Ladder operators and quasinormal modes in Banados-Teitelboim-Zanelli black holes

Based on 2205.15610 (T.K. and Masashi Kimura)

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Outline

1. Introduction

2. Review: Mass ladder operators

3. Review: Quasinormal modes

4. Quasinormal modes in Banados-Teitelboim-Zanell spacetimes

5. Mass ladder operators in Banados-Teitelboim-Zanell spacetimes

6. Shift of quasinormal mode frequencies

Black hole perturbations

- Test fields in BH spacetimes play an important role in the understanding of phenomena in the strong-gravity regime

  e.g., gravitational waves from binary black holes,
  time evolution of ultralight bosons around black holes,
  linear stabilities of black hole spacetimes,
  relaxation phenomena within AdS/CFT, …

- In many problems, master equations take a form of Schrodinger eq:

$$\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right]\phi = 0$$

- Mathematical tools in quantum mechanics can be useful in BH perturbation theory

  e.g., ladder operators

Ladder operators

- In quantum mechanics,

  ladder operators allow to relate the different energy eigenvalues

- Ladder operators in curved spacetimes: ***Mass ladder operators***

[Cardoso *et al*, 2017]
[Cardoso *et al*, 2018]

$$\left[\Box - \mu^2\right]\Phi = 0 \quad \Rightarrow \quad \left[\Box - \left(\mu^2 + \delta\mu^2\right)\right] D\Phi = 0$$

Mass ladder operator D maps a Klein-Gordon field onto another Klein-Gordon field

This is constructed from spacetime conformal symmetry

Question:
Does a mass ladder operator keep physics determined by boundary conditions?

This work: Application of mass ladder operators to black hole physics

Outline

Review: Mass Ladder operators from spacetime conformal symmetry

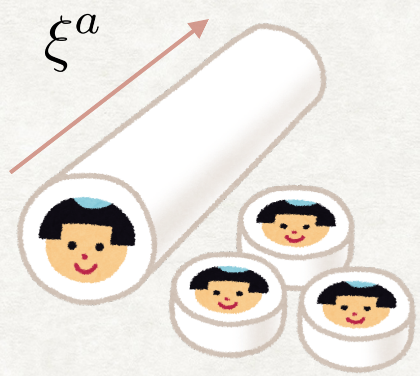
- Material: spacetime conformal symmetry

[Wald's textbook]

Definition: spacetime symmetry

Spacetime $(\mathcal{M}, g_{ab})$ possesses symmetry
iff g_{ab} admits an isometry defined by $\varphi_t : \mathcal{M} \to \mathcal{M}$ such that $\varphi_t^* g_{ab} = g_{ab}$

Isometry group is generated by $x^a \to \bar{x}^a - \xi^a$
along a ***Killing vector field*** that satisfies $\mathcal{L}_\xi g_{ab} = 0$

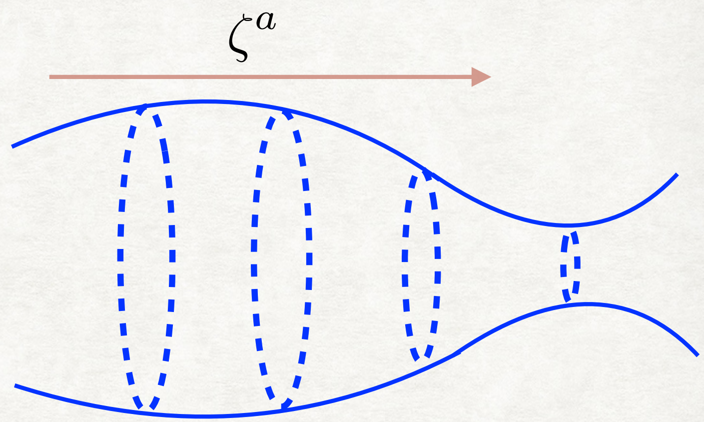


Kintaro-ame

Review: Mass Ladder operators from spacetime conformal symmetry

- What is spacetime conformal symmetry?

  Generalization of spacetime symmetry



[Wald's textbook]

Definition: spacetime conformal symmetry

Spacetime $(\mathcal{M}, g_{ab})$ possesses conformal symmetry
iff g_{ab} admits a conformal isometry defined by $\varphi_t : \mathcal{M} \to \mathcal{M}$
such that $\varphi_t^* g_{ab} = \exp(2Q)\, g_{ab}$, where Q is a function on $\mathcal{M}$

Conformal isometry group is generated by $x^a \to \bar{x}^a - \zeta^a$
along a ***conformal Killing vector field*** that satisfies $\mathcal{L}_\zeta g_{ab} = 2Q g_{ab}$

∗ Conformal Killing vector field is
a Killing vector field of $\tilde{g}_{ab} = \Omega^2 g_{ab}$ with $Q = -2\zeta^a \nabla_a \ln \Omega$

Construction of mass Ladder operators

[Cardoso *et al*, 2017]
[Cardoso *et al*, 2018]

- "Closed" condition for conformal Killing vector fields: $\nabla_a \zeta_b = \nabla_b \zeta_a$

- Assumption: $R^a{}_b \zeta^b = \chi (n-1) \zeta^a, \quad \chi \in \mathbb{R}$

  e.g., n-dimensional (anti-) de Sitter spacetimes with $\chi = \dfrac{\Lambda}{n-1}$

- ***Mass ladder operator*** : $D_k := \mathcal{L}_\zeta - \dfrac{k}{n} \left(\nabla_\sigma \zeta^\sigma \right), \ k \in \mathbb{R}$

- Commutation relation holds: $[\Box, D_k] = \chi \left(2k + n - 2\right) D_k + 2Q \left(\Box + \chi k \left(k + n - 1\right)\right)$

$$\Rightarrow D_{k-2} \left[\Box - \mu^2\right] \Phi = \left[\Box - \left(\mu^2 + \delta\mu^2\right)\right] D_k \Phi, \qquad \begin{aligned} \mu^2 &= -\chi k(k+n-1), \\ \delta\mu^2 &= \chi(2k+n-2) \end{aligned}$$

If Φ is a Klein-Gordon field with μ^2,
$D_k \Phi$ is also a Klein-Gordon field but with $\mu^2 + \delta\mu^2$

Note: Mass ladder operator is an onto map

Mass Ladder operators

[Cardoso *et al*, 2017]
[Cardoso *et al*, 2018]

- Mass ladder operator connects Klein-Gordon fields with different mass squared:

$$\left[\Box - \mu^2\right]\Phi = 0 \longrightarrow \left[\Box - \left(\mu^2 + \delta\mu^2\right)\right] D_k \Phi = 0$$

$$\boxed{\mu^2 = -\,\chi k(k+n-1),}$$
$$\delta\mu^2 = \chi(2k+n-2)$$

- k is required to be real, so $\boxed{}$ leads to inequalities:

$$\mu^2 \geq \frac{\chi}{4}(n-1)^2 \quad (\text{for} \quad \chi < 0)\,, \quad \mu^2 \leq \frac{\chi}{4}(n-1)^2 \quad (\text{for} \quad \chi > 0)$$

Note: In AdS case $(\chi < 0)$, the lower bound coincides with the BF bound

- When parametrizing $\mu^2 = -\chi\nu(\nu+n-1)\ \ (\nu \geq -1)$,

$\boxed{}$ leads to two solutions, $k_+ = -n+1-\nu, \quad k_- = \nu$

$$\left[\Box + \chi\nu(\nu+n-1)\right]\Phi = 0$$

$$\longrightarrow \left[\Box + \chi\tilde{\nu}(\tilde{\nu}+n-1)\right] D_{k_+}\Phi = 0$$

