Classification of rank one 5d $\mathcal{N}=1$ & 6d (1,0) SCFTs

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Motivation:

Try to classify 5d & 6d $C(1,0)$ SCFTs by compactifying them to 4d & look at their SW solutions on Coulomb branch.

Review of $\mathcal{N}=2$ Coulomb branch

Coulomb branch: a branch of the moduli space, and the low energy effective theory at generic point is given by

$U(1)^r$ gauge theory

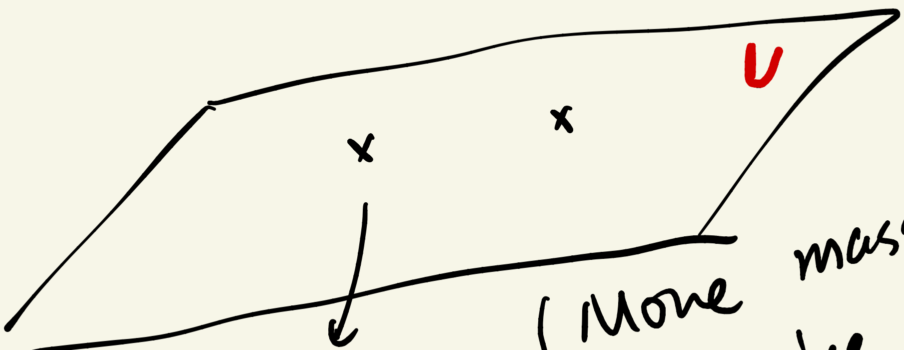
r = called the rank of the theory

In general, the low energy theory at generic point could have more components:

1) free hypermultiplets (i.e $N=2^*$) (Extendend Coulomb branch)

2) Strongly-Coupled SCFT which does not have a Coulomb branch (Rare)

However, there are special points on the Coulomb branch, where the low energy theory is quite complicated



physical origin: $\left\{ \begin{array}{l} \text{More massless particles,} \\ \text{(massive BPS particles become massless)} \end{array} \right.$

The low energy theory at special points could be

a) a IR free theory

i.e. $U(1)$ & N hypermultiplets

non-abelian gauge theory coupled with matter

b) SCFT

The goal of finding the Coulomb branch solution

a) determine the effective coupling for the abelian gauge theory,

b) determine the low energy theory at the special points, (most difficult)

c) The central charge for BPS particles

For SU(2) gauge theory, Seiberg & Witten solved above problem by finding a SW curve $F(x, y, u, m) = 0$, and a SW differential λ, and one can find the low energy theory as follows:

1) For a given u, m, if the curve is smooth, then it is a generic vacua, and the photon coupling is given by the <u>complex structure</u> of F!

2) if the curve is singular, then it is a special vacua,

they agree that there are more massless particles at this point

3) one can use SW differential λ to calculate the period integral

$$a = \int_A \lambda \,, \qquad a^D = \int_B \lambda$$

$$m_i = \int_{\Omega_i} \lambda$$

and the central charge for BPS particle is given as

$$Z = n a + m a^D + \sum_i S_i m_i$$

(so finding SW solution, one need to find a family of curves, & a SW differential)

Some remarks: The crucial physical insights are the electric-magnetic duality for the low energy theory, and this fact is implemented in geometric picture automatically!

$\Rightarrow$ the complex structure of the elliptic curve has a $SL(2,\mathbb{Z})$ action!

The above solution should be intepreted in following more general terms (D.X, D.X, Zhang)

a) The cohomology group of the

$H^1(F, \mathbb{C})$ has dimension

$CH^1(F, \mathbb{Z})$ also useful)

$2\gamma + f$,

At generic points, we have a flat holomorphic vector bundles

To describe physics of $N=2$ theory, one need more structures on $H^1(F,\mathbb{C})$ of generic point,

a) we need to distinguish the electric-magnetic part & flavor part $\Rightarrow$ we need a Mixed Hodge structure

Mixed Hodge structure on a vector space H

Two filtrations (i.e)

