Wilson loops & Topological Strings

A/B-model approaches to 5d Wilson loops

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Introduction

- Topological strings: A $N = (2{,}2)$ supersymmetric non-linear sigma model from world sheet Σ to target space X

$$\phi_i : \Sigma \to X$$

When the target space X is a Calabi-Yau 3-fold, topological string theory is the most interesting that higher genus free energies are non-trivial

- There are two types of topological twists, they give A-model and B-model. The topological string partition function

$$Z = \exp(\sum_{g=0}^{\infty} g_s^{2g-2} F_g(t_i))$$

where t_i are Kähler moduli in the case of A-model, and complex structure moduli in the case of B-model

Introduction

- Topological strings on a CY3 X, have a close connection with M-theory on the background $\mathbb{R}^4 \times S^1 \times X$
- The M2-branes winding on holomorphic 2-cycles in X give BPS particles. The number of the BPS particles are BPS invariants or Gopakumar-Vafa(GV) invariants, which are captured by the A-model topological string free energies.
- They are related to Gromov-Witten invariants by a transformation.
- Low energy theory is a supergravity theory on $\mathbb{R}^4 \times S^1$. When X is non-compact, the low energy theory is 5d N=1 supersymmetric gauge theory on $\mathbb{R}^4 \times S^1$ with 8 supercharges

Introduction

- Mirror symmetry relates topological A-model on manifold X to topological B-model on its mirror manifold.
- Some very difficult mathematical problems of enumerate geometry can be easily solved by topological B-model methods
- The solutions to the Picard-Fuchs operator for quintic:

$$L = \theta^4 - 5^5 z \prod_{k=1}^{4} (\theta - a_k), \quad a_k = \frac{k}{5}, k = 1, 2, 3, 4.$$

$$\Pi = X_0 \begin{pmatrix} 1 \\ t \\ \partial_t \mathcal{F} \\ 2\mathcal{F} - t\partial_t \mathcal{F} \end{pmatrix} \quad \propto \quad \Pi_0 = \begin{pmatrix} f_0(z) \\ f_0(z)\log(z) + f_1(z) \\ \frac{1}{2} f_0(z)\log^2(z) + f_1(z)\log(z) + f_2(z) \\ \frac{1}{6} f_0(z)\log(z)^3 + \frac{1}{2} f_1(z)\log(z)^2 + f_2(z)\log(z) + f_3(z) \end{pmatrix}$$

Introduction

Mirror map

$$t(z) = \log(z) + 770z + 717825z^2 + \frac{3225308000z^3}{3} + \frac{3947314570625z^4}{2} + 4062154117561404z^5 + \mathcal{O}(z^6)$$

- The coefficients in the mirror map are not always integers, but the coefficients in the inverse of the complex structure are always integers ($Q = e^t$)

$$\frac{1}{z} = \frac{1}{Q} + 770 + 421375Q + 274007500Q^2 + 236982309375Q^3 + 251719793608904Q^4 + \mathcal{O}(Q^5)$$

Introduction

- For a K3 surface, $\dfrac{1}{z}$ gives Thompson series related to moonshine. [Lian and Yau, 94′]
- Local (non-compact) Calabi-Yau
  - Local $\mathbb{P}^1 \times \mathbb{P}^1$, the low energy theory is 5d pure SU(2),

$$\frac{1}{z^{\frac{1}{2}}} = \frac{1}{\sqrt{Q}} + 2\sqrt{Q} + 3Q^{3/2} + 10Q^{5/2} + 49Q^{7/2} + 288Q^{9/2} + \cdots$$

BPS particles with Wilson loop operator

Introduction

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- Local Calabi-Yau
  - Local $\mathbb{P}^1 \times \mathbb{P}^1$, the low energy theory is 5d pure SU(2), [Lian and Yau, 94']

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$$\boxed{\frac{1}{\sqrt{Q}} + \sqrt{Q}} + m\sqrt{Q} + \cdots$$

BPS particles with Wilson loop operator

Outline

- Topological strings on local CY3 / N=1 SQFT in 5d
- Wilson loops in SQFT in 5d
  - Perspective in M-theory
  - p-q five-brane description, refined topological vertex
  - Relation to quantum (refined?) curves.
- B-model approach to 5d Wilson loops
  - Wilson loop free energies are quasi-modular forms
- Applications to quantum/refined periods.

5d BPS partition function

- 5d $N=1$ SYM on Omega-deformed $\mathbb{R}^4_{\epsilon_{1,2}} \times S^1$
  - Nekrasov's partition function, well-defined in mathematics
- M-theory compactified on non-compact Calabi-Yau three-fold X
- The BPS states are captured by M2-branes winding on 2-cycles $C \in H_2(X,\mathbb{Z})$
- Schwinger integral for dynamic electric particles [Gopakumar and Vafa, 98′]

$$F_{\rm BPS} = \sum_{C\in H_2(X,\mathbb{Z})} \sum_{j_L,j_R} N^C_{j_L,j_R} \int_0^\infty \frac{ds}{s} \frac{\mathrm{Tr}_{(j_L,j_R)}(-1)^{\sigma_L+\sigma_R} e^{-sm} e^{-2se(\sigma_L F_+ + \sigma_R F_-)}}{(2\sinh(seF_1/2))(2\sinh(seF_2/2))},$$

↓

Spectrum of dynamic operator

$e^{-(n+1/2)\beta}$

BPS invariants

$$F_{\text{BPS}}(t_C, \epsilon_1, \epsilon_2) = \sum_{C\in H_2(X,\mathbb{Z})} \sum_{n\in\mathbb{Z}} \sum_{j_L, j_R} (-1)^{2j_L+2j_R} N^C_{j_L, j_R} \frac{\chi_{j_L}(n\epsilon_-)\chi_{j_R}(n\epsilon_+)}{n(2\sinh(n\epsilon_1/2))(2\sinh(n\epsilon_2/2))} e^{-nt_C},$$

refined BPS invariants

positive integers

With the spin (j_L, j_R) in the representation of $SU(2)_L \times SU(2)_R = SO(4)$ Lorentz symmetry of $\mathbb{R}^4$

BPS invariants

$$F_{\rm BPS}(t_C,\epsilon_1,\epsilon_2)=\sum_{C\in H_2(X,\mathbb{Z})}\sum_{n\in\mathbb{Z}}\sum_{j_L,j_R}(-1)^{2j_L+2j_R}N^C_{j_L,j_R}\frac{\chi_{j_L}(n\epsilon_-)\chi_{j_R}(n\epsilon_+)}{n(2\sinh(n\epsilon_1/2))(2\sinh(n\epsilon_2/2))}e^{-nt_C},$$

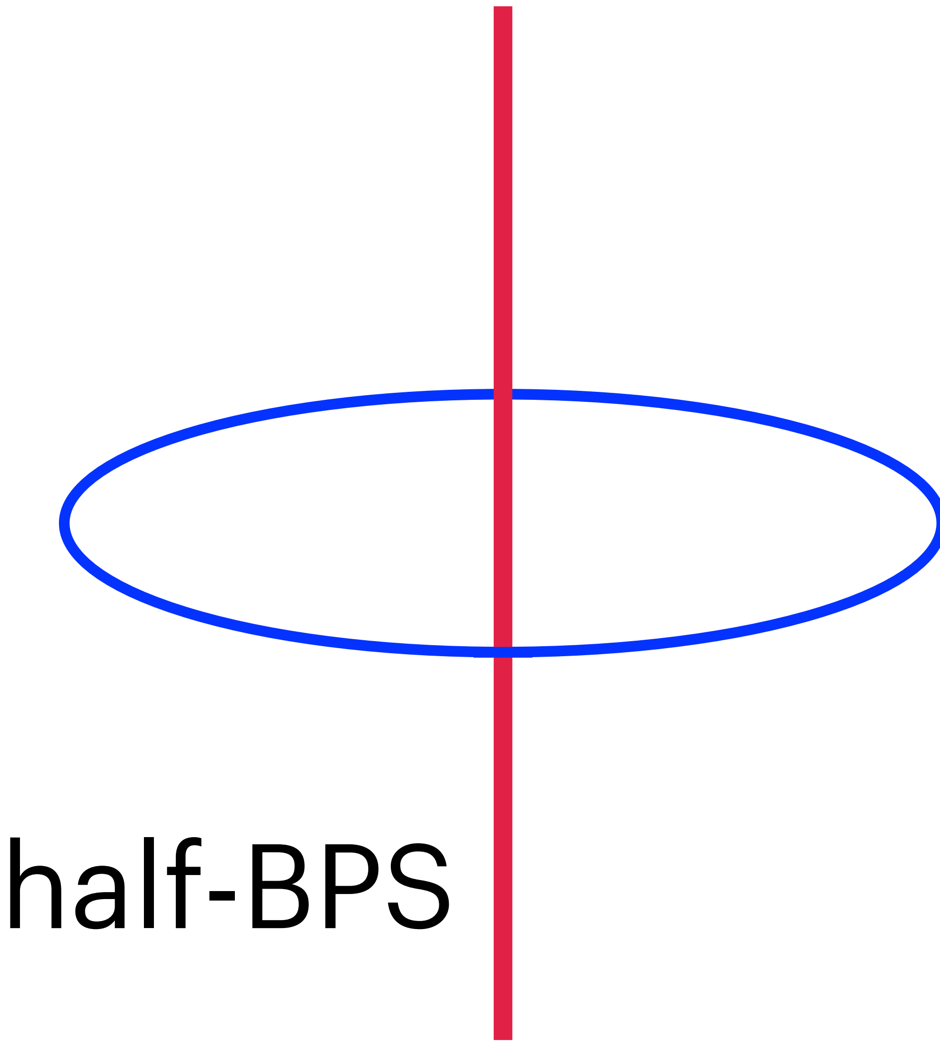
refined BPS invariants

positive integers

$$F_{\rm BPS}(t_C,\epsilon_1,\epsilon_2)=\sum_{n,g}(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}F^{(n,g)}_{\rm BPS}(t_C)$$

Refined Topological string amplitude/free energy

Half-BPS Wilson loop operator



- 5d $N=1$ SYM on $\mathbb{R}^4\times S^1$
- Put a heavy, stationery quark at the origin of space $\mathbb{R}^4$, by inserting a half-BPS operator

$$W_{\mathbf{r}} = \mathrm{Tr}_{\mathbf{r}}\mathcal{T}\exp\left(i\oint_{S^1} dt(A_0(t)-\varphi(t))\right)$$

- Labeled by a representation $\mathbf{r}$ of the gauge group.
- Goal: expectation value of the half-BPS Wilson loop operator $\langle W_{\mathbf{r}}\rangle$
- In the 4d limit, Chiral operators [Losev, Marshakov and Nekrasov, 03']

Brane realization [Tong and Wong]