$\tilde{\nu} = \nu + 1$ mass raising (lowering) for $\chi < 0\ (> 0)$

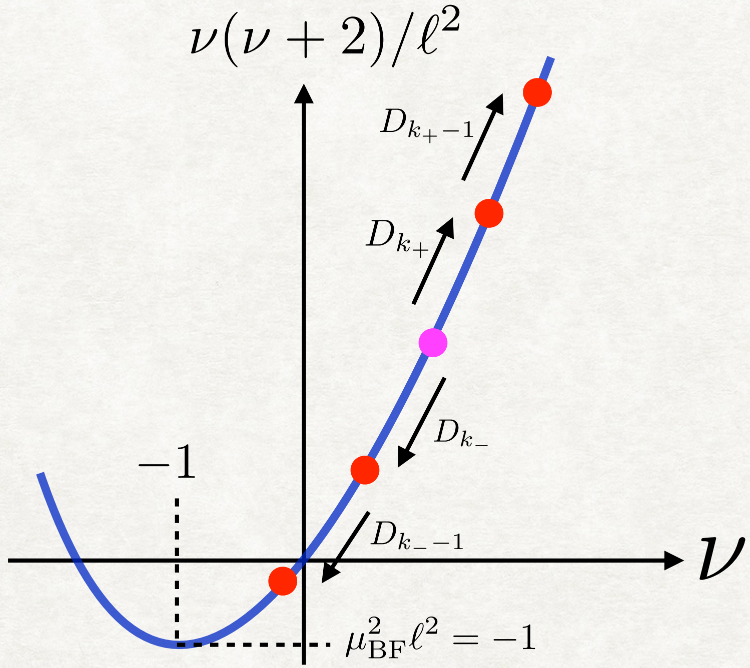
$$\longrightarrow \left[\Box + \chi\tilde{\nu}(\tilde{\nu}+n-1)\right] D_{k_-}\Phi = 0$$

$\tilde{\nu} = \nu - 1$ mass lowering (raising) for $\chi < 0\ (> 0)$

Mass shift

Example: $\left[\Box - \mu^2\right]\Phi = 0$ on AdS_3 with length scale $\ell := \sqrt{-1/\Lambda}$

Mass parametrization: $\mu^2 = \nu(\nu+2)/\ell^2 \;\; (\nu \geq -1)$



Mass ladder operators $D_{k_\pm}$ make mass squared raise or lower

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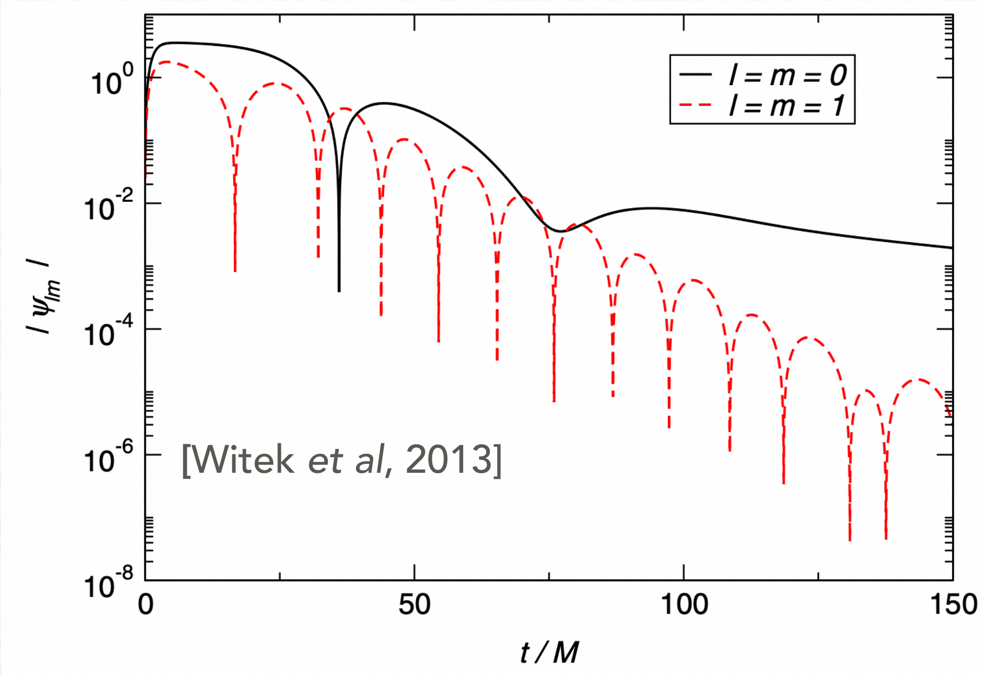
Quasinormal modes

- Quasinormal modes (QNMs) describe

  characteristic dynamics of test fields on BH spacetimes as a linear response



- Many applications:

  modeling ringdown gravitational waveforms, [Giesler *et al*, 2019]
  analysis of relaxation phenomena within AdS/CFT, [Horowitz and Hubeny, 2000]
  linear mode stability of BH spacetimes,… [Regge and Wheeler, 1957]

Brief review: QNMs in asymptotically flat BH spacetimes

- Equation for linear perturbations on asymptotically flat background:

$$\left[\frac{d^2}{dx^2}+\omega^2-V(x)\right]\phi\left(x;\omega\right)=0$$

e.g., spin-s field perturbation on Schwarzschild backgrounds

$$\left[\frac{d^2}{dx^2}+\omega^2-V^{(s)}\right]\Phi^{(s)}=0, \quad \frac{dr}{dx}=1-\frac{r_H}{r}$$

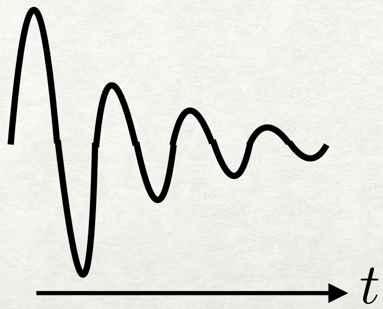
$$V^{(s)}=\left(1-\frac{r_H}{r}\right)\left(\frac{\ell(\ell+1)}{r^2}+\frac{\left(1-s^2\right)r_H}{r^3}\right),$$

- Appropriate boundary conditions at the horizon $x=-\infty$ and infinity $x=\infty$,

$$\lim_{x\to-\infty}\phi = e^{-i\omega x} \quad \text{(purely ingoing)}$$

$$\lim_{x\to+\infty}\phi = e^{i\omega x} \quad \text{(purely outgoing)}$$

define QNMs and (a discrete set of) QNM frequencies



- QNM frequencies are complex, $\phi\propto e^{-i\mathrm{Re}[\omega]t}e^{\mathrm{Im}[\omega]t}$

Brief review: QNMs as poles of Green's function

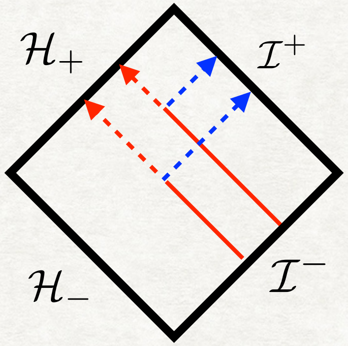
- Equation for linear perturbations in time domain: $\left[\partial_x^2 - \partial_t^2 - V(x)\right]\phi(t,x) = 0$

  Laplace transform leads to $\left[\frac{d^2}{dx^2} + \omega^2 - V(x)\right]\hat{\phi}(x;\omega) = \left[i\omega\phi - \partial_t\phi\right]\Big|_{t=0}$

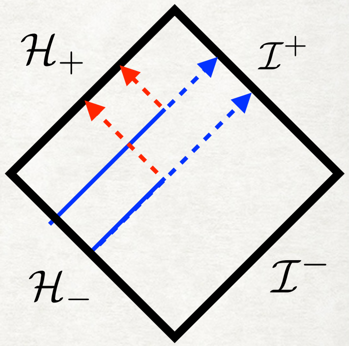
- Two homogeneous sols. such that

$$\hat{\phi}_+ \simeq \begin{cases} e^{-i\omega t}, & x \to -\infty, \\ a_1(\omega)\, e^{-i\omega t} + a_2(\omega)\, e^{+i\omega t}, & x \to +\infty \end{cases}$$

$$\hat{\phi}_- \simeq \begin{cases} b_1(\omega)\, e^{-i\omega t} + b_2(\omega)\, e^{+i\omega t}, & x \to -\infty \\ e^{+i\omega t}, & x \to +\infty, \end{cases}$$



$\hat{\phi}_+$: in-sol.