weight filtration

$$\{0\} = W^0 \subset W^1 \subset W^2 = H$$

Hodge filtration

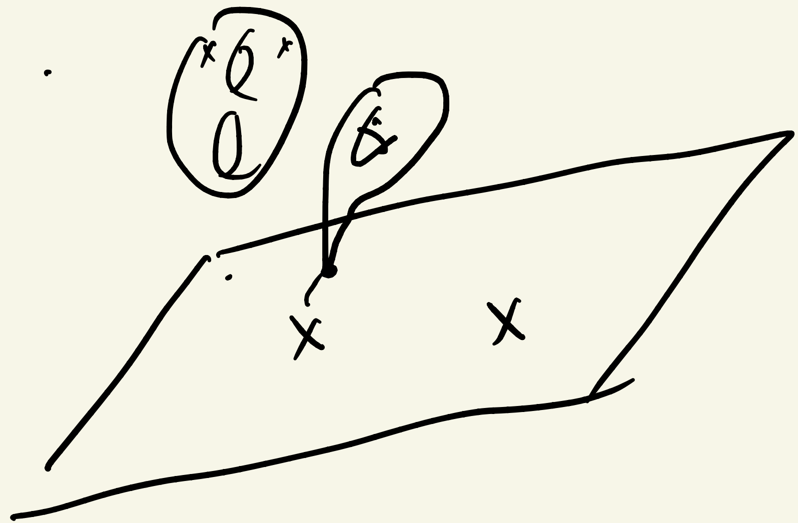
$$H = F^0 \subset F^1$$

We have two spaces

$$Gr^{1}_{W} = W^{1}/W^{0}$$ (electric-magnetic part)

$$Gr^{2}_{W} = W^{2}/W^{1}$$ (flavor-part)

To ensue the positivity of the photon coupling, one also need a polarization on $H^1(F, \mathbb{C})$.



b) In the original SW description, the monodromy group around singularity

M

is important. However, one has little to say about the singularity.

In my previous work, we show that one can define a vector space H, (whose rank is the same as $2r+f$), and another vector space called (vanishing cycles)

One can get some useful information from these new structures

a) one can define a set of rational numbers

$$(n_1, \ldots\ldots n_s)$$

from which one can find the <u>Coulomb branch spectrum</u>

These rational numbers satisfy

$$\lambda_k = \exp(2\pi i w_k)$$

here λ_k is the eigenvalue!!

There are three vector spaces

H_{dim}, H_{van}, H_s defined at the

singularity

previously, only this one,

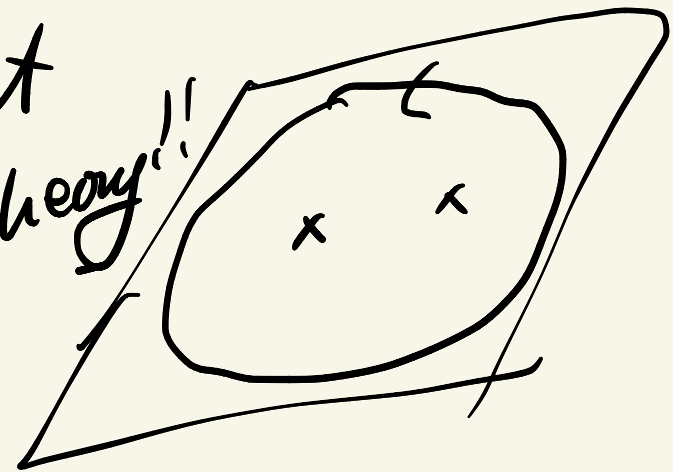
& all of them carry Mixed Hodge structure

c) one can do the similar calculation

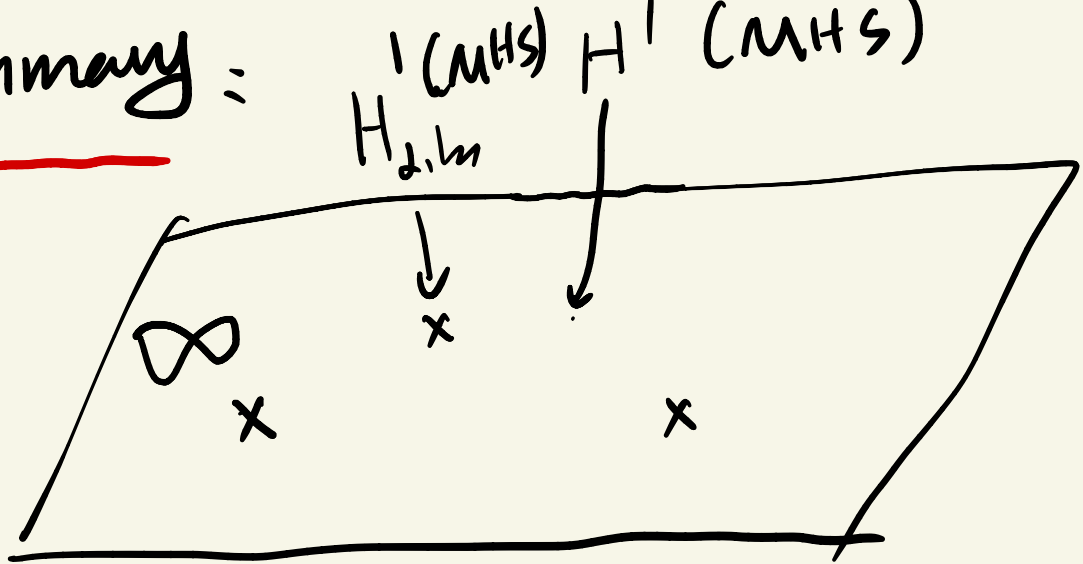
at $U \to \infty$, $\Rightarrow$ we can

get information about

the UV theory!!



