z}_N)$ reps

Fun with Finite Group

- If $N = \prod_i p_i^{\lambda_i}$ with p_i prime, then:

$$\mathrm{SL}(2, \mathbb{Z}_N) = \prod_i \mathrm{SL}(2, \mathbb{Z}_{p_i^{\lambda_i}})$$

- Understand representation theory of $\mathrm{SL}(2, \mathbb{Z}_{p^\lambda})$, then tensor them together

Easily accessed from e.g., GAP

- Assumption:

1. characters transform as irreducible representation

Not for degenerate theory where $\exists (i, j), h_i - h_j \in \mathbb{Z}$

2. p^λ cannot be reduced further

Not for irreps that are pulled back from irreps of $\mathrm{SL}(2, \mathbb{Z}_{p^{\lambda-1}})$

Lesson from Finite Group

- Define exponents α_i as $\chi_i \sim q^{\alpha_i}(1+\cdots)$, i.e., $\alpha_i = h_i - \dfrac{c}{24}$ $(0 \le i \le n-1)$

- For given n, the set of $\{\alpha_i \pmod 1\}$is always finite, independent of l and can be determined

- When $n = 2$

N	Exponents mod 1
2	$\{0, \frac{1}{2}\}$
6	$\{\frac{2}{3}, \frac{1}{6}\}$, $\{\frac{1}{3}, \frac{5}{6}\}$
8	$\{\frac{1}{8}, \frac{3}{8}\}$, $\{\frac{5}{8}, \frac{7}{8}\}$
12	$\{\frac{1}{4}, \frac{11}{12}\}$, $\{\frac{3}{4}, \frac{5}{12}\}$ $\{\frac{1}{4}, \frac{7}{12}\}$, $\{\frac{3}{4}, \frac{1}{12}\}$ $\{\frac{1}{12}, \frac{5}{12}\}$, $\{\frac{7}{12}, \frac{11}{12}\}$
20	$\{\frac{1}{20}, \frac{9}{20}\}$, $\{\frac{3}{20}, \frac{7}{20}\}$, $\{\frac{11}{20}, \frac{19}{20}\}$, $\{\frac{13}{20}, \frac{17}{20}\}$
24	$\{\frac{11}{24}, \frac{17}{24}\}$, $\{\frac{5}{24}, \frac{23}{24}\}$ $\{\frac{1}{24}, \frac{19}{24}\}$, $\{\frac{7}{24}, \frac{13}{24}\}$
60	$\{\frac{11}{60}, \frac{59}{60}\}$, $\{\frac{17}{60}, \frac{53}{60}\}$, $\{\frac{23}{60}, \frac{47}{60}\}$, $\{\frac{29}{60}, \frac{41}{60}\}$ $\{\frac{1}{60}, \frac{49}{60}\}$, $\{\frac{7}{60}, \frac{43}{60}\}$, $\{\frac{19}{60}, \frac{31}{60}\}$, $\{\frac{13}{60}, \frac{37}{60}\}$

- Extend the classification to $n \le 5,\ l < 6$ [Kaidi, Lin, Parra-Martinez 2107.13557]

Integrality for Γ_ϑ ?

- Conjecture (is it?):

If coefficients of a vector-valued modular form of Γ_ϑ are all integral,
each component is invariant under a fixed principal congruence subgroup $\Gamma(N)$ with N even

Smallest N possible

For super MTC under technical assumption, proved by [Bonderson, Rowell, Zhang, Wang 1704.02041]

- $T^N \in \Gamma(N) := \left\{ \begin{pmatrix} a & b \\ d & e \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}), N \in \mathbb{N}, \begin{pmatrix} a & b \\ d & e \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\} < \Gamma_\theta = \langle T^2, S \rangle$

subgroup

- Action of Γ_ϑ is reduced to: $\Gamma_\theta/\Gamma(N) < \mathrm{SL}(2,\mathbb{Z}_N)$