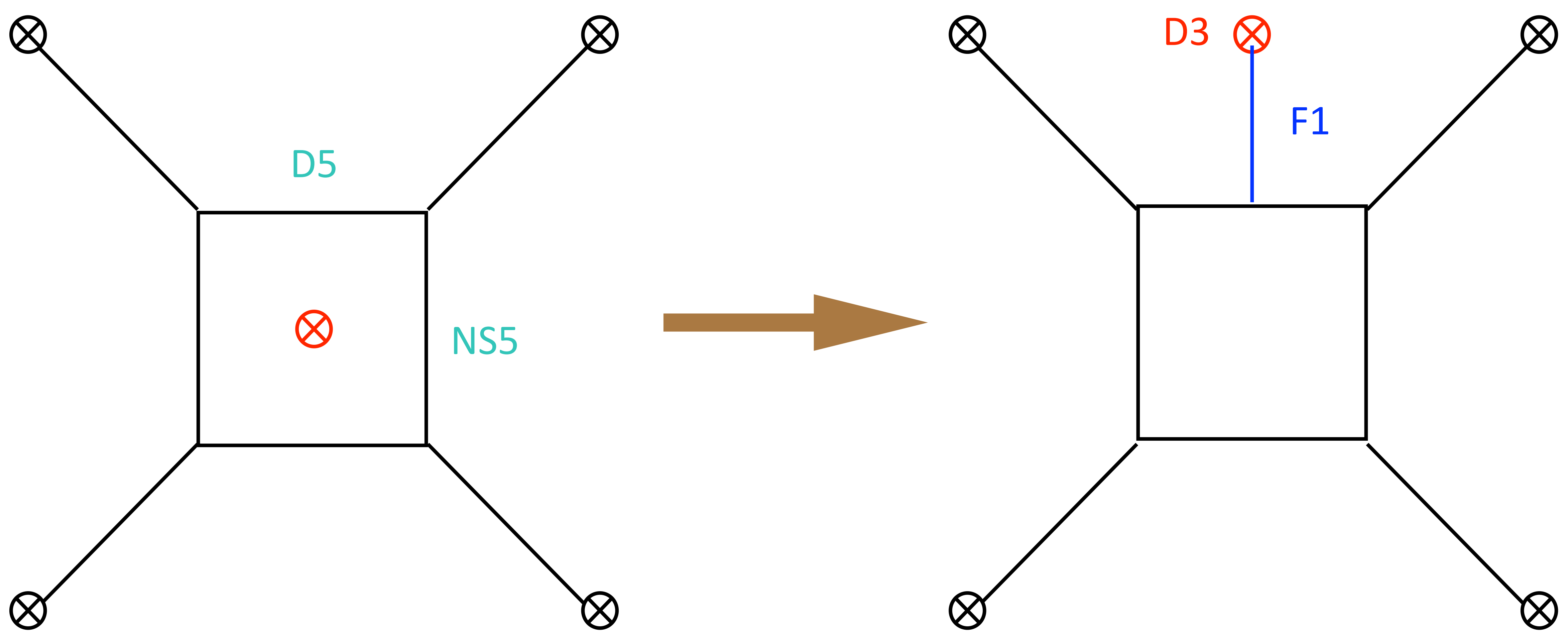
- Half-BPS brane bound states
- D3 branes in type IIB (fermonic quarks)
- F1 string with fixed end point on D3(stationery)

S^1 $\mathbb{R}^4$

	0	1	2	3	4	5	6	7	8	9
D5	×	×	×	×	×	×				
NS5	×	×	×	×	×		×			
F1	×						×			
D3	×	○	○	○	○			×	×	×

Codimension 4 defect

No-dynamic on $\mathbb{R}^4$



Dyson-Schwinger equation [Nekrasov, 15']

- Consider a single D3 defect-brane and define the partition function to be $\chi(x)$
- Introduce $\mathcal{Y}(x)$ observable
- Non-perturbative Dyson-Schwinger equation (for pure SU(N))

$$\mathcal{Y}(x) + \Lambda^{2N} \frac{1}{\mathcal{Y}(x+\epsilon_1+\epsilon_2)} = \chi(x)$$

- The LHS is called qq-character
- The RHS is a Laurent polynomial of $X = e^x$, with the coefficients to be Wilson loops
- In the Nekrasov-Shatashvili (NS) limit $\epsilon_2 \to 0$, quantum Seiberg-Witten curve
- classical Seiberg-Witten curve $\epsilon_{1,2} \to 0$

Example pure SU(2)

- For pure SU(2) case

$$\mathcal{Y}(x) + \Lambda^4 \frac{1}{\mathcal{Y}(x+\epsilon_1+\epsilon_2)} = -X - \frac{1}{X} + \langle W_{\mathbf{2}}^{SU(2)} \rangle$$

- Quantum curve

$$\widehat{Y} + \frac{1}{\widehat{Y}} + \widehat{X} + \frac{1}{\widehat{X}} = H, \quad H = \lim_{\epsilon_2 \to 0} \langle W_{\mathbf{2}}^{SU(2)} \rangle, \qquad \widehat{X}\widehat{Y} = e^{\epsilon_1} \widehat{X}\widehat{Y}$$

- Classical curve (Spectral curve for relativistic Toda chain)

$$Y + \frac{1}{Y} + X + \frac{1}{X} = u, \quad u = \lim_{\epsilon_{1,2} \to 0} \langle W_{\mathbf{2}}^{SU(2)} \rangle$$

Mirror geometry of local CY3

- For local CY3's, the mirror geometry

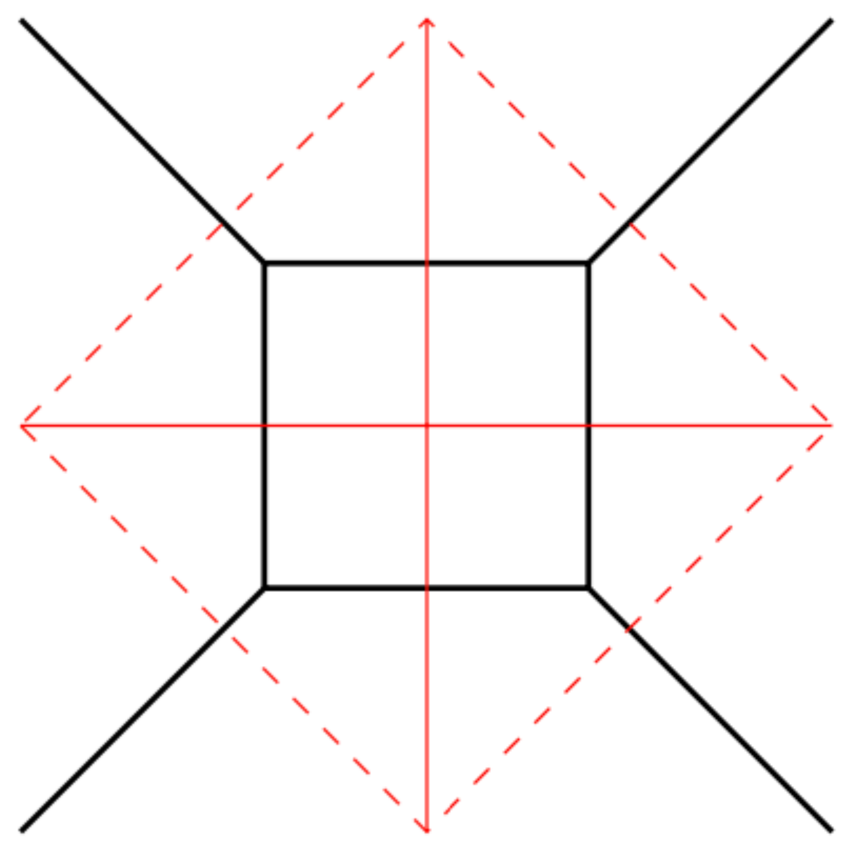
$$st = H(X, Y, u_i)$$

- All the information of periods can be translated to the periods of the mirror curve

$$0 = H(X, Y, u_i)$$

- For local $\mathbb{P}^1 \times \mathbb{P}^1$, the mirror curve is

$$Y + \frac{1}{Y} + X + \frac{1}{X} = u$$



where u is the complex structure parameter dual to the compact divisor

Mirror geometry of local CY3

- The mirror curve of local $\mathbb{P}^1 \times \mathbb{P}^1$ coincides with the classical SW curve of pure SU(2)

$$Y + \frac{1}{Y} + X + \frac{1}{X} = u, \quad u = \lim_{\epsilon_{1,2} \to 0} \langle W_{\mathbf{2}}^{SU(2)} \rangle$$

- The statement here can be generalized to arbitrary local toric CY3's and even non-toric cases
    - E.g. for SU(N), the complex structure parameter of the Sasaki-Einstein manifold $Y^{N,0}$

$$u_i = \prod_j z_j^{C_{ij}^{-1}} = \lim_{\epsilon_{1,2} \to 0} \langle W^{SU(N)}_{\underset{i\,\text{th}}{(00\cdots010\cdots0)}} \rangle$$

Half-BPS Wilson loop operator

- The source of the Wilson loop are heavy stationery quarks which can be fermionic or bosonic

- From M-theory perspective, bosonic quarks (electric particles) are generated by M2-branes winding on curves in CY3.
  - heavy: curves with infinite volume, non-compact
  - stationery: the curves are fixed as background, without dynamic

Half-BPS Wilson loop operator

- Schwinger integral for dynamic electric particles [Gopakumar and Vafa, 98']

$$F_{\text{BPS}} = \sum_{C \in H_2(X,\mathbb{Z})} \sum_{j_L, j_R} N^C_{j_L, j_R} \int_0^\infty \frac{ds}{s} \frac{\text{Tr}_{(j_L, j_R)} (-1)^{\sigma_L + \sigma_R} e^{-sm} e^{-2se(\sigma_L F_+ + \sigma_R F_-)}}{(2\sinh(seF_1/2))(2\sinh(seF_2/2))},$$

- Schwinger integral for stationery electric particles

$$F_{\text{BPS}} = \sum_{C \in H_2(X,\mathbb{Z})} \sum_{j_L, j_R} N^C_{j_L, j_R} \int_0^\infty \frac{ds}{s} \frac{\text{Tr}_{(j_L, j_R)} (-1)^{\sigma_L + \sigma_R} e^{-sm} e^{-2se(\sigma_L F_+ + \sigma_R F_-)}}{\cancel{(2\sinh(seF_1/2))(2\sinh(seF_2/2))}},$$

No dynamic term!