Summary:



(They form a so-called mixed Hodge module)

Let's now try to use the above SW solution to classify 4d $N=2$ SCFT (Argyres et al).

The first simplification is that it appears that the monodromy group acts on weight 2 part in a simple way:

the monodromy group acting on weight 2 part is trivial!

The second simplification is that the monodromy satisfies following condition (quasi-unipotent)

$$(T^k - 1)^2 = 0,$$

namely the maximal size of Jordan block is TWO !!

So we could just look at weight one part, and furthermore, we focus on just one case

$\Rightarrow$ So H^1 has just two dimensional and the monodromy group satisfies the condition $(T^k - 1)^2 = 0$

We also assume that there is an integral structure, so monodromy group is in $SL(2,\mathbb{Z})$, so the local monodromy is classified by the conjugacy class of $SL(2,\mathbb{Z})$ satisfying $(T^k-1)^2=0$

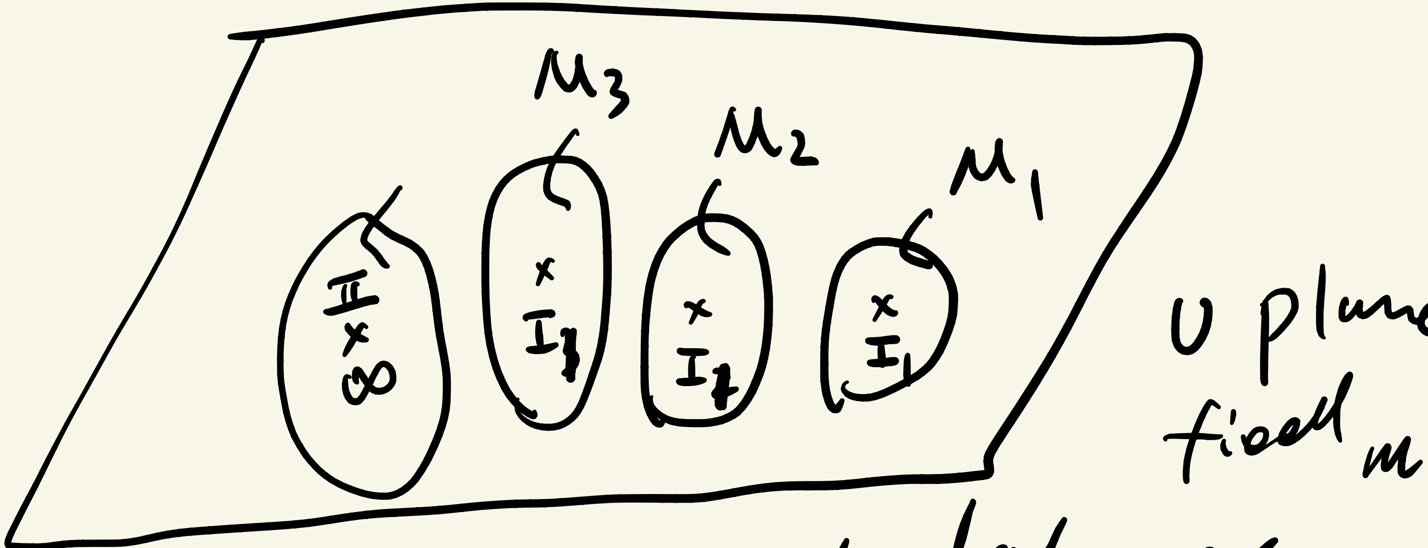
Such conjugacy class has been classified as follows

$$I_n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \qquad II = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \qquad II^* \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

$$III = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad III^* \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$I_n^* = \begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix} \qquad IV = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \qquad IV^* \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

Let's emphasize that, by looking at the weight one part, we lose some information, (as most literature did).



So we get topological data as shown above!

At generic point, we also have a holomorphic function $\tau(v)$ (which can be defined using Hodge filtration on H^1),

and is multi-valued !! & defined up to $SL(2,\mathbb{Z})$ transformation !!