- Similar conjecture for $\widetilde{\mathrm{NS}}$ sector: $\Gamma^0(2)$ and R sector: $\Gamma_0(2)$

More Fun with Finite Group

- If $N = 2^k \times \prod_i p_i^{\lambda_i}$ with p_i odd prime, then:

$$\Gamma_\theta/\Gamma(N) = \Gamma_\theta/\Gamma(2^k) \times \prod_i \mathrm{SL}(2, \mathbb{Z}_{p_i^{\lambda_i}})$$

- New ingredient: understand representation theory of $\Gamma_\theta/\Gamma(2^k)$

[Cho, Kim, Seo, You, 2210.03681]

- How to identify it?

$\Gamma_\theta/\Gamma(2^k)$ is a 2-Sylow subgroup of $\mathrm{SL}(2,\mathbb{Z}_{2^k})$!

Order is a pure power of 2

Precisely three of them

$\Gamma_\theta/\Gamma(2^k)$, $\Gamma^0(2)/\Gamma(2^k)$ and $\Gamma_0(2)/\Gamma(2^k)$

- Same assumptions: non-degeneracy, irreducibility of representations

More Lesson from Finite Group

- Irrep $\mathscr{R}_i \Longleftrightarrow$ character function $\chi_i(g) = \mathrm{Tr}_{\mathscr{R}_i}\, g$

- For given $\Gamma_\theta/\Gamma(2^k)$, extract eigenvalues of some elements

$\langle T^2, S\rangle$

① $g_m = T^{2m}$ for $m = 0, 1, \cdots, 2^{k-1}-1$: T^2 acts diagonally

knowing all values $\chi_{\mathscr{R}_i}(g_m) \Longleftrightarrow$ knowing eigenvalues of $T^2 = e^{2\pi i(2\alpha_k^{\mathrm{NS}})}$ $2\alpha_k^{\mathrm{NS}} = 2h_k^{\mathrm{NS}} - \frac{c}{12}$ (mod 1)

② $h_j = (ST)^{-1}T^j(ST)$ for $j = 0, 1, \cdots, 2^k - 1$: Lift $\mathscr{R}_i$ from $\Gamma_\theta/\Gamma(2^k)$ to $\mathrm{SL}(2,\mathbb{Z}_{2^k})$

ST Conjugates R to NS

knowing all values $\chi_{\mathscr{R}_i}(h_j) \Longleftrightarrow$ knowing eigenvalues of T in $\Gamma_0(2)/\Gamma(2^k) = e^{2\pi i\, \alpha_k^{\mathrm{R}}}$

$\alpha_k^{\mathrm{R}} = h_k^{\mathrm{R}} - \frac{c}{24}$ (mod 1)

Example: $\Gamma_\theta/\Gamma(4)$

```
gap> Display(CharacterTable(sub2));
CT1

     2  4  3  3  3  3  3  4  4  3  4

       1a 4a 4b 4c 4d 2a 2b 2c 2d 2e

X.1     1  1  1  1  1  1  1  1  1  1
X.2     1 -1  1 -1  1 -1  1  1 -1  1
X.3     1 -1 -1 -1 -1  1  1  1  1  1
X.4     1  1 -1  1 -1 -1  1  1 -1  1
X.5     1  A -A -A  A  1  1 -1 -1 -1
X.6     1 -A -A  A  A -1  1 -1  1 -1
X.7     1 -A  A  A -A  1  1 -1 -1 -1
X.8     1  A  A -A -A -1  1 -1  1 -1
X.9     2  .  .  .  .  . -2  2  . -2
X.10    2  .  .  .  .  . -2 -2  .  2

A = E(4)
  = Sqrt(-1) = i
```

id $\quad (ST)^{-1}T(ST) \quad T^2 \quad -\mathrm{id}$

Pulled back from $\Gamma_\theta/\Gamma(2)$

Result

[ZD, K. Lee, S. Lee, L.Li, 2210.06805]

- Finiteness at given rank: When N is large, low dimensional irreps are all pullbacks