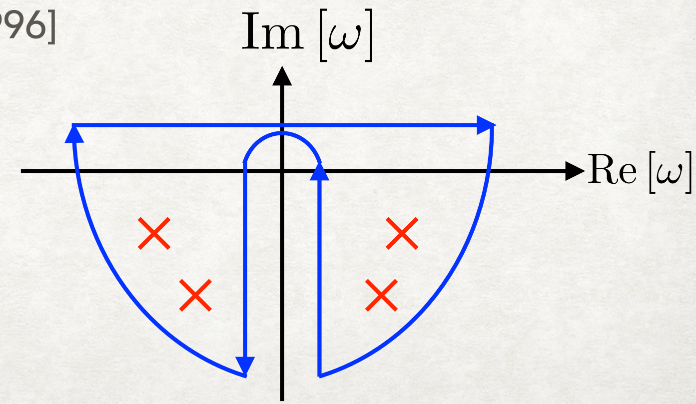
$\hat{\phi}_-$: up-sol.

- Green's function in frequency domain: [Leaver,1988] [Anderson,1996]

$$\hat{G}(x, x'; \omega) = \frac{\hat{\phi}_+(x;\omega)\,\hat{\phi}_-(x';\omega)}{W(\omega)}$$

  where W is a Wronskian of $\hat{\phi}_+$ and $\hat{\phi}_-$

  QNM frequencies are determined by $W = 0$



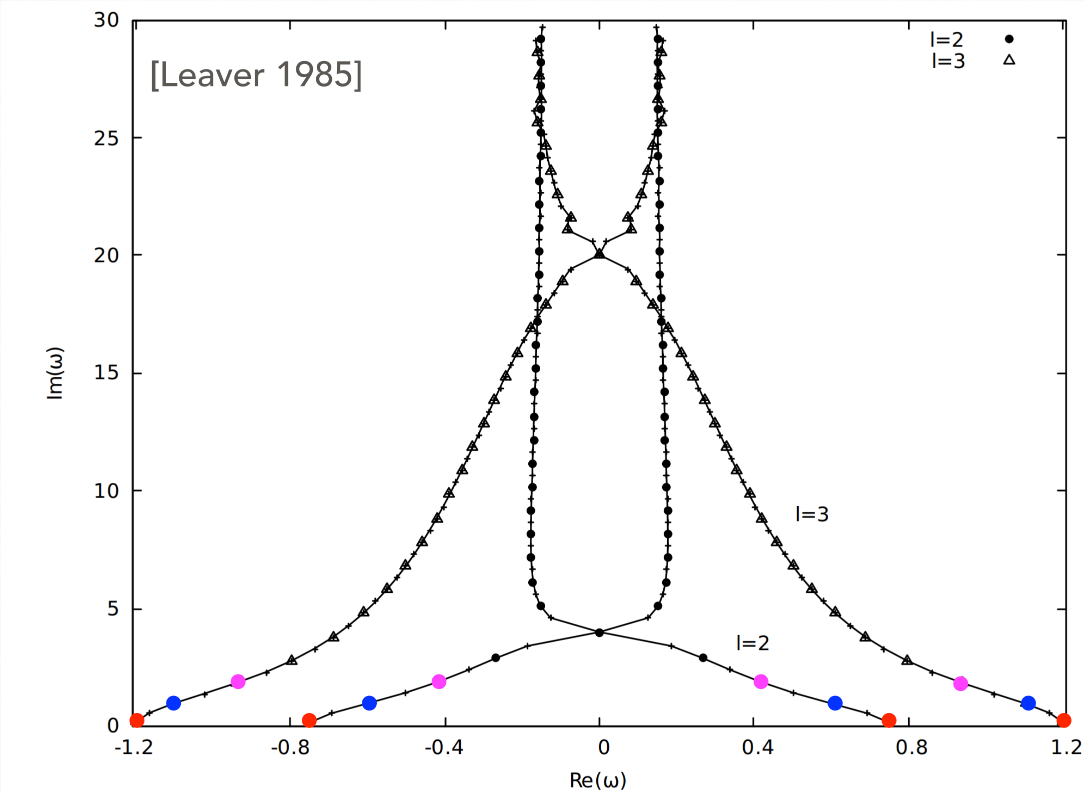
Brief review: Overtones

- QNM takes a form: $\Phi_{\rm QNM} = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{n=0}^{\infty} \phi_{\ell m n}(x) e^{-i\omega_{\ell m n} t} Y_{\ell m}(\theta, \varphi)$

  For each mode with ℓ ,
  there exists a discrete set of modes labeled by $n (= 0, 1, 2, \cdots)$: ***overtones***

- Index of overtones is defined in the order from the smallest value of $|{\rm Im}[\omega]|$

  $n = 0$: fundamental mode, $n = 1$: 1st overtones, $n = 2$: 2nd overtones,...



QNMs in AdS BH spacetimes

- QNMs in AdS BH spacetimes can be defined in the same manner [Berti *et al*, 2009]

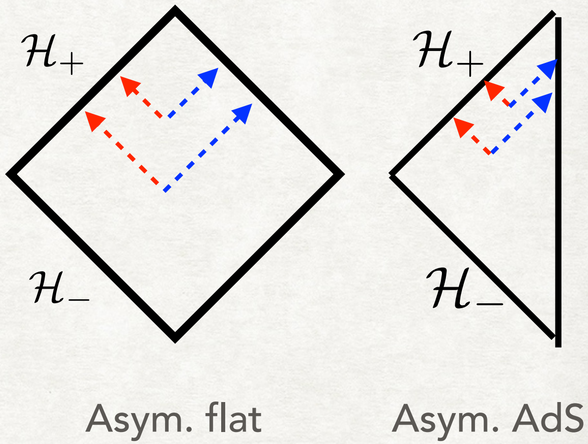
- Variety of boundary condition at infinity exists due to the asymptotic structure

e.g., $\lim_{x\to\infty}\phi = A + \frac{B}{x^3}$ for massless scalar field

$A = 0$: Dirichlet condition

$B = 0$: Neumann condition

$B = \kappa A, \quad \kappa \in \mathcal{R}$: Robin condition



⇨ Rich structure of QNM dynamics appears in AdS (as will be seen later)

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Question and our work

Mass ladder operator works in curved spacetimes with conformal symmetry:

$$\left[\Box - \mu^2\right]\Phi = 0 \quad \Rightarrow \quad \left[\Box - \left(\mu^2 + \delta\mu^2\right)\right] D\Phi = 0$$

Question:
Does a mass ladder operator keep physics determined by boundary conditions?

We study QNMs of a massive Klein-Gordon field
in Banados-Teitelboim-Zanelli black hole spacetimes

Why BTZ?: BTZ spacetime is the simplest system,
in which QNMs and mass ladder operators can be exactly derived

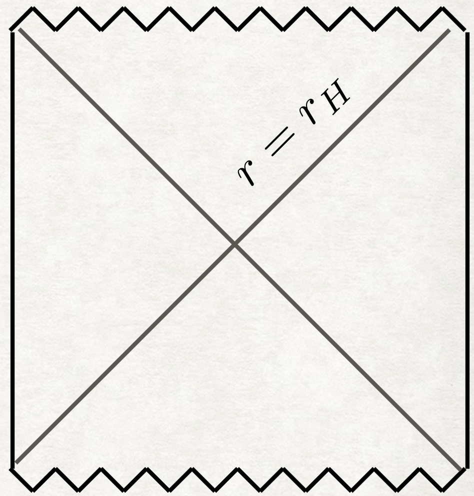
Static Banados-Teitelboim-Zanelli spacetime

- BTZ geometry describes asymptotically AdS black hole spacetimes in 3 dim.