- We first add dynamic electric particles and then absorb the dynamic term by defining the effective mass

$$M \equiv \frac{e^{-t_C}}{2\sinh(\epsilon_1/2) \cdot 2\sinh(\epsilon_2/2)}$$

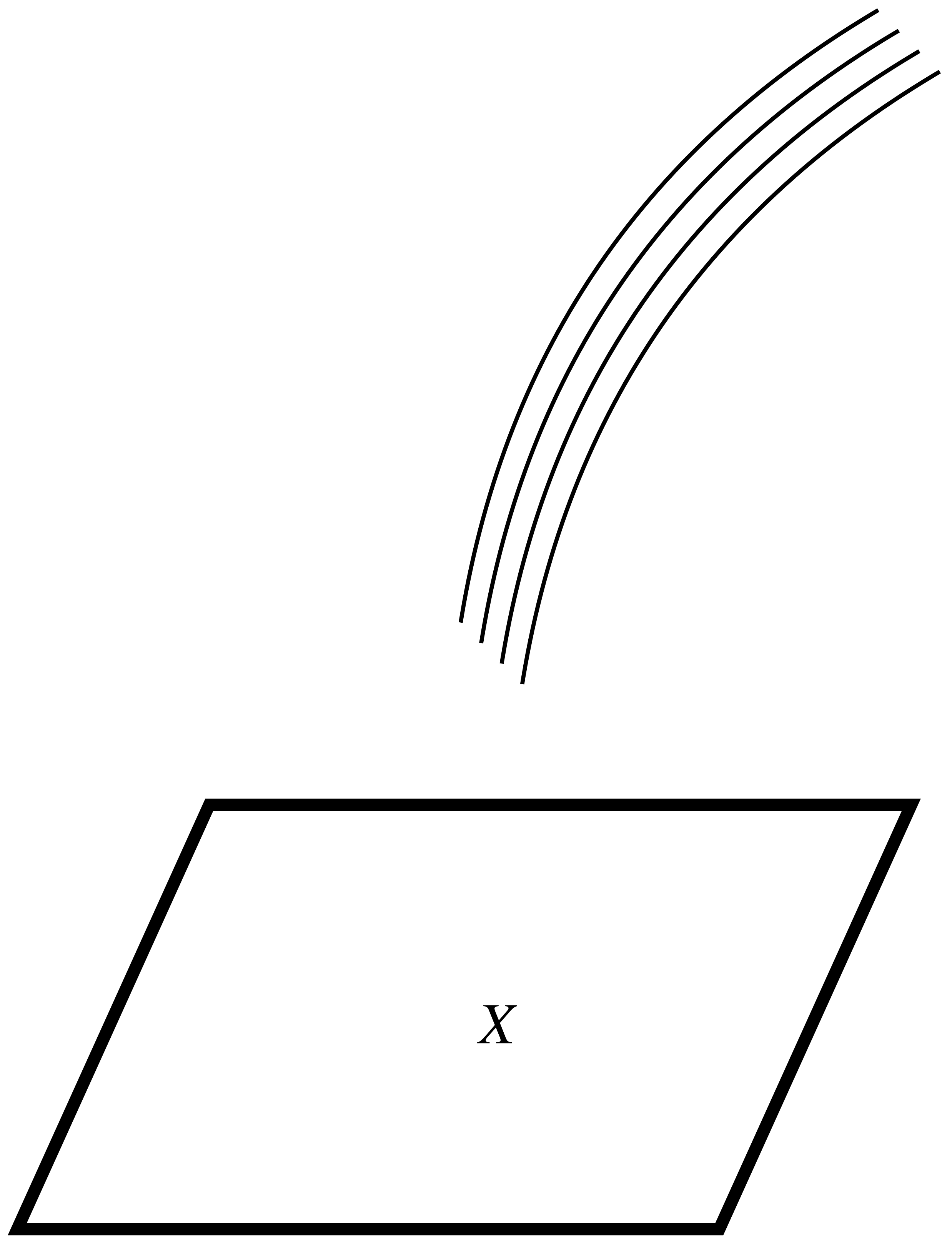
Half-BPS Wilson loop operator

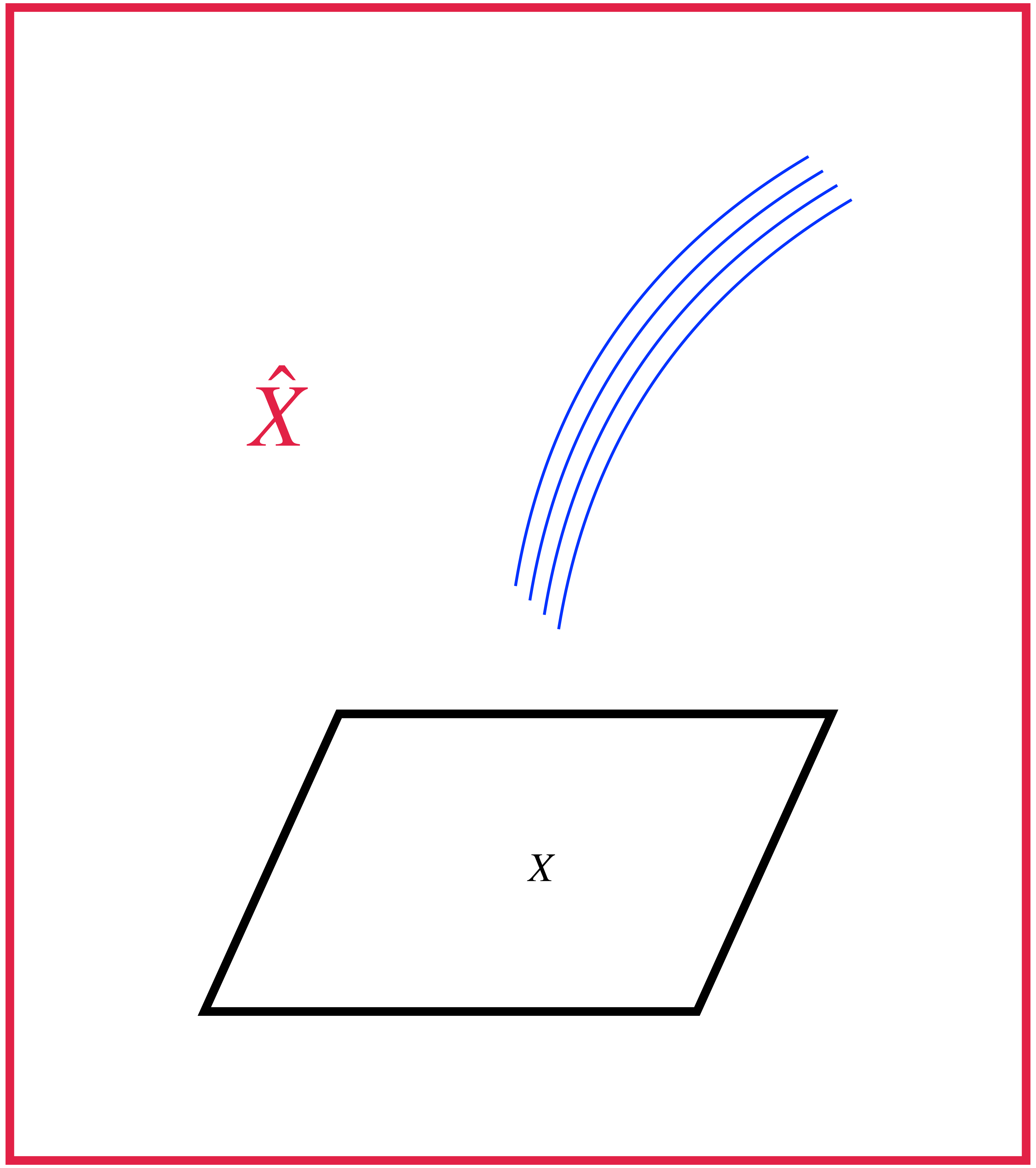
- The additional non-compact curve C as primitive curve [Kim, Kim, Kim,21'], it generates stationery electric particles.

- The particle mass is extremely heavy, that only the leading term contributes.
  - There is no state like $e^{-2t_{\mathrm{C}}}, e^{-3t_{\mathrm{C}}}, \cdots$

- The representation $\mathbf{r}$ generated by a primitive curve is not decomposable

Half-BPS Wilson loop operator





Half-BPS Wilson loop operator

- Denote the additional curve C as primitive curve, the particle mass is extremely heavy, that only the leading term contributes, that we have the Wilson loop partition function

$$Z_{\rm BPS}^{\rm Wilson} = \exp\left(\mathcal{F}_{\rm BPS}(t_C,\epsilon_1,\epsilon_2) + \mathcal{F}_{\rm BPS}^{\rm Wilson}(t_{\mathsf{C}},t_C,\epsilon_1,\epsilon_2)\right)$$
$$\sim Z_{\rm BPS}(t_C,\epsilon_1,\epsilon_2)\left(1+\sum_{C\in H_2(X,\mathbb{Z})}\sum_{j_L,j_R}(-1)^{2j_L+2j_R}N_{j_L,j_R}^{C,\mathsf{C}}\chi_{j_L}(\epsilon_-)\chi_{j_R}(\epsilon_+)e^{-t_C}M\right)$$

↓

$$Z_{W_{\mathbf{r}}} = \exp\left(\mathcal{F}_{\rm BPS}(t_C,\epsilon_1,\epsilon_2)\right)\left(\sum_{C\in H_2(X,\mathbb{Z})}\sum_{j_L,j_R}(-1)^{2j_L+2j_R}\tilde{N}_{j_L,j_R}^{C}\chi_{j_L}(\epsilon_-)\chi_{j_R}(\epsilon_+)e^{-t_C}\right)$$

- The representation $\mathbf{r}$ generated by a primitive curve is not decomposable

Half-BPS Wilson loop operator

- For SU(2), $\mathbf{2}$ is not decomposable which is generated by a primitive curve
- $\mathbf{3} = \mathbf{2} \otimes \mathbf{2} - \mathbf{1}$ is decomposable
- The tensor product of non-decomposable representations are generated by multiple primitive curves, e.g. $\mathbf{2} \otimes \mathbf{2}$ are generated by $\mathsf{C}_1, \mathsf{C}_2$ which generate $\mathbf{2}$ and $\mathbf{2}$ respectively.
- Define the BPS sector

$$\mathcal{F}_{\mathrm{BPS},\{\mathsf{C}_1,\cdots,\mathsf{C}_l\}} = \sum_{C\in H_2(X,\mathbb{Z})} \sum_{j_L,j_R} (-1)^{2j_L+2j_R} N_{j_L,j_R}^{C,\mathsf{C}_1,\cdots,\mathsf{C}_l} \chi_{j_L}(\epsilon_-)\chi_{j_R}(\epsilon_+) e^{-t_C}$$

as the amplitude of the curve class $C + \mathsf{C}_1 + \cdots + \mathsf{C}_l$ for any $C \in H_2(X,\mathbb{Z})$