(we can form a j invariant)

Classification strategy

1) We look at the consistent configuration on the U plane. There are some global constraints. It can be best understood using the correspondence with

rational elliptic surface !!

The monodromy data & the j invariant gives us a rational elliptic surface !!

Moreover, all the rational elliptic surfac has been classified!

(Persson's list)

$\Rightarrow$ So we have a data base! (finite set)

2) we require the theory is UV complete, and this put the constrant on the possible singular fiber at ∞ !! this condition puts constrant on the eigenvalues of the monodromy group

As we discussed earlier, one can define a set of rational number which is actually determined by the eigenvalues (in one case

$$\lambda = e^{\frac{2\pi i}{k}}$$

the conclusion is that, the fiber at infinity can not be

I_n type!!

3) the third constraint is that there is only one-dimensional deformation space for the singularity at the bulk!
(so-called undeformable singularity)
(This terminology is a bit confusing)

This constraint immediately
remove the choice of II, III, IV
type singularity at the bulk !!
$\Rightarrow$ the Coulomb branch operator
has dimension < 2, so a relevant
deformation is possible !!

4) A final constraint is called Dirac quantization. Namely, for a I_n singularity, we assume that there is a $\sqrt{n}\,(1,0)$ BPS particle which would become massless,

the Dirac pairing is defined

as $(p_1, q_1) \cdot (p_2, q_2)$

$$= p_1 q_2 - q_1 p_2$$

so

$I_2 I_1 I_1$ ✗

$I_4 I_1 I_1$ ✓

I Can now use all above constraints to classify the theories. We still do not have control over those strange singularities

(I_n^*, II^*, III^*, IV^*), so

We do the following

1) First, we classify configurations using only I_n fibers.

2) We use the so-called discrete gauging, which in the geometric terms, is just

the base change (possibly with quardratic twists)

$$g_n : z \to z^n$$

and one can get a different rational elliptic surface. If we use the <u>previous configurations with only I_n fibres</u>, we should get new theories!

Here is the result:

$\underline{I_1 \text{ series}} = (II, I_1^{10}), (III, I_1^{9}) \ (IV, I_1^{8})$

$(I_0^{*}, I_1^{6}) \ (IV^{*}, I_1^{4}) \ (III^{*}, I_1^{3}) \ (II^{*}, I_1^{2})$

$\underline{I_2 \text{ series}} = (I_0^{*}, I_2^{3})$

$I_4 \text{ series} = (II, I_4 I_1^{5}) \ (III, I_4 I_1^{4})$

$(IV, I_4 I_1^{3})$

$(\mathrm{I}_0^*, \mathrm{I}_4 \mathrm{I}_1^2)$ $\quad$ $(\mathrm{II}, \mathrm{I}_4^2 \mathrm{I}_1)$

$\mathbb{Z}_4$ covering $\quad (\mathrm{IV}^*, \mathrm{I}_1^4) \to (\mathrm{II}, \mathrm{III}^* \mathrm{I}_1)$

$\mathbb{Z}_3$ covering $\quad (\mathrm{I}_0^*, \mathrm{I}_1^6) \to \underline{(\mathrm{II}, \mathrm{IV}^* \mathrm{I}_1^2)}$

$(\mathrm{III}^*, \mathrm{I}_1^3) \to \underline{(\mathrm{III}, \mathrm{IV}^* \mathrm{I}_1)}$

$(\mathrm{I}_0^*, \mathrm{I}_2^3) \to \underline{(\mathrm{II}, \mathrm{IV}^* \mathrm{I}_2)}$

$\mathbb{Z}_2$ covering

$$(IV,\ I_4 I_1^4) \to (II,\ I_2^* I_1^2)$$

$$(IV,\ I_1^8) \to (II,\ I_0^* I_1^4)$$

$$(I_0^*,\ I_4 I_1^2) \to (III,\ I_2^* I_1)$$

$$(I_0^*,\ I_2^3) \to (III,\ I_1^* I_2)$$

$$(I_0^*,\ I_1^6) \to (III,\ I_0^* I_1^3)$$

$$(IV^*, I_1^4) \to (IV, I_0^* I_1^2)$$

To understand those theories further, I find the D7 brane constructions are very useful!!

Such D7 brane configuration is very helpful in finding the

- flavor symmetry
- BPS quiver
- one-form symmetry, etc !!