- Line element in (t, r, φ) coordinates: [Banados, Teitelboim, and Zanelli, 1992]

$$ds^2 = -N^2(r)\,dt^2 + \frac{1}{N^2(r)}dr^2 + r^2 d\varphi^2, \quad N^2(r) = \frac{r^2 - r_H^2}{\ell^2}$$

$$-\infty < t < \infty, \ r_H < r < \infty, \quad 0 \le \varphi < 2\pi$$



- Horizon is located at $r = r_H$ such that $N^2(r_H) = 0$

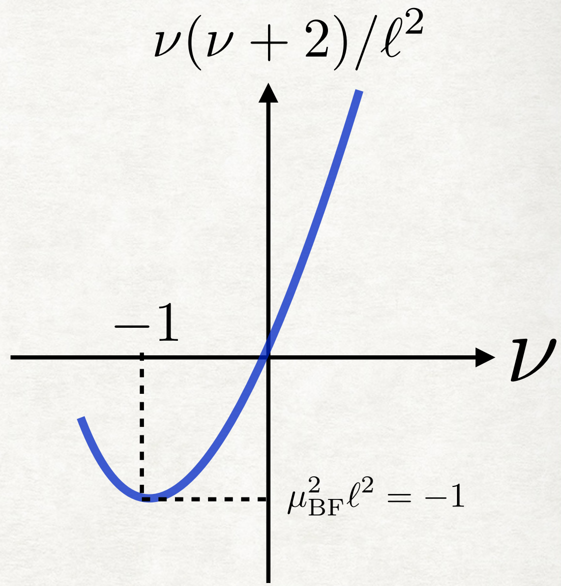
- locally isometric to AdS_3

＊BTZ BH can rotate but for simplicity we consider the static version

Massive Klein-Gordon fields

- Massive Klein-Gordon field: $\left[\nabla_\mu \nabla^\mu - \frac{\nu(\nu+2)}{\ell^2}\right]\Phi = 0$

$$\nu \geq -1$$



- Expanding the field: $\Phi = \sum_m \phi_m(r) e^{-i\omega_m t} e^{im\varphi}$

$$\phi'' + \left(\frac{1}{r} + \frac{(N^2)'}{N^2}\right)\phi' + \frac{1}{N^2}\left(\frac{\omega^2}{N^2} - \frac{m^2}{r^2} - \frac{\nu(\nu+2)}{\ell^2}\right)\phi = 0$$

- Appropriate B.C. selects a discrete set of eigenvalues, i.e., QNM frequencies

At the horizon: Ingoing-wave condition

$$\phi = A\left(1 - \frac{r_H^2}{r^2}\right)^{-\frac{i\omega\ell^2}{2r_H}} \left(\frac{r_H}{r}\right)^{\nu+2} {}_2F_1\left(a, b; c; 1 - r_H^2/r^2\right)$$

$$a = \frac{\nu+2}{2} - i\frac{\ell}{2r_H}(\omega\ell - m)$$
$$b = \frac{\nu+2}{2} - i\frac{\ell}{2r_H}(\omega\ell + m)$$
$$c = 1 - i\frac{\omega\ell^2}{r_H}$$

QNMs in BTZ spacetimes

- B.C. at infinity: Dirichlet B.C. for $\nu > -1$ $(\mu^2 > \mu_{\rm BF}^2)$

$$a = \frac{\nu+2}{2} - i\frac{\ell}{2r_H}(\omega\ell - m)$$
$$b = \frac{\nu+2}{2} - i\frac{\ell}{2r_H}(\omega\ell + m)$$
$$c = 1 - i\frac{\omega\ell^2}{r_H}$$

$$\phi = A_{\rm I}(\omega)\left(\frac{r_H}{r}\right)^{-\nu}[1+\cdots] + A_{\rm II}(\omega)\left(\frac{r_H}{r}\right)^{\nu+2}[1+\cdots]$$

$A_{\rm I}(\omega)\left(:= A\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right)$ vanishes

if $a = -n$ or $b = -n$ for $n = 0, 1, 2, \cdots$ due to $1/\Gamma(-n) = 0$

- QNMs frequencies: $$\omega_{\rm D} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n+2+\nu)$$ [Cardoso and Lemos, 2001]

$$\phi = A\left(1-\frac{r_H^2}{r^2}\right)^{-i\frac{\omega_{\rm D}\ell^2}{2r_H}}\left(\frac{r_H}{r}\right)^{2+\nu}\sum_{k=0}^{n}\frac{(a)_k(b)_k}{k!(c)_k}\left(1-\frac{r_H^2}{r^2}\right)^k$$ where $(\xi)_k \equiv \Gamma(z+k)/\Gamma(z)$

Imaginary parts are negative, indicating linear mode stability

QNMs in BTZ spacetimes: Other boundary condition

- B.C. at infinity: Neumann B.C. for $\nu > -1$ $(\mu^2 > \mu_{\rm BF}^2)$

$$\phi = A_{\rm I}(\omega)\left(\frac{r_H}{r}\right)^{-\nu}[1+\cdots] + A_{\rm II}(\omega)\left(\frac{r_H}{r}\right)^{\nu+2}[1+\cdots]$$

⇨ $$\omega_{\rm N} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n-\nu)$$

Imaginary part can be nonnegative if $\nu \geq 0$ $(\mu^2 \geq 0)$,
indicating linear mode instability
due to the presence of non-normalizable mode

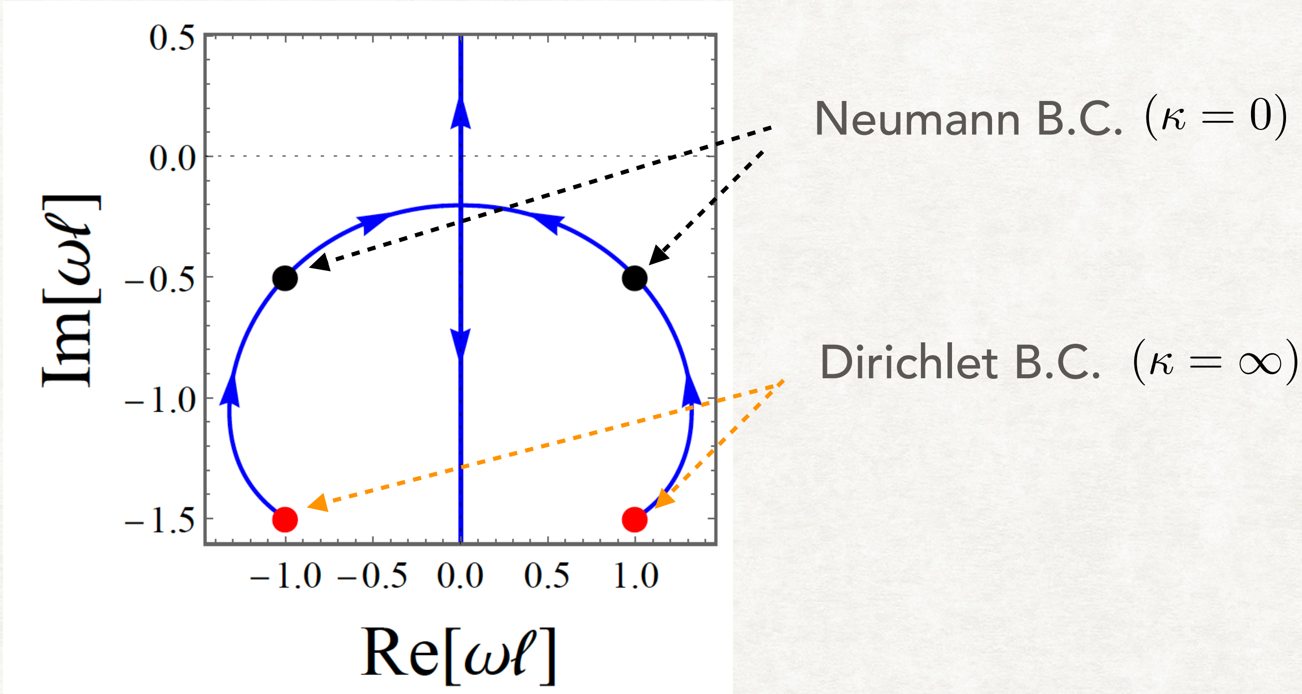
- B.C. at infinity: Robin B.C. for $-1 < \nu < 0$ $(\mu_{\rm BF}^2 < \mu^2 < 0)$

$$A_{\rm II}/A_{\rm I} = \kappa \quad (\kappa \in \mathbb{R})$$

including the Dirichlet $\kappa = \infty$ $(A_{\rm I} = 0)$ and Neumann B.C. $\kappa = 0$ $(A_{\rm II} = 0)$