Half-BPS Wilson loop operator

- The free energy has an additional term

$$\mathcal{F}^{\text{Wilson}}_{\text{BPS}}(t_{\mathsf{C}}, t_{C}, \epsilon_1, \epsilon_2) = \mathcal{F}_{\text{BPS},\{\mathsf{C}_1\}} M_1 + \mathcal{F}_{\text{BPS},\{\mathsf{C}_2\}} M_2 + \mathcal{I} \cdot \mathcal{F}_{\text{BPS},\{\mathsf{C}_1,\mathsf{C}_2\}} M_1 M_2 + \mathcal{O}(M^2_{1,2})$$

where

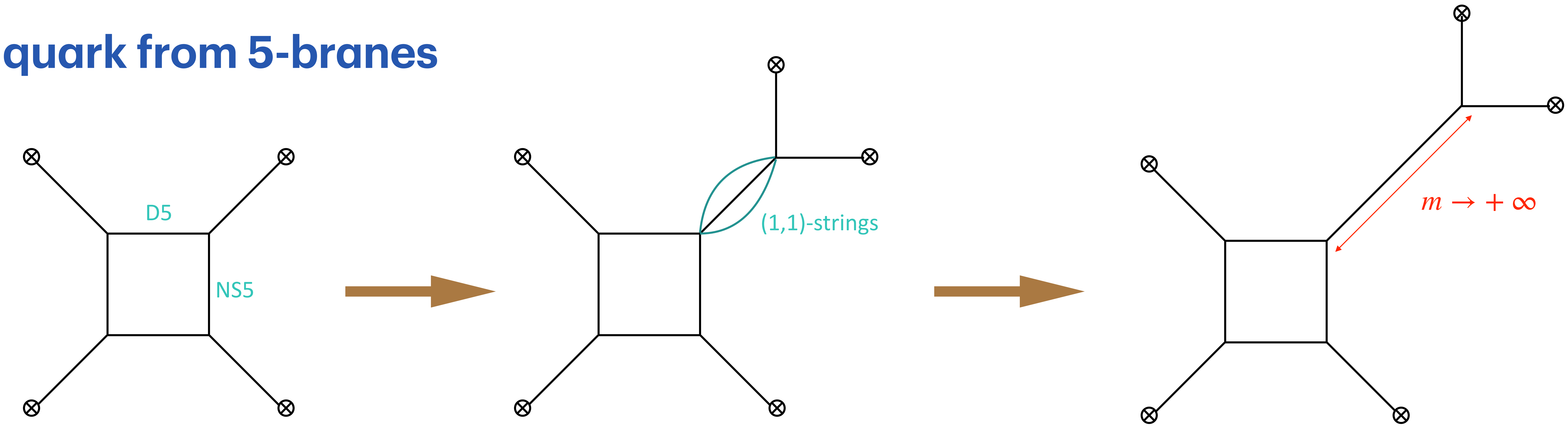
$$\mathcal{I} \equiv 2\sinh(\epsilon_1/2) \cdot 2\sinh(\epsilon_2/2)$$

In the massive limit $M_{1,2} \to 0$, the Wilson loop expectation value has the BPS expansion in terms with BPS sectors.

$$\langle W_{\mathbf{r}_1 \otimes \mathbf{r}_2} \rangle = \mathcal{F}_{\text{BPS},\{\mathsf{C}_1\}} \mathcal{F}_{\text{BPS},\{\mathsf{C}_2\}} + \mathcal{I} \cdot \mathcal{F}_{\text{BPS},\{\mathsf{C}_1,\mathsf{C}_2\}}$$

In each BPS sector, the BPS invariants are always positive integers!

Bosonic quark from 5-branes



Pure SU(2) | SU(2) with $N_f = 1$ | Extremely heavy limit

effect mass of an additional bosonic quark

$$Z^{SU(2),N_f=1} = Z^{SU(2)} + \langle W_{\mathbf{2}}^{SU(2)} \rangle \frac{e^{-m}}{2\sinh(\epsilon_1/2)\cdot 2\sinh(\epsilon_2/2)} + \mathcal{O}(e^{-2m})$$

Wilson loop expectation value

Momentum term from dynamics

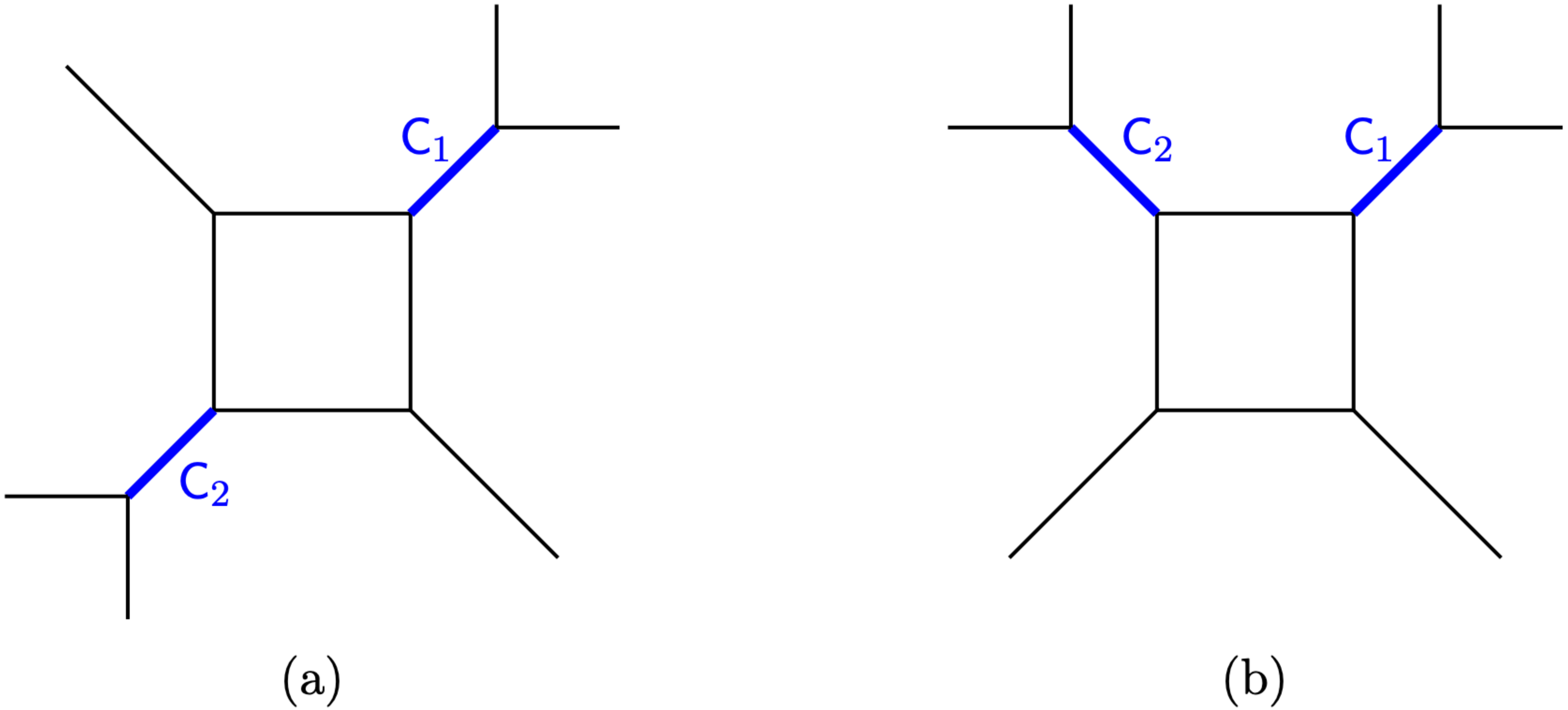
Bosonic quark from 5-branes

$\mathbf{d}$	$\oplus \widetilde{N}^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$	$\mathbf{d}$	$\oplus \widetilde{N}^{\mathbf{d}}_{j_l,j_r}(j_l,j_r)$
$(1,1)$	$(0,0)$	$(1,3)$	$(0,1)$
$(1,5)$	$(0,2)$	$(1,7)$	$(0,3)$
$(1,9)$	$(0,4)$	$(2,5)$	$(0,2)$
$(2,7)$	$(0,2)\oplus 2(0,3)\oplus(1/2,7/2)$	$(2,9)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus$ $(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$
$(3,7)$	$(0,3)$	$(3,9)$	$(0,2)\oplus 2(0,3)\oplus 3(0,4)\oplus$ $(1/2,7/2)\oplus 2(1/2,9/2)\oplus(1,5)$

Table 1. BPS Spectrum of $SU(2)$ Wilson loop expectation value in the representation $\mathbf{2}$ for the curve class $d_1 m_0 + d_2\phi$ with $d_1 = 1,2,3$, and $d_2 \le 9$.

$$Z_{W_{\mathbf{2}}} = Z^{SU(2)} \times \langle W_{\mathbf{2}} \rangle = Z^{SU(2)}_{\text{pert}} \sum_{\mu_1,\mu_2} \left(\mathfrak{q}\sqrt{\frac{q}{t}}\right)^{|\mu_1|+|\mu_2|} \cdot \frac{\text{Ch}_{\mu_1\mu_2}(Q_F,t,q)}{\prod_{i,j=1}^{2} N_{\mu_i\mu_j}(Q_{ij};t,q)}$$