- For rank $n \leq 4$, we determine all possible N for $\Gamma(N)$ and irreps:
    - $n = 1,\ N \in \{2,4,6,8,12,16,24,48\}$, 48 irreps
    - $n = 2,\ N \in \{4,8,12,16,20,24,32,40,48,60,80,96,120,240\}$, 300 irreps
    - $n = 3,\ N \in \{6,10,12,14,20,24,28,30,40,42,48,56,60,80,84,112,120,168,240,336\}$, 208 irreps
    - $n = 4,\ N \in \{8,10,12,16,18,20,24,28,30,32,36,40,48,56,60,64,72,80,84,96,112,120,144,160,168,192,240,336,480\}$, 1206 irreps

$$\Downarrow$$

$2n$ exponents $\{2\alpha_i^{\mathrm{NS}}, \alpha_i^{\mathrm{R}}\}$ mod 1

Rank 1

$n = 1,\ N \in \{2,4,6,8,12,16,24,48\}$, 48 irreps

- Exponents mod 1 $\{2\alpha^{\mathrm{NS}}, \alpha^{\mathrm{R}}\}$ can be grouped into two families

Ruled out by anomaly argument $\longleftarrow$

~~1~~	$\{\frac{i}{24}, \frac{12-i}{24}\},$	$0 \le i \le 23$
2	$\{\frac{i}{24}, \frac{24-i}{24}\},$	$0 \le i \le 23$

$\downarrow$

MW fermions!

$$f^{\mathrm{NS}} = q^{-\frac{1}{48}}(1 + q^{\frac{1}{2}} + \cdots)$$
$$f^{\mathrm{R}} = \sqrt{2}q^{\frac{1}{24}}(1 + q^{1} + \cdots)$$
($i = 23$)

- For higher n, exponents mod 1 can also be grouped into families with exponent differ by $\{\frac{i}{24}, -\frac{i}{24}\}$.

We can always tensor a theory with decoupled free fermions which shifts the exponent

(But theories with exponents in the same family may be total unrelated)

Classification

① For each pair of exponents <u>mod 1</u> $\{2\alpha_i^{\mathrm{NS}}, \alpha_i^{\mathrm{R}}\}$, generate a list of <u>real</u> exponents s.t. $|h_i^{\mathrm{NS,R}}|$ < fixed bound

Exponents $\{2\alpha_i^{\mathrm{NS}}, \alpha_i^{\mathrm{R}}\}$ constrains the parameters in MLDE, when $l=0$ determine it up to $n=4$

Transform

$$\begin{aligned}
&\left[D_\tau^n + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a\right] f_i^{\mathrm{NS}}(\tau) = 0, \quad f_i^{\mathrm{NS}} \sim q^{\alpha_i^{\mathrm{NS}}} \\
&\quad \big\updownarrow \mathcal{T} \\
&\left[D_\tau^n + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a\right] f_i^{\widetilde{\mathrm{NS}}}(\tau) = 0, \quad f_i^{\widetilde{\mathrm{NS}}} \sim q^{\alpha_i^{\mathrm{NS}}} \\
&\quad \big\updownarrow \mathcal{S} \\
&\left[D_\tau^n + \sum_{a=0}^{n-1} \phi_a(\tau) D_\tau^a\right] f_i^{\mathrm{R}}(\tau) = 0, \quad f_i^{\mathrm{R}} \sim q^{\alpha_i^{\mathrm{R}}}
\end{aligned}$$

② Expand the solutions to high powers in q and impose other physical constraints

Confirm two character results, and bootstrap new rank three and four candidate theories

Summary and Future Directions

- Initiate the program of classifying fermionic rational CFTs from fermionic MLDE
- Classify solutions with low rank, identify many solutions and study their properties
- Explore the consequence of integrality

- Identify or disprove all solutions from holomorphic bootstrap
- Reconstruct S matrix? Make contact with MTC classification?
- Connection to 4d SCFTs, e.g. flavored MLDE

[Pan, Wang, Zheng, 2104.12180 & 2207.10463]