In this sense, Robin B.C. is more general and admits rich structure

Fundamental modes in Robin condition



As κ decreases, the trajectories approach the imaginary axis, eventually intersect, and split into two parts

[Ishibashi and Wald, 2004]

[TK and Harada, 2021]

There exist growing modes,
indicating linear mode instability due to the boundary condition

QNMs in BTZ spacetimes: BF bound case

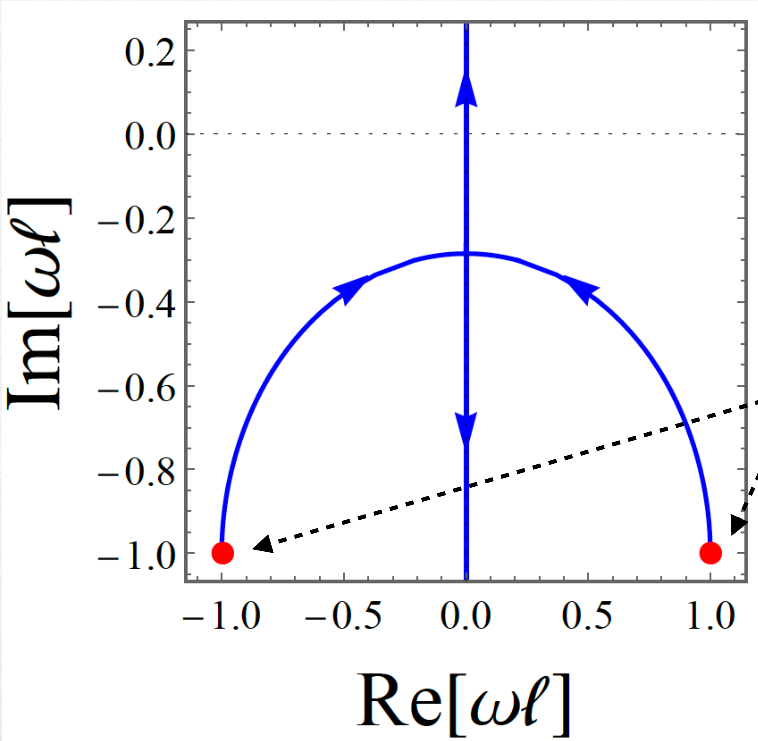
- B.C. at infinity: Dirichelt-Neumann B.C. for $\nu = -1 \ (\mu^2 = \mu_{\rm BF}^2)$ [Ishibashi and Wald, 2004]

$$\phi(r) = A_{\rm I,BF}\frac{r_H}{r} + A_{\rm II,BF}\frac{r_H}{r}\ln\left(\frac{r_H}{r}\right) + \mathcal{O}\left(1/r^3\right),$$

$$\Rightarrow \quad \omega_{\rm DN} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n+1)$$

- B.C. at infinity: Robin B.C. for $\nu = -1 \ (\mu^2 = \mu_{\rm BF}^2)$

$$A_{\rm II,BF}/A_{\rm I,BF} = 1/\kappa_{\rm BF} \quad (\kappa_{\rm BF} \in \mathbb{R})$$



Dirichlet - Neumann B.C. $(\kappa_{\rm BF} \to -\infty)$

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Mass ladder operators in BTZ spacetimes

- Mass ladder operators: $D_{i,k} := \mathcal{L}_{\zeta_i} - \frac{k}{3}\nabla_\mu \zeta_i^\mu, \quad k \in \mathbb{R}$

- Four independent CCKVs $(i = 0, 1, 2, 3)$ exist in the BTZ spacetime, thus:

$$D_{0,k} = e^{\frac{r_H}{\ell^2}t}\left(\frac{1}{\sqrt{r^2 - {r_H}^2}}\partial_t - \frac{r\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H}\partial_r + k\frac{\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H}\right),$$

$$D_{1,k} = e^{-\frac{r_H}{\ell^2}t}\left(\frac{1}{\sqrt{r^2 - {r_H}^2}}\partial_t + \frac{r\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H}\partial_r - k\frac{\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H}\right),$$

$$D_{2,k} = e^{\frac{r_H}{\ell}\varphi}\left(\frac{r^2 - {r_H}^2}{\ell r_H}\partial_r + \frac{1}{r}\partial_\varphi - k\frac{r}{\ell r_H}\right),$$

$$D_{3,k} = e^{-\frac{r_H}{\ell}\varphi}\left(-\frac{r^2 - {r_H}^2}{\ell r_H}\partial_r + \frac{1}{r}\partial_\varphi + k\frac{r}{\ell r_H}\right).$$

Acting on $\Phi \propto e^{-i\omega t}e^{im\varphi}$, the factors $e^{\pm\frac{r_H^2}{\ell^2}t}$ shift $\omega \to \omega \pm i r_H/\ell^2$

while $e^{\pm\frac{r_H}{\ell}\varphi}$ break the periodicity to φ

⇨ We mainly focus on $D_{0,k}, D_{1,k}$ ($D_{2,k}, D_{3,k}$ still work by multiple action)

Mass ladder operators in BTZ spacetimes

- Commutation relation: $[\nabla_\mu \nabla^\mu, D_{i,k}] = -\frac{2k+1}{\ell^2} D_{i,k} + \frac{2}{3}(\nabla_\mu \zeta_i^\mu) \left[\nabla_\mu \nabla^\mu - \frac{k(k+2)}{\ell^2} \right],$

When choosing $k = k_+ := -2 - \nu$

$$D_{i,k_+ - 2} \left[\nabla_\mu \nabla^\mu - \frac{\nu(\nu+2)}{\ell^2} \right] \Phi = \left[\nabla_\mu \nabla^\mu - \frac{(\nu+1)(\nu+3)}{\ell^2} \right] D_{i,k_+} \Phi.$$

When choosing $k = k_- := \nu$

$$D_{i,k_- - 2} \left[\nabla_\mu \nabla^\mu - \frac{\nu(\nu+2)}{\ell^2} \right] \Phi = \left[\nabla_\mu \nabla^\mu - \frac{(\nu-1)(\nu+1)}{\ell^2} \right] D_{i,k_-} \Phi.$$

For the massive Klein-Gordon field Φ with $\nu(\nu+2)/\ell^2$

$D_{i,k_\pm}\Phi$ is also that with $\tilde{\nu}(\tilde{\nu}+2)/\ell^2 \quad (\tilde{\nu} = \nu \pm 1)$

Mass ladder operators and QNM boundary conditions

- QNM with Dirichlet B.C.: $\Phi = A\left(1-\frac{r_H^2}{r^2}\right)^{-i\frac{\ell^2}{2r_H}\omega_{\rm D}}\left(\frac{r_H}{r}\right)^{2+\nu}{}_2F_1\left(a,b;c;1-r_H^2/r^2\right)e^{-i\omega_{\rm D}t+im\varphi},$

$$\omega_{\rm D} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n+2+\nu)$$

$$\Phi|_{r\simeq r_H} = 2^{-i\frac{\ell^2}{2r_H}\omega_{\rm D}}A\left(\frac{r-r_H}{r_H}\right)^{-i\frac{\ell^2}{2r_H}\omega_{\rm D}}\left[1+\mathcal{O}(r-r_H)\right]e^{-i\omega_{\rm D}t+im\varphi}.$$ (Ingoing wave)