Bosonic quark from 5-branes

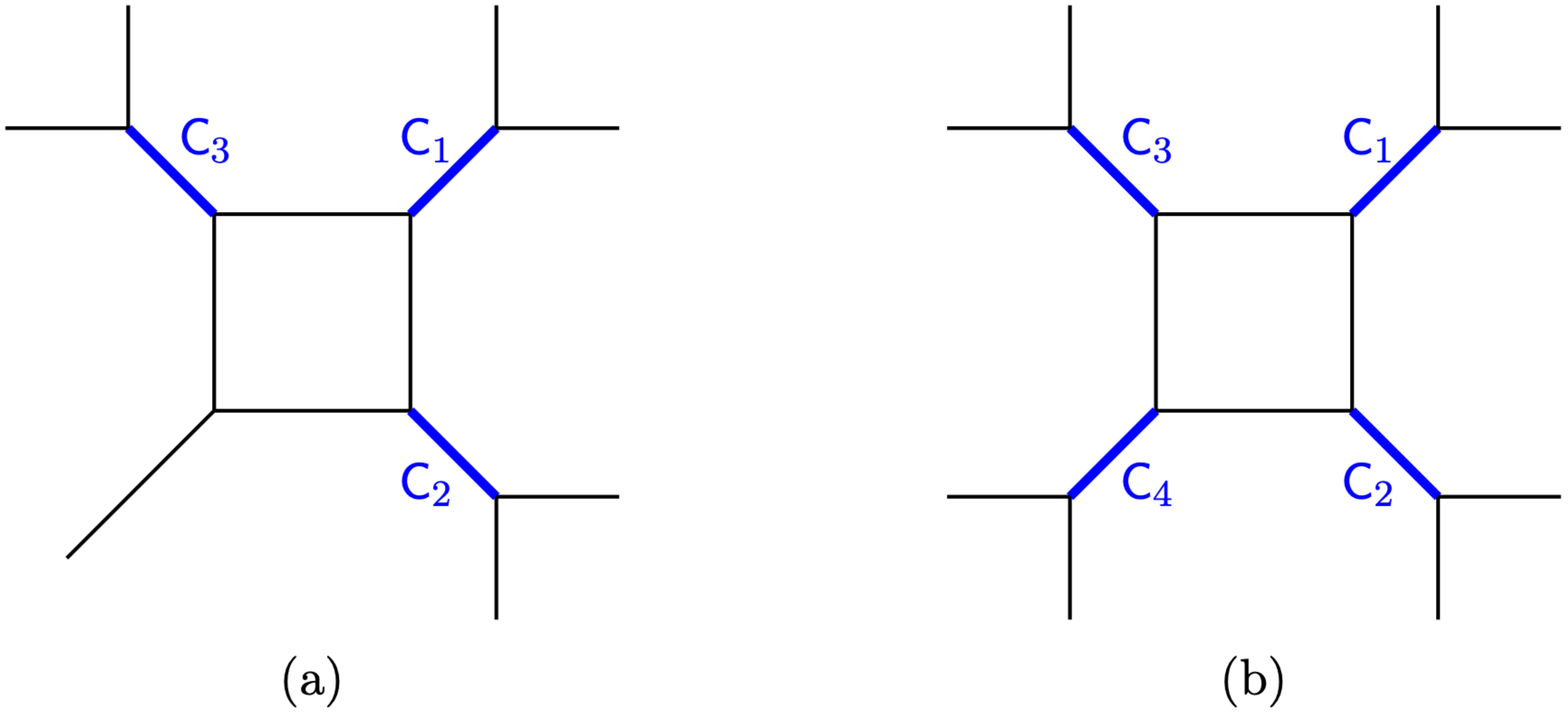


Bosonic quark from 5-branes

$$Z_{W^{(\mathrm{a})}_{\mathbf{2}\otimes\mathbf{2}}} = Z^{SU(2)}_{\mathrm{pert}} \sum_{\mu_1,\mu_2} \left(\mathfrak{q}\sqrt{\frac{q}{t}}\right)^{|\mu_1|+|\mu_2|} \cdot \frac{\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)\mathrm{Ch}_{\mu_2^t\mu_1^t}(Q_F,q,t)}{\prod_{i,j=1}^{2} N_{\mu_i\mu_j}(Q_{ij};t,q)}$$

$$Z_{W^{(\mathrm{b})}_{\mathbf{2}\otimes\mathbf{2}}} = Z^{SU(2)}_{\mathrm{pert}} \sum_{\mu_1,\mu_2} \left(\mathfrak{q}\sqrt{\frac{q}{t}}\right)^{|\mu_1|+|\mu_2|} \cdot \frac{1}{\prod_{i,j=1}^{2} N_{\mu_i\mu_j}(Q_{ij};t,q)} \times \left(\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)^2 + (1-q)(1-1/t)\mathfrak{q}\right)$$

Bosonic quark from 5-branes



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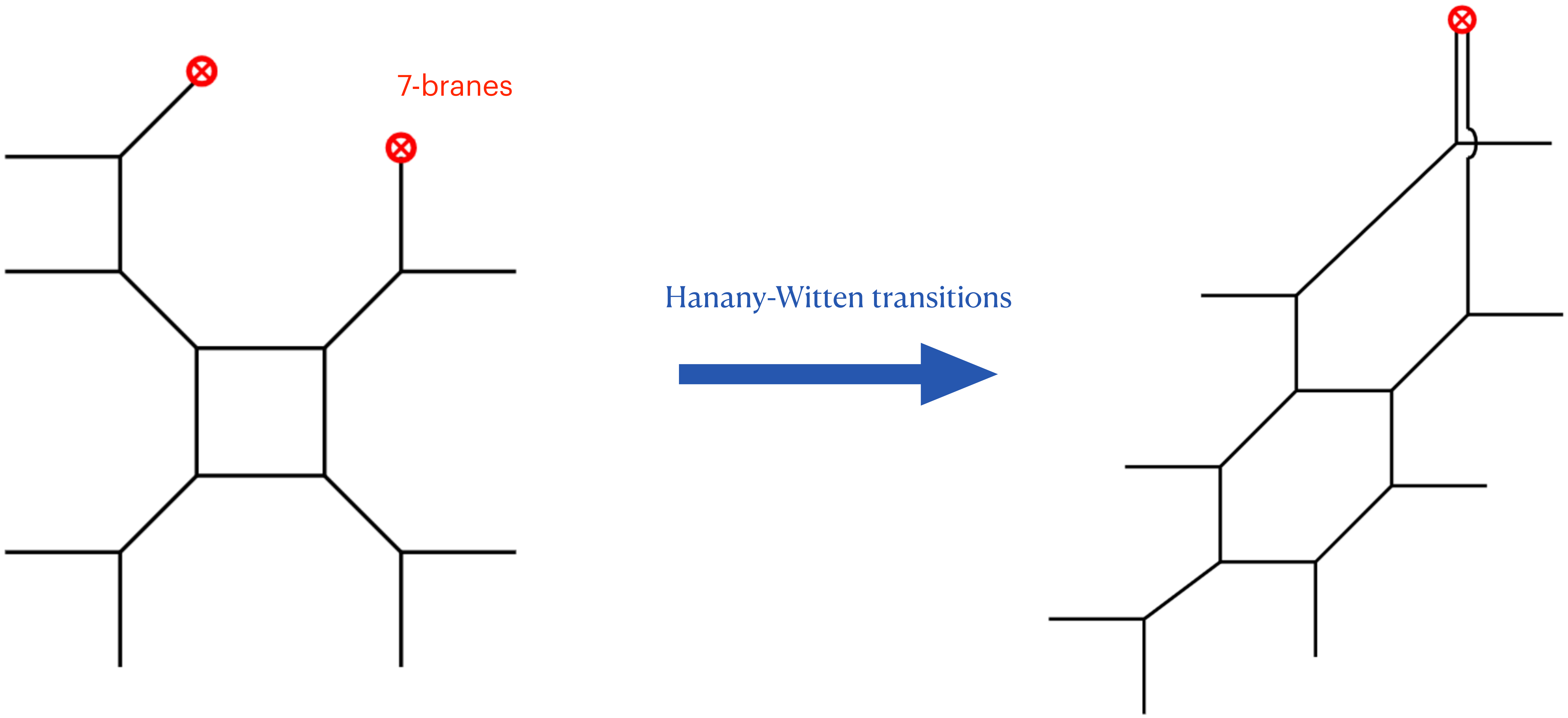
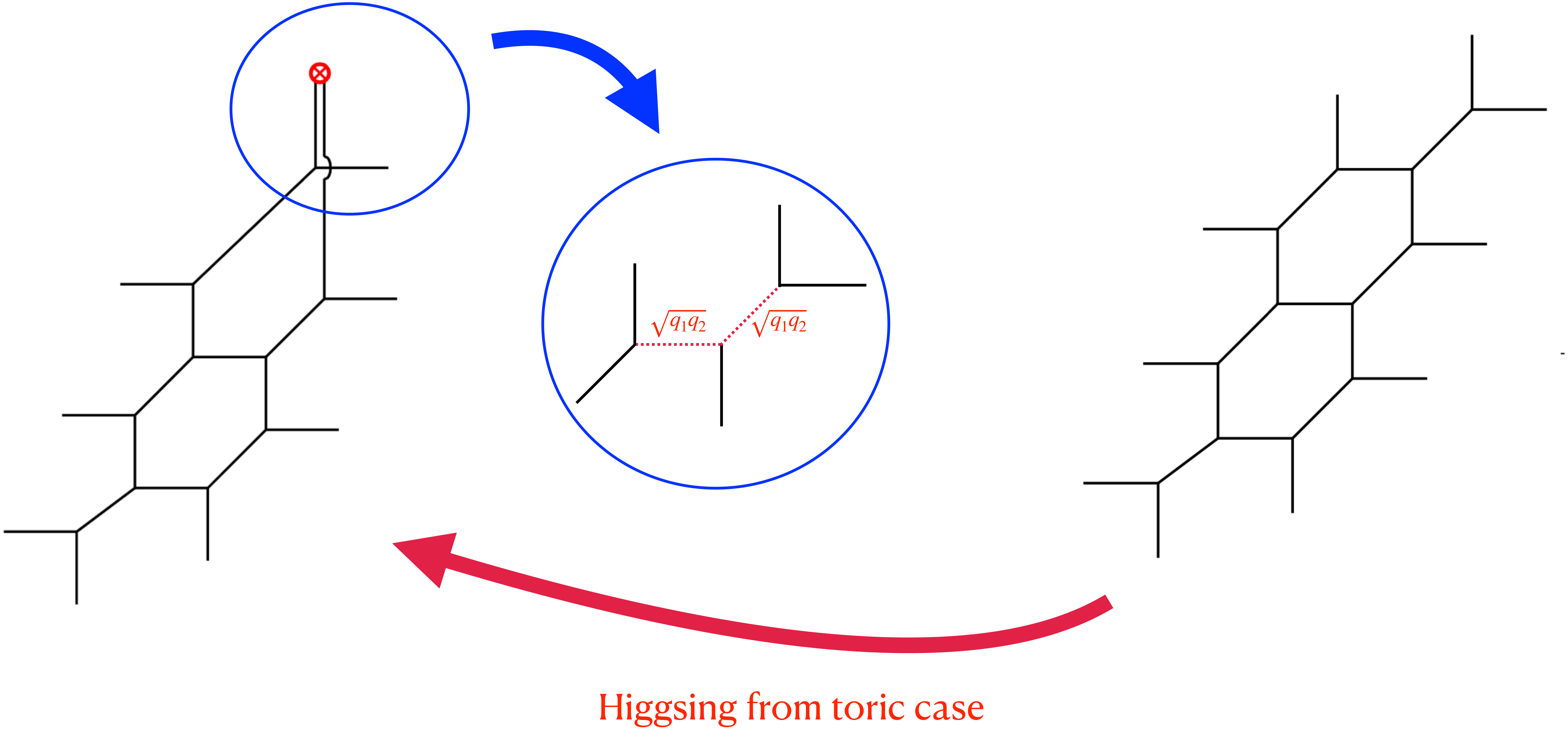