A (p,q) 7 brane is characterized by the monodromy

$$K_{[p,q]} = \begin{pmatrix} 1-pq & -p^2 \\ q^2 & 1+pq \end{pmatrix}$$

The basic branes are

$$A = [1, 0]$$

$$B = [1, -1]$$

$$C = [1, 1]$$

These D7 brane can engineer Kodaira singularities

I_n : A^n 	II, III, IV, $A^n C$

I_n^* : $A^{n+4} BC$

$II^* \ III^* \ IV^*$: $A^n BC^2$, $n = 7, 6, 5$

For all the previous theories, we can find a brane configuration

A^8BCBC fundamental one

trivial total monodromy

Example:

$$\left(X_{[4,1]}X_{[-3,-1]}\right)A^{8}X_{[2,-1]}C$$

Infinity

$$(\mathrm{II},\ \mathrm{I}_1^{10})$$

Now let's consider classification of 5d $N=1$ & 6d $(1,0)$ theory.

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we put 5d $N=1$ theory on a circle, and so at low energy, we get effectively a 4d $N=2$ theory!

such theory also have a Coulomb branch & the solutions are the same!! There are two important differences:

1) the silgular fiber at infinity

Can only be I_n type!!

2) The BPS particle can carry

Windly number charge n!

So the dimsion of charge lattice

is $\boxed{2r+f+1}$!

The classificatin strategy is the same: we find

I_1 series $=$ (I_9, I_1^3), $(I_8, I_1^4)_a$ $(I_8, I_1^4)_b$

(I_7, I_1^5) (I_6, I_1^6), (I_5, I_1^7) (I_4, I_1^8)

(I_3, I_1^9) (I_2, I_1^{10}) (I_1, I_1^{11})

These are the familiar rank one 5d SCFTs !! (defined as M theory on cone over del Pezzo surfaces)

I_2 series: (I_2, I_2^5) (I_4, I_2^4)

I_3 series: (I_3, I_3^3)

I_4 series:

$(I_1, I_4 I_1^7)$ $(I_2, I_4 I_1^6)$ $(I_3, I_4 I_1^5)$

$(I_4, I_4 I_1^4)_a$ $(I_4, I_4 I_1^4)_b$ $(I_5, I_4 I_1^3)$

$(I_1, I_4^2 I_1^3)_a$ $(I_2, I_4^2 I_1^2)$

Z_6 covering

$$(I_6, I_1^6) \to (I_1, II^* I_1)$$

Z_4 covering

$$(I_8, I_1^4) \to (I_2, III^* I_1)$$

$$(I_4, I_2^4) \to (I_1, III^* I_2)$$

$$(I_4, I_1^8) \to (I_1, III^* I_1^2)$$

Z_3 covering

$$(I_9, I_1^3) \to (I_3, IV^* I_1)$$

$$(I_3, I_3^3) \to (I_1, IV^* I_3)$$

$$(I_6, I_1^6) \to (I_2, IV^* I_1^2)$$

$$(I_3, I_1^9) \to (I_1, IV^* I_1^3)$$

$\mathbb{Z}_2$ Covers

$(I_8, I_1^4) \to (I_4, I_0^* I_1^2)$

$(I_4, I_2^4) \to (I_2, I_0^* I_2^2)$

$(I_2, I_4 I_1^6) \to (I_1, I_2^* I_1^3)$

$(I_4, I_1^8) \to (I_2, I_0^* I_1^4)$

$(I_6, I_1^6) \to (I_3, I_0^* I_1^3)$

$(I_4, I_4 I_1^4) \to (I_2, I_2^* I_1^2)$

$(I_2, I_2^5) \to (I_1, I_1^* I_2^2)$

$(I_2, I_1^{10}) \to (I_1, I_0^* I_1^5)$

6d C(1,0)

we can put 6d theory on T^2 &

get low energy 4d $\mathcal{N}=2$ theory.

Two differences

a): The fiber at infinity is

I_0 !!

b): Two more charges for BPS particles

I_1 series: I_1^{12} (E_8 SCFT)

I_2 series: I_2^{6}

I_3 series: I_3^{4}

I_4 series: $I_4 I_1^{8}$ $(I_4^2 I_1^4)_a$ $(I_4^2 I_1^4)_b$

$\mathbb{Z}_6$ covering: $I_1^{12} \to II^* I_1^2$

$\mathbb{Z}_4$ covering: $I_1^{12} \to III^* I_1^3$

$\mathbb{Z}_3$ covering

$$I_1^{12} \to IV^* I_1^4$$

$\mathbb{Z}_2$ covering

$$I_4 I_1^8 \to I_2^* I_1^4$$

$$I_2^6 \to I_0^* I_2^3$$

$$I_4^2 I_1^4 \to I_0^* I_4 I_1^2$$

$$I_1^{12} \to I_0^* I_1^6$$

Many of these theories deserve further study!!