- Acting $D_{0,k_\pm}, D_{1,k_\pm}$ on the QNMs, at $r\to r_H$

$$D_{0,k_\pm}\Phi = c_{0,k_\pm}\left(\frac{r-r_H}{r_H}\right)^{-i\frac{\ell^2}{2r_H}\left(\omega+i\frac{r_H}{\ell^2}\right)}\left[1+\mathcal{O}(r-r_H)\right]e^{-i\left(\omega+i\frac{r_H}{\ell^2}\right)t+im\varphi},$$

$$D_{1,k_\pm}\Phi = c_{1,k_\pm}\left(\frac{r-r_H}{r_H}\right)^{-i\frac{\ell^2}{2r_H}\left(\omega-i\frac{r_H}{\ell^2}\right)}\left[1+\mathcal{O}(r-r_H)\right]e^{-i\left(\omega-i\frac{r_H}{\ell^2}\right)t+im\varphi}.$$

Mass ladder operators keep the ingoing-wave condition

Mass ladder operators and QNM boundary conditions

$$\Phi|_{r\simeq\infty} = A_{\rm II}\left(\frac{r_H}{r}\right)^{2+\nu}\left[1+\mathcal{O}(1/r^2)\right]e^{-i\omega t+im\varphi} \quad \text{(Dirichlet B.C.)}$$

- Acting D_{0,k_+}, D_{1,k_+} on the QNMs, at $r\to\infty$

$$D_{0,k_+}\Phi = c^{(\rm D)}_{0,k_+}\left(\frac{r_H}{r}\right)^{3+\nu}\left[1+\mathcal{O}\left(1/r^2\right)\right]\ e^{-i\left(\omega+i\frac{r_H}{\ell^2}\right)t+im\varphi},$$
$$D_{1,k_+}\Phi = c^{(\rm D)}_{1,k_+}\left(\frac{r_H}{r}\right)^{3+\nu}\left[1+\mathcal{O}\left(1/r^2\right)\right]\ e^{-i\left(\omega-i\frac{r_H}{\ell^2}\right)t+im\varphi}.$$

Asymptotic behaviors of the Klein-Gordon field with $\tilde{\nu}\left(\tilde{\nu}+2\right)/\ell^2 \quad (\tilde{\nu}=\nu+1)$

Mass ladder operators D_{0,k_+}, D_{1,k_+} keep the Dirichlet B.C.

Mass ladder operators and QNM boundary conditions

$$\Phi|_{r\simeq\infty} = A_{\rm II}\left(\frac{r_H}{r}\right)^{2+\nu}\left[1+\mathcal{O}(1/r^2)\right]e^{-i\omega t+im\varphi} \quad \text{(Dirichlet B.C.)}$$

- Acting D_{0,k_-}, D_{1,k_-} on the QNMs, at $r\to\infty$

$$D_{0,k_-}\Phi = c^{\rm (D)}_{0,k_-}\left(\frac{r_H}{r}\right)^{1+\nu}\left[1+\mathcal{O}\left(1/r^2\right)\right]\ e^{-i\left(\omega+i\frac{r_H}{\ell^2}\right)t+im\varphi},$$

$$D_{1,k_-}\Phi = c^{\rm (D)}_{1,k_-}\left(\frac{r_H}{r}\right)^{1+\nu}\left[1+\mathcal{O}\left(1/r^2\right)\right]\ e^{-i\left(\omega-i\frac{r_H}{\ell^2}\right)t+im\varphi}.$$

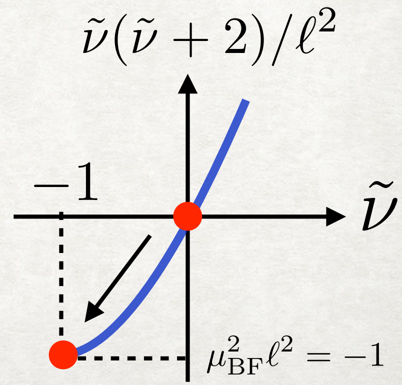
- For $\nu>0$ $(\mu^2>0)$:

  Asymptotic behaviors of the Klein-Gordon field with $\tilde{\nu}\left(\tilde{\nu}+2\right)/\ell^2$ $(\tilde{\nu}=\nu-1)$

  Mass ladder operators D_{0,k_-}, D_{1,k_-} keep the Dirichlet B.C.

- For $\nu=0$ $(\mu^2=0)$: $\tilde{\nu}=-1$ corresponds to $\mu^2_{\rm BF}\ell^2=-1$

  Dirichlet B.C. changes to the Dirichlet-Neumann B.C.



Mass ladder operators and QNM boundary conditions

$$D_{0,k_-}\Phi = c^{(\rm D)}_{0,k_-}\left(\frac{r_H}{r}\right)^{1+\nu}\left[1+\mathcal{O}\left(1/r^2\right)\right]\ e^{-i\left(\omega+i\frac{r_H}{\ell^2}\right)t+im\varphi},$$

$$D_{1,k_-}\Phi = c^{(\rm D)}_{1,k_-}\left(\frac{r_H}{r}\right)^{1+\nu}\left[1+\mathcal{O}\left(1/r^2\right)\right]\ e^{-i\left(\omega-i\frac{r_H}{\ell^2}\right)t+im\varphi}.$$

For $-1<\nu<0\ (\mu^2_{\rm BF}<\mu^2<0)$:

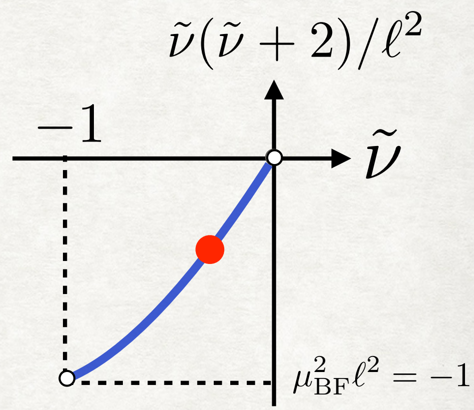
The above corresponds to $A_{\rm II}(\omega)=0$ (Neumann B.C.)

of the asymptotic behavior

$$\phi = A_{\rm I}\left(\omega\right)\left(\frac{r_H}{r}\right)^{-\tilde{\nu}}\left[1+\cdots\right]+A_{\rm II}\left(\omega\right)\left(\frac{r_H}{r}\right)^{\tilde{\nu}+2}\left[1+\cdots\right]$$

with $\tilde{\nu}=|\nu|-1$

$(-1<\tilde{\nu}<0)$



Dirichlet B.C. changes to Neumann B.C.

Brief summary: Changes of QNM boundary conditions

- At the horizon:

Mass ladder operators $D_{0,k_\pm}, D_{1,k_\pm}$ keep the ingoing-wave condition

- At infinity:

Mass ladder operators D_{0,k_+}, D_{1,k_+}

Dirichlet B.C. $\longrightarrow$ Dirichlet B.C.

Mass ladder operators D_{0,k_-}, D_{1,k_-}

For $\nu > 0 \ (\mu^2 > 0)$	Dirichlet B.C.	$\longrightarrow$	Dirichlet B.C.
For $\nu = 0 \ (\mu^2 = 0)$	Dirichlet B.C.	$\longrightarrow$	Dirichlet-Neumann B.C.
For $-1 < \nu < 0 \ (\mu^2_{\rm BF} < \mu^2 < 0)$	Dirichlet B.C.	$\longrightarrow$	Neumann B.C.

Brief summary: Neumann case

- At infinity:

Mass ladder operators D_{0,k_+}, D_{1,k_+}

Neumann B.C. $\longrightarrow$ Neumann B.C.

Mass ladder operators D_{0,k_-}, D_{1,k_-}

For $\nu > 0$ $(\mu^2 > 0)$ Neumann B.C. $\longrightarrow$ Neumann B.C.

For $\nu = 0$ $(\mu^2 = 0)$ Neumann B.C. $\longrightarrow$ Dirichlet-Neumann B.C.