Bosonic quark from 5-branes

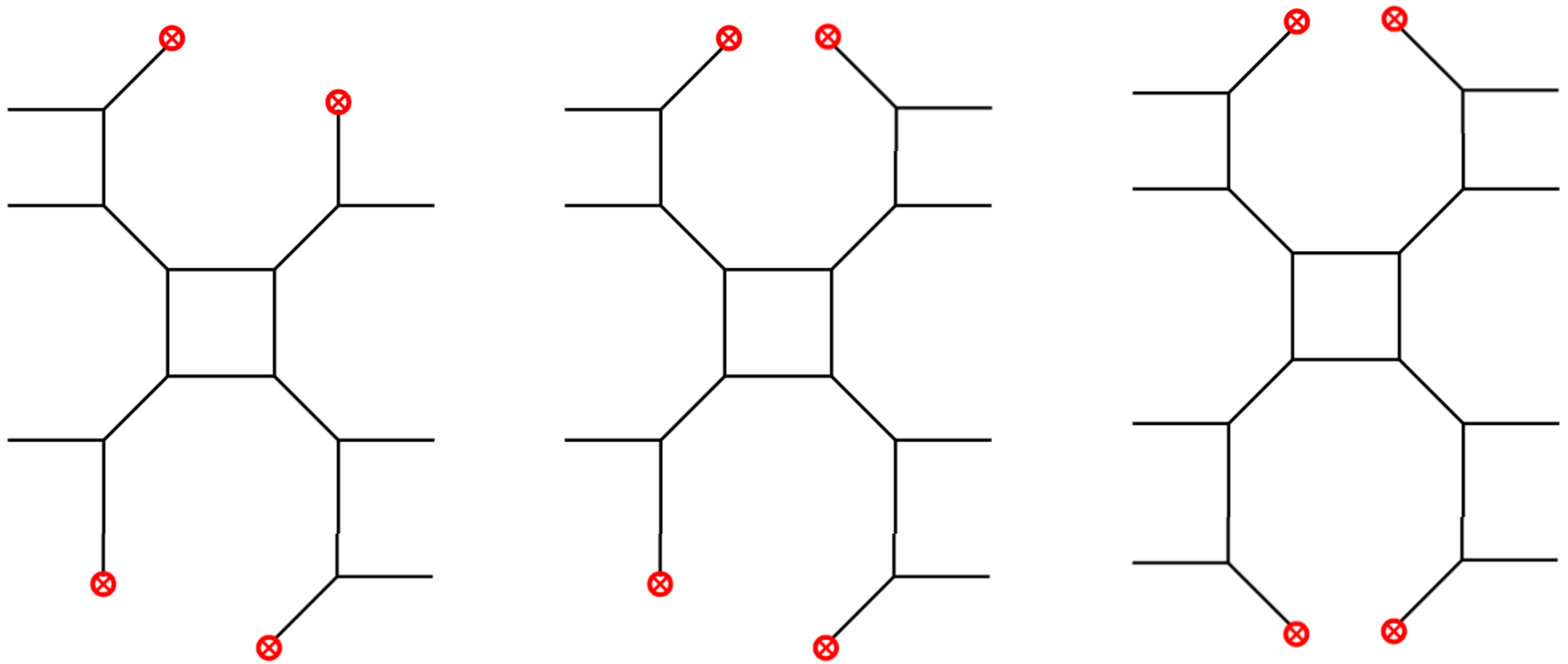


Wilson loops as 1-strings ending on D5 branes

$$Z_{W_{\mathbf{2}\otimes 5}} = Z_{\text{pert}}^{SU(2)} \sum_{\mu_1,\mu_2} \left(\mathfrak{q}\sqrt{\frac{q}{t}}\right)^{|\mu_1|+|\mu_2|} \cdot \frac{1}{\prod_{i,j=1}^{2} N_{\mu_i\mu_j}(Q_{ij};t,q)}$$

$$\times \Big[\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)^3\mathrm{Ch}_{\mu_2^t\mu_1^t}(Q_F,q,t)^2 + (1-t)(1-1/q)\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)^3$$
$$- \mathfrak{q}(1-q)^2(1-1/t)^2\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)\mathrm{Ch}_{\mu_2^t\mu_1^t}(Q_F,q,t)^2$$
$$+ 3\mathfrak{q}(1-q)(1-1/t)\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)\mathrm{Ch}_{\mu_2^t\mu_1^t}(Q_F,q,t)^2$$
$$+ 3\mathfrak{q}^2(1-q)^2(1-t^2)/t/q\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)$$
$$-(1-t)^3(1-q)^3/q/t^2\left(2\mathfrak{q}\mathrm{Ch}_{\mu_2^t\mu_1^t}(Q_F,q,t) - \mathfrak{q}^2\mathrm{Ch}_{\mu_1\mu_2}(Q_F,t,q)\right)\Big]$$

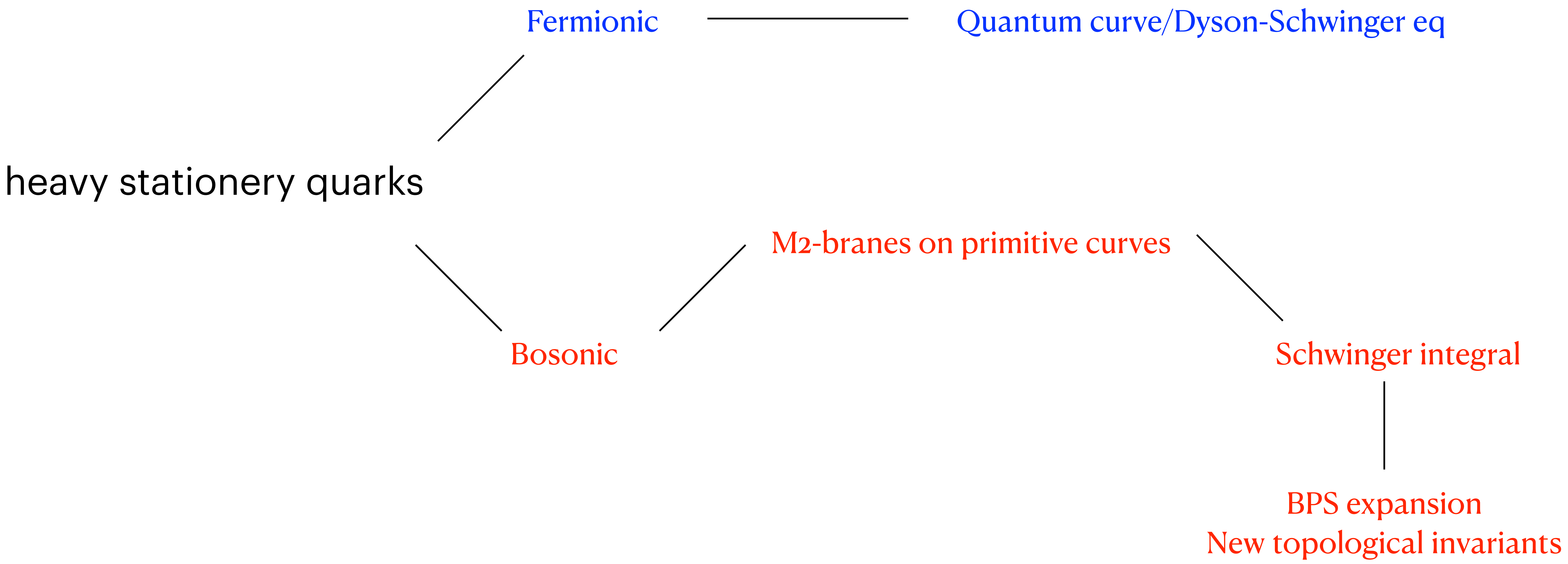
Wilson loops as 1-strings ending on D5 branes



Tao(道) web diagram

[Kim,Taki,Yagi,15']

Summary: 5d half-BPS Wilson loop



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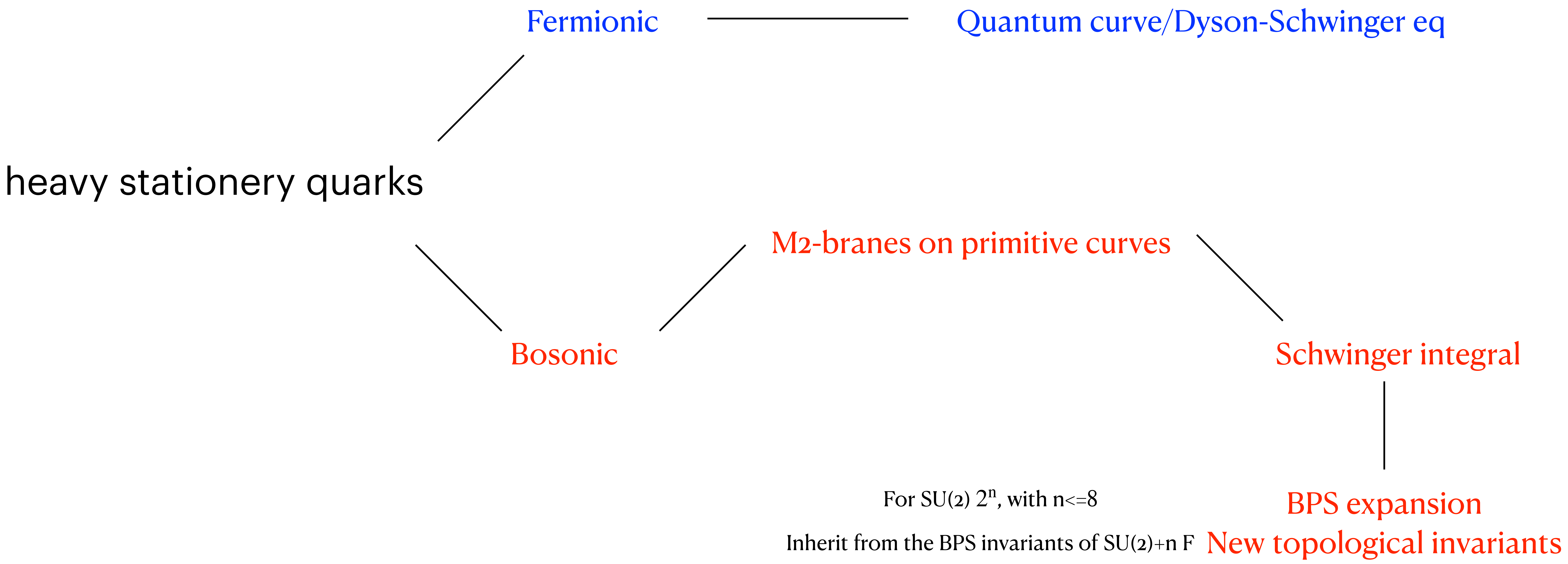


Summary: 5d half-BPS Wilson loop

heavy stationery quarks

Fermionic — Quantum curve/Dyson-Schwinger eq

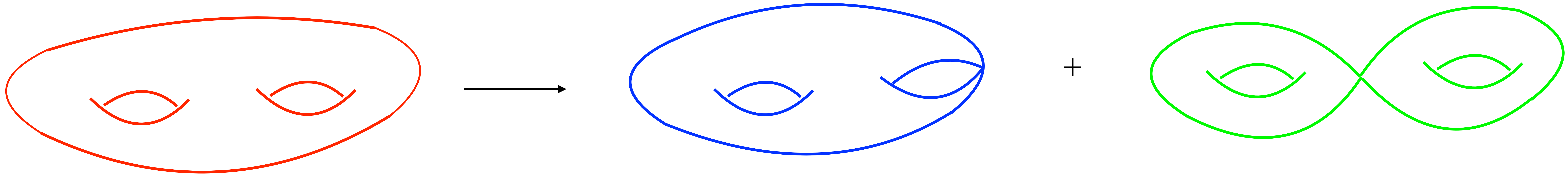
Bosonic — M2-branes on primitive curves — Schwinger integral

BPS expansion
New topological invariants

For SU(2) 2^n, with $n>8$

NEW!