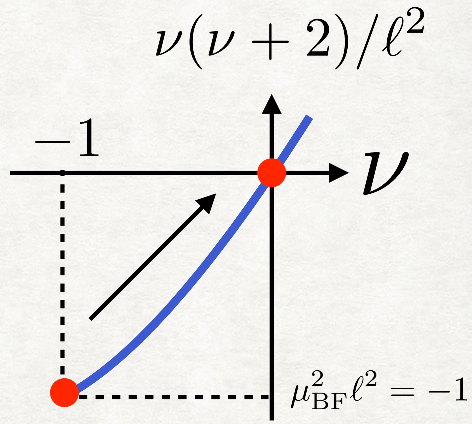
For $-1 < \nu < 0$ $(\mu^2_{\mathrm{BF}} < \mu^2 < 0)$ Neumann B.C. $\longrightarrow$ Dirichlet B.C.

Brief summary: Dirichlet-Neumann, Robin cases

- Dirichlet-Neumann case $\nu = -1 \ (\mu^2 = \mu_{\rm BF}^2)$:

  Mass ladder operators $D_{0,k_+}, D_{1,k_+} \quad (k_+ = k_-)$

  Dirichlet-Neumann B.C. ⟶ Dirichlet B.C.



- Robin boundary condition is kept

  but the resulting boundary condition parameter is complex

Outline

1. Introduction

2. Review: Mass ladder operators

3. Review: Quasinormal modes

4. Quasinormal modes in Banados-Teitelboim-Zanell spacetimes

5. Mass ladder operators in Banados-Teitelboim-Zanell spacetimes

6. Shift of quasinormal mode frequencies

QNM frequency shift

- Frequency shift from expressions of mass ladder operators

$$D_{0,k} = e^{\frac{r_H}{\ell^2} t} \left(\frac{1}{\sqrt{r^2 - {r_H}^2}} \partial_t - \frac{r\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H} \partial_r + k \frac{\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H} \right),$$

$$D_{1,k} = e^{-\frac{r_H}{\ell^2} t} \left(\frac{1}{\sqrt{r^2 - {r_H}^2}} \partial_t + \frac{r\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H} \partial_r - k \frac{\sqrt{r^2 - {r_H}^2}}{\ell^2 r_H} \right),$$

The factors $e^{\pm \frac{r_H^2}{\ell^2} t}$ suggest $\omega \to \omega \pm i r_H / \ell^2$

- Example: Shift by D_{0,k_+}

Original QNM frequency with Dirichlet B.C.: $\omega_{\mathrm{D}} = \pm \frac{m}{\ell} - i \frac{r_H}{\ell^2} (2n + 2 + \nu)$

Mass parameter shift: $\nu \to \tilde{\nu} = \nu + 1$

QNM frequency shift: $\omega_{\mathrm{D}} \to \tilde{\omega}_{\mathrm{D}} = \pm \frac{m}{\ell} - i \frac{r_H}{\ell^2} \left[2\underline{(n-1)} + 2 + \tilde{\nu} \right]$

$n \to n - 1$

Note: no "negative overtones" are generated, $D_{0,k_+}[\text{fundamental mode}] = 0$

QNM frequency shift

Operators	D_{0,k_+}	D_{0,k_-}	D_{1,k_+}	D_{1,k_-}
Frequencies	$\omega_{\rm D}(\nu+1,n-1)$	$\omega_{\rm D}(\nu-1,n)\ (\nu>0)$ $\omega_{\rm DN}(n)\ (\nu=0)$ $\omega_{\rm N}(\|\nu\|-1,n)\ (-1<\nu<0)$	$\omega_{\rm D}(\nu+1,n)$	$\omega_{\rm D}(\nu-1,n+1)\ (\nu>0)$ $\omega_{\rm DN}(n+1)\ (\nu=0)$ $\omega_{\rm N}(\|\nu\|-1,n+1)\ (-1<\nu<0)$

where
$$\omega_{\rm D}(\nu,n)=\pm\frac{m}{\ell}-i\frac{r_H}{\ell^2}(2n+2+\nu)\,,\quad \text{(Dirichlet B.C.)}$$
$$\omega_{\rm N}(\nu,n)=\pm\frac{m}{\ell}-i\frac{r_H}{\ell^2}(2n-\nu)\,,\quad \text{(Neumann B.C.)}$$
$$\omega_{\rm DN}(n)=\pm\frac{m}{\ell}-i\frac{r_H}{\ell^2}(2n+1)\,,\quad \text{(Dirichlet-Neumann B.C.)}$$

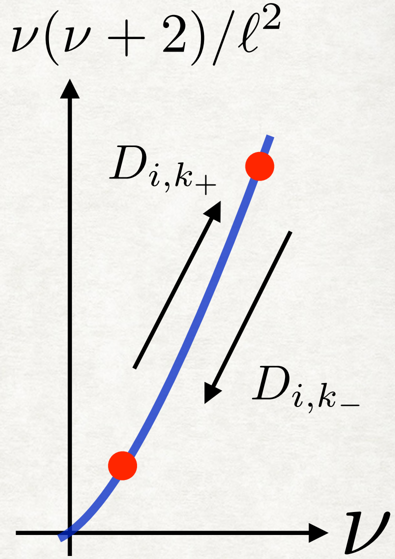
Mass ladder operators change not only mass squared but also indices of overtones

Overtone shift by multiple actions

- Multiple actions $D_{0,k_+-1}D_{0,k_-}$ or $D_{1,k_+-1}D_{1,k_-}$ keep mass squared but QNM frequencies are shifted:

$$\tilde{\omega} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}\left[2\left(n-1\right)+2+\nu\right] \quad \text{for} \quad D_{0,k_+-1}D_{0,k_-}$$

$$\tilde{\omega} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}\left[2\left(n+1\right)+2+\nu\right] \quad \text{for} \quad D_{1,k_+-1}D_{1,k_-}$$



Note: no "negative overtones" are generated from the fundamental mode

All overtones can be generated from the fundamental mode

Regular solutions generated by multiple actions of $D_{2,k_\pm}$, $D_{3,k_\pm}$

$$D_{2,k} = e^{\frac{r_H}{\ell}\varphi}\left(\frac{r^2 - {r_H}^2}{\ell r_H}\partial_r + \frac{1}{r}\partial_\varphi - k\frac{r}{\ell r_H}\right),$$

$$D_{3,k} = e^{-\frac{r_H}{\ell}\varphi}\left(-\frac{r^2 - {r_H}^2}{\ell r_H}\partial_r + \frac{1}{r}\partial_\varphi + k\frac{r}{\ell r_H}\right).$$

Factors $e^{\pm\frac{r_H}{\ell}\varphi}$ break the periodicity to φ ;
thus, the single action of $D_{2,k_\pm}$, $D_{3,k_\pm}$ fails to generate a regular solution

- Multiple actions can remove those singular factors,

e.g., $D_{2,k_+-1}D_{3,k_+}$

$$\tilde{\omega} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}\left[2\left(n-1\right) + 2 + \left(\nu+2\right)\right]$$ for Dirichlet B.C.

$$\tilde{\omega} = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}\left[2\left(n+1\right) - \left(\nu+2\right)\right]$$ for Neumann B.C.