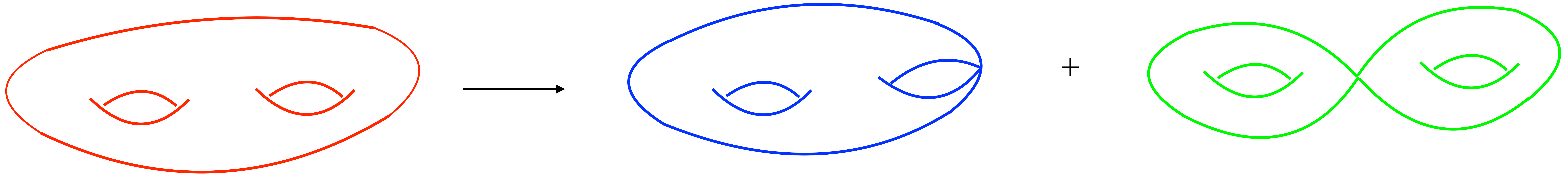
Holomorphic anomaly equation [BCOV]



$$\bar{\partial}_{\bar{i}} \mathscr{F}_g = \frac{1}{2} \bar{C}_{\bar{i}}^{jk} \left(D_j D_k \mathscr{F}_{g-1} + \sum_{g'=1}^{g-1} D_j \mathscr{F}_{g'} \cdot D_k \mathscr{F}_{g-g'} \right)$$

- $\mathscr{F}_g(t, \bar{t})$: Genus g topological string amplitude, which is a section of line bundle of the geometry

Holomorphic anomaly equation



$$\frac{\partial}{\partial S^{ij}}\mathcal{F}_g = \frac{1}{2}\left(D_j D_k \mathcal{F}_{g-1} + \sum_{g'=1}^{g-1} D_j \mathcal{F}_{g'} \cdot D_k \mathcal{F}_{g-g'}\right)$$

- $\mathcal{F}_g = \lim_{\mathrm{Im}\, t \to \infty} \mathscr{F}_g$: holomorphic limit

- $S^{ij} = \frac{1}{C_{ijk}} \partial_{t_k} \mathcal{F}_1 + \text{holo.func.}$: propagator

$$\mathcal{F}_g = \int dS^{ij} \frac{1}{2}\left(D_j D_k \mathcal{F}_{g-1} + \sum_{g'=1}^{g-1} D_j \mathcal{F}_{g'} \cdot D_k \mathcal{F}_{g-g'}\right) + \text{holomorphic ambiguity}$$

Direct integration method

- Effective coupling: $\tau_{ij} = \partial_i \partial_j F^{(0,0)}$
- S^{ij} is proportional to quasi-modular form. e.g. Eisenstein series $E_2(\tau)$

$$\mathcal{F}_g = \int dS^{ij} \frac{1}{2}\left(D_j D_k \mathcal{F}_{g-1} + \sum_{g'=1}^{g-1} D_j \mathcal{F}_{g'} \cdot D_k \mathcal{F}_{g-g'}\right) + \text{holomorphic ambiguity}$$

Rational function of complex structure parameters (modular function)

Direct integration method

- Holomorphic anomaly equation works for *all* Calabi-Yau three folds
- F_g is a weight zero meromorphic quasi-modular form
- Usually hard to solve the holomorphic ambiguity
- F_g can be explicitly expanded to *arbitrary* order at *any* point in the CY moduli space
    - e.g. MUM, orbifold, conifold point
- For non-compact CY3, Refined holomorphic anomaly equation [Huang and Klemm, 10'] [Krefl and Walcher 10']

$$\frac{\partial \mathcal{F}^{(n,g)}}{\partial S^{ij}} = \frac{1}{2}\left(D_i D_j \mathcal{F}^{(n,g-1)} + {\sum_{n',g'}}' D_i \mathcal{F}^{(n',g')} \cdot D_j \mathcal{F}^{(n-n',g-g')} \right), \quad n+g \geq 2.$$

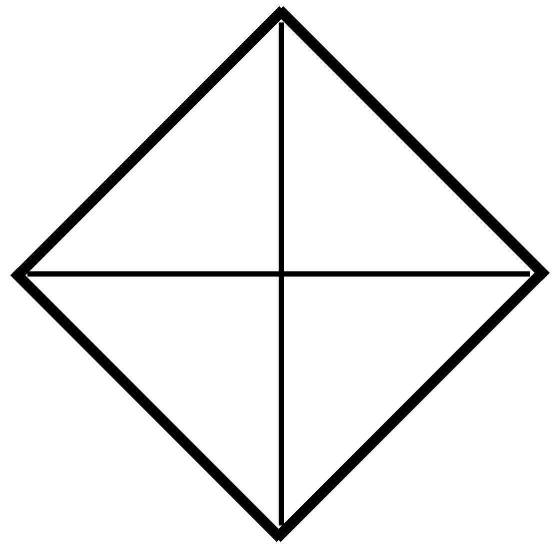
Refined holomorphic anomaly equation

- Holomorphic ambiguities can be completely solved from gap condition near conifold point and regularity near orbifold point to arbitrary (n, g)
- For local $\mathbb{P}^1 \times \mathbb{P}^1$

$$F^{(0,2)} = \frac{5S^3}{24z^6(-1+16z)^2} + \frac{S^2\left(-2160z^2+21600z^3\right)}{12960z^6(-1+16z)^2} + \frac{S\left(585z^4-12960z^5+80640z^6\right)}{12960z^6(-1+16z)^2} + \frac{-55z^6+1884z^7-24000z^8+110592z^9}{12960z^6(-1+16z)^2},$$

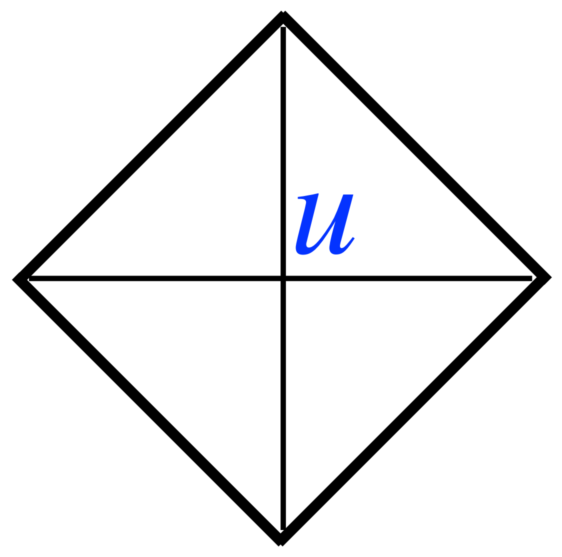
$$F^{(1,1)} = \frac{S^2(90-720z)}{2160z^4(-1+16z)^2} + \frac{S\left(-45z^2+600z^3-7680z^4\right)}{2160z^4(-1+16z)^2} + \frac{5z^4-108z^5+2560z^6-21504z^7}{2160z^4(-1+16z)^2},$$

$$F^{(2,0)} = \frac{S\left(15-240z+960z^2\right)}{4320z^2(-1+16z)^2} + \frac{-5z^2+164z^3-3200z^4+10752z^5}{4320z^2(-1+16z)^2}$$



Local $\mathbb{P}^1 \times \mathbb{P}^1$

Mirror curve & Seiberg-Witten curve & Wilson loops



Local $\mathbb{P}^1 \times \mathbb{P}^1$

$D = \{u = 0\}$

Seiberg-Witten curve

Mirror curve

$$Y + \frac{1}{Y} + X + \frac{1}{X} - u = 0$$

Wilson loop

Complex structure parameter

$$\langle W_{\mathbf{r}} \rangle \quad = \quad u = z^{C^{-1}}$$

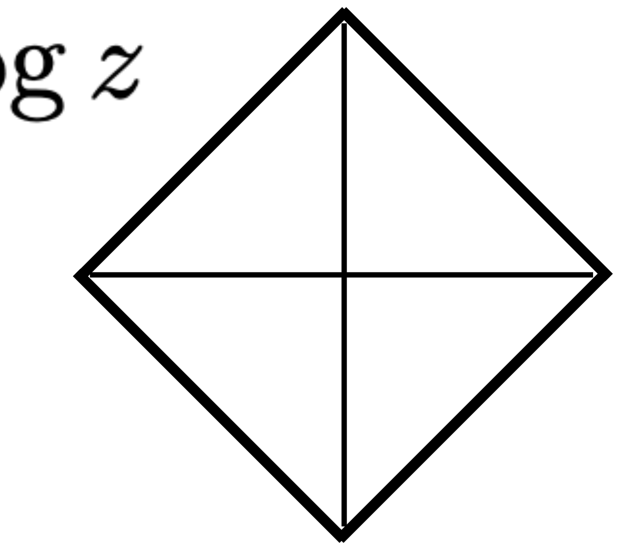
$C = -2$: intersection matrix

e.g. SU(2), $u = \frac{1}{z^{1/2}}$

Refined holomorphic anomaly equation for Wilson loops

Conjecture
$$\frac{\partial F_{\mathbf{r}}^{(n,g)}}{\partial S^{ij}} = \frac{1}{2}\left(D_i D_j F_{\mathbf{r}}^{(n,g-1)} + \sum_{n',g'}{}' D_i F_{\mathbf{r}}^{(n',g')} \cdot D_j F_{\mathbf{r}}^{(n-n',g-g')}\right).$$

- $F_{\mathbf{r}} = \log Z_{W_{\mathbf{r}}} = \sum_{n,g} (\epsilon_1+\epsilon_2)^{2n} (\epsilon_1\epsilon_2)^{g-1} F_{\mathbf{r}}^{(n,g)}$
- $\mathcal{W}_{\mathbf{r}} = \log\langle W_{\mathbf{r}}\rangle = \log\frac{Z_{W_{\mathbf{r}}}}{Z} = \sum_{n,g} (\epsilon_1+\epsilon_2)^{2n} (\epsilon_1\epsilon_2)^{g-1} \mathcal{W}_{\mathbf{r}}^{(n,g)}$
- Properties
  1. pole free from $\epsilon_{1,2}$: $\mathcal{W}_{\mathbf{r}}^{(n,g=0)} = 0$
  2. Seiberg-Witten curve/Mirror curve correspondence : $\mathcal{W}_{\mathbf{r}}^{(0,1)} = -C^{-1}\log z$
     - For SU(2), $\mathcal{W}_{\mathbf{r}=\mathbf{2}}^{(0,1)} = -\frac{1}{2}\log z$
  3. Tensor product : $\mathcal{W}_{\mathbf{r}_1\otimes\mathbf{r}_2}^{(0,1)} = \mathcal{W}_{\mathbf{r}_1}^{(0,1)} + \mathcal{W}_{\mathbf{r}_2}^{(0,1)}$
  4. Direct sum: $\exp(\mathcal{W}_{\mathbf{r}_1\oplus\mathbf{r}_2}^{(0,1)}) = \exp(\mathcal{W}_{\mathbf{r}_1}^{(0,1)}) + \exp(\mathcal{W}_{\mathbf{r}_2}^{(0,1)})$



Local $\mathbb{P}^1\times\mathbb{P}^1$

Refined holomorphic anomaly equation for Wilson loops

- Holomorphic ambiguities can be completely solved from gap condition near conifold point and regularity near orbifold point for representation $\mathbf{r} = \mathbf{2}$

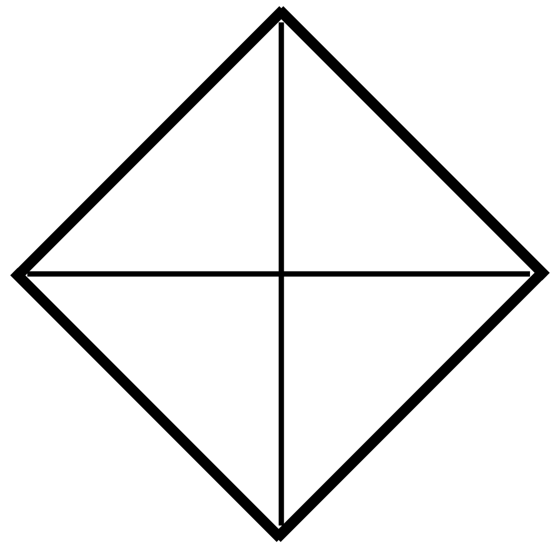
$$W_{\mathbf{2}\otimes l}^{(1,1)} = \frac{l}{72z^2\Delta}(3S - 24Sz - z^2 + 20z^3),$$

$$W_{\mathbf{2}}^{(0,2)} = \frac{9S^2 + 24Sz^3 - z^4 + 16z^5}{36z^4\Delta},$$

$$W_{\mathbf{2}\otimes\mathbf{2}}^{(0,2)} = \frac{18S^2 + 9Sz^2 - 96Sz^3 - 5z^4 + 116z^5 - 576z^6}{36z^4\Delta},$$

$$W_{\mathbf{2}\otimes\mathbf{2}\otimes\mathbf{2}}^{(0,2)} = \frac{9S^2 + 9Sz^2 - 120Sz^3 - 4z^4 + 100z^5 - 576z^6}{12z^4\Delta},$$

$$\vdots$$



Local $\mathbb{P}^1 \times \mathbb{P}^1$

Refined holomorphic anomaly equation for Wilson loops

- By computing the amplitudes to higher enough genus, we recover the BPS invariants from topological vertex

$2j_L \backslash 2j_R$	0
0	1

$d = -\frac{1}{2}$

$2j_L \backslash 2j_R$	0
0	2

$d = \frac{1}{2}$

$2j_L \backslash 2j_R$	0	1	2
0			1

$d = \frac{3}{2}$

$2j_L \backslash 2j_R$	0	1	2	3	4
0					2

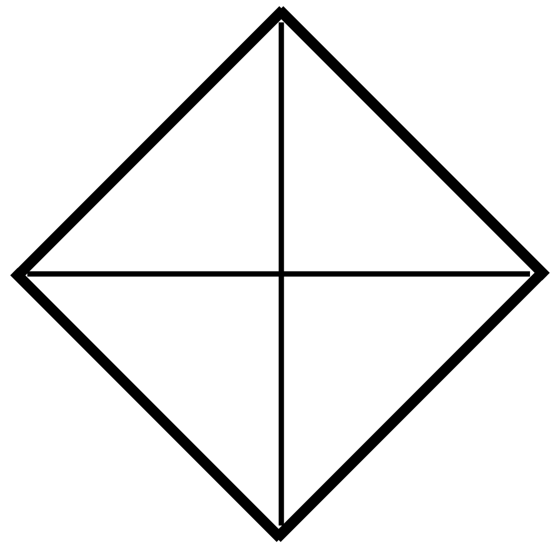
$d = \frac{5}{2}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7
0					1		4	
1								1

$d = \frac{7}{2}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7	8	9	10
0					2		4		8		
1								2		4	
2											2

$d = \frac{9}{2}$



Local $\mathbb{P}^1 \times \mathbb{P}^1$

Refined holomorphic anomaly equation for Wilson loops

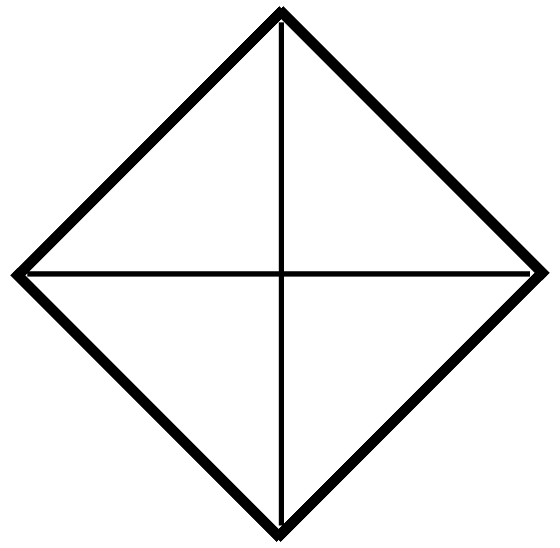
- By computing the amplitudes to higher enough genus, we recover the BPS invariants from topological vertex

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0			1		4		9		12		17		1		
1						1		5		10		12		1	
2									1		5		9		
3												1		4	
4															1

$d = \frac{11}{2}$

$2j_L \backslash 2j_R$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
0	2		4		12		20		32		36		40		6		2		
1				2		6		16		28		38		38		8			
2							2		6		18		28		32		4		
3										2		6		16		20		2	
4													2		6		12		
5																2		4	
6																			2

$d = \frac{13}{2}$



Local $\mathbb{P}^1 \times \mathbb{P}^1$

Refined holomorphic anomaly equation for Wilson loops

- In the NS-limit

$$\frac{\partial \mathcal{W}_{\mathbf{r}}^{(n,1)}}{\partial S^{ij}} = \sum_{n'=0}^{n-1} D_i \mathcal{W}_{\mathbf{r}}^{(n',1)} \cdot D_j \mathcal{F}^{(n-n',0)}$$

- additive

$$\mathcal{W}_{\mathbf{r}_{i_1}\otimes\cdots\otimes\mathbf{r}_{i_l}}^{(n,1)} = \mathcal{W}_{\mathbf{r}_1}^{(n,1)} + \cdots + \mathcal{W}_{\mathbf{r}_l}^{(n,1)}, \quad n \geq 0,$$

- In the NS limit, the Wilson loop expectation value of tensor products is equal to the product of the Wilson loop expectation values of each representations.
    - Hamiltonians are commutative

Other applications

- Quantum period from WKB method of a quantum mechanic system

$$Y+\frac{1}{Y}+X+\frac{1}{X}-u=0$$

$$\Rightarrow \quad \mathcal{W}_{\mathbf{r}}^{(0,1)}(t(z))=\sum_{n=0}^{\infty}\mathcal{W}_{\mathbf{r}}^{(n,1)}(t(z;\hbar))\,\hbar^{2n}$$

$\log u$ Quantum Wilson loop

- One can solve

$$t(z;\hbar)=[1+\sum_{i=1}^{\infty}\hbar^{2i}\mathcal{D}_{2i}]t(z)$$

$$S^{ij}=\frac{1}{C_{ijk}}\partial_{t_k}\mathcal{F}_1+\text{holo.func.}$$

- Results agree with [Huang, Klemm, Reuter and Schiereck, 14′]

Other applications

- Refined period

$$\mathcal{W}_{\mathbf{r}}^{(0,1)}(t(z)) = \sum_{n,g=0}^{\infty} \mathcal{W}_{\mathbf{r}}^{(n,g)}(t(z;\epsilon_1,\epsilon_2))\,(\epsilon_1+\epsilon_2)^{2n}(\epsilon_1\epsilon_2)^{g-1}$$

- One can solve

$$t(z;\epsilon_1,\epsilon_2) = \left[1 + \sum_{\substack{i,j=1,\\ i+j>0}}^{\infty} (\epsilon_1+\epsilon_2)^{2i}(\epsilon_1\epsilon_2)^{j}\mathcal{D}_{i,j}\right] t(z)$$

Other applications

- Local $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{D}_{i,j}$ is still a second differential operator, but depends on the propagator S

$$\mathcal{D}_{0,1} = \frac{1}{6z^2}[(-3S + z^2 - 32z^3)\Theta + (3S + 4z^2 - 64z^3)\Theta^2],$$

$$\begin{aligned}\mathcal{D}_{0,2} = \frac{1}{12960z^8\Delta^2}[(&-3240S^4 + 1080S^3z^2 + 34560S^3z^3 - 2295S^2z^4 + 79920S^2z^5 \\ &- 1725Sz^6 - 1105920S^2z^6 + 125280Sz^7 + 830z^8 - 3098880Sz^8 - 78920z^9 \\ &+ 25436160Sz^9 + 2891616z^{10} - 48084480z^{11} + 306118656z^{12})\Theta \\ &+ (3240S^4 - 1080S^3z^2 - 34560S^3z^3 + 2295S^2z^4 - 79920S^2z^5 + 1725Sz^6 \\ &+ 1105920S^2z^6 - 125280Sz^7 + 2545z^8 + 3098880Sz^8 - 218944z^9 \\ &- 25436160Sz^9 + 7271616z^{10} - 108942336z^{11} + 612237312z^{12})\Theta^2],\end{aligned}$$

where $\Theta = z\partial_z$

Thank You