e.g., $D_{2,k_--1}D_{3,k_+}$

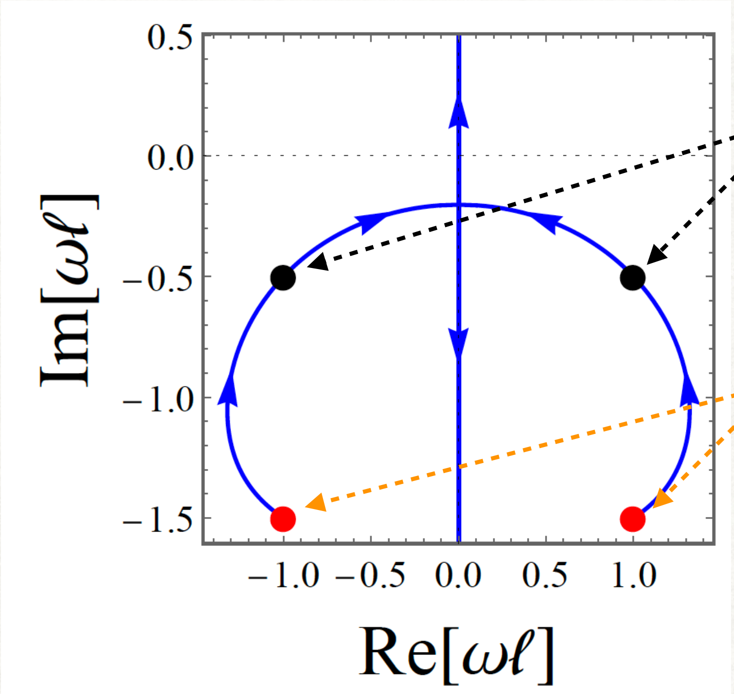
QNM frequency does not change
In fact, $D_{2,k_--1}D_{3,k_+} = \mathcal{L}_\xi$ is a symmetry operator

Other boundary condition: Robin case

[Ishibashi and Wald, 2004]

- Robin B.C. $(-1 < \nu < 0)$: $A_{\mathrm{II}}/A_{\mathrm{I}} = \kappa \ (\kappa \in \mathbb{R})$,

$$\phi \simeq A_{\mathrm{I}}(\omega)\left(\frac{r_H}{r}\right)^{-\nu} + A_{\mathrm{II}}(\omega)\left(\frac{r_H}{r}\right)^{\nu+2}$$



Neumann B.C. $(\kappa = 0)$

$$\omega_{\mathrm{N}}(0) = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}[-\nu]$$

Dirichlet B.C. $(\kappa = \infty)$

$$\omega_{\mathrm{D}}(0) = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}[2+\nu]$$

- Acting the mass ladder operators, $\omega \to \tilde{\omega} = \omega \pm i r_H/\ell^2$

$$\kappa \to \tilde{\kappa} \ \text{(complex value)}$$

At least, for κ so that ω is purely imaginary,
$\tilde{\kappa}$ is real, and $\tilde{\omega}$ is also purely imaginary

Summary

We have studied QNMs of massive Klein-Gordon fields in static BTZ spacetimes in terms of mass ladder operators

- Ladder operator is a useful tool to understand mathematical properties of QNMs

- Mass ladder operator change not only mass squared but also QNM frequencies

- In particular, an index of overtones are shifted

- All overtones can be generated from fundamental modes by their multiple actions

Other boundary condition: Neumann case

Operators	D_{0,k_+}	D_{0,k_-}	D_{1,k_+}	D_{1,k_-}
Frequencies	$\omega_{\rm N}(\nu+1,n)$	$\omega_{\rm N}(\nu-1,n-1)\ (\nu>0)$ $\omega_{\rm DN}(n-1)\ (\nu=0)$ $\omega_{\rm D}(\lvert\nu\rvert-1,n-1)\ (-1<\nu<0)$	$\omega_{\rm N}(\nu+1,n+1)$	$\omega_{\rm N}(\nu-1,n)\ (\nu>0)$ $\omega_{\rm DN}(n)\ (\nu=0)$ $\omega_{\rm D}(\lvert\nu\rvert-1,n)\ (-1<\nu<0)$

where
$$\omega_{\rm D}\left(\nu,n\right)=\pm\frac{m}{\ell}-i\frac{r_H}{\ell^2}\left(2n+2+\nu\right),\quad \text{(Dirichlet B.C.)}$$
$$\omega_{\rm N}\left(\nu,n\right)=\pm\frac{m}{\ell}-i\frac{r_H}{\ell^2}\left(2n-\nu\right),\quad \text{(Neumann B.C.)}$$
$$\omega_{\rm DN}\left(n\right)=\pm\frac{m}{\ell}-i\frac{r_H}{\ell^2}\left(2n+1\right),\quad \text{(Dirichlet-Neumann B.C.)}$$

BF bound case

Operators	$D_{0,-1}$	$D_{1,-1}$	D_0^{BF}
Frequencies	$\omega_{\mathrm{D}}(0, n-1)$	$\omega_{\mathrm{D}}(0, n)$	$\omega_{\mathrm{DN}}(0) \to \omega_{\mathrm{N}}(0,0)$

$$\omega_{\mathrm{D}}(\nu, n) = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n+2+\nu)\,, \quad \text{(Dirichlet B.C.)}$$

$$\omega_{\mathrm{N}}(\nu, n) = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n-\nu)\,, \quad \text{(Neumann B.C.)}$$

$$\omega_{\mathrm{DN}}(n) = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}(2n+1)\,, \quad \text{(Dirichlet-Neumann B.C.)}$$

- We find new mass ladder operator only for the fundamental mode in BF bound:

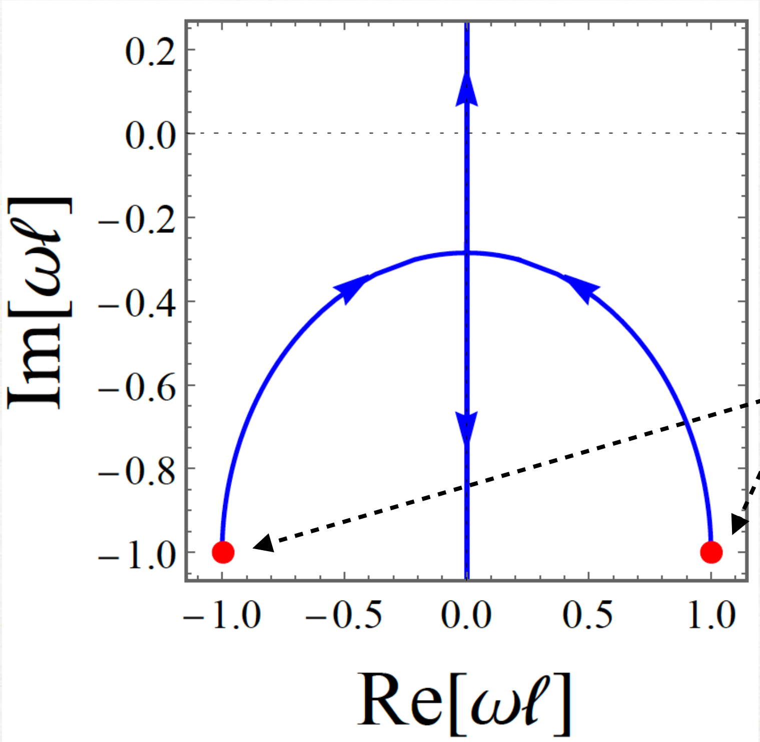
$$D_0^{\mathrm{BF}}\Phi_0 := (\nabla_\mu \zeta_0^\mu)\Phi_0 = e^{r_H t/\ell^2}\frac{\sqrt{r^2 - r_H^2}}{\ell^2 r_H}\Phi_0,$$

Other boundary condition: Robin case with BF bound

[Ishibashi and Wald, 2004]

- Robin B.C. $(\nu = -1)$: $A_{\rm II,BF}/A_{\rm I,BF} = 1/\kappa_{\rm BF}$ $(\kappa_{\rm BF} \in \mathbb{R})$,

$$\phi(r) = A_{\rm I,BF}\frac{r_H}{r} + A_{\rm II,BF}\frac{r_H}{r}\ln\left(\frac{r_H}{r}\right) + \mathcal{O}\left(1/r^3\right),$$



Dirichlet - Neumann B.C. $(\kappa_{\rm BF} \to -\infty)$

$$\omega_{\rm DN}(0) = \pm\frac{m}{\ell} - i\frac{r_H}{\ell^2}$$

- Acting the mass ladder operators, $\omega \to \tilde{\omega} = \omega \pm i r_H/\ell^2$

$$\kappa_{\rm BF} \to \tilde{\kappa}_{\rm BF} \ \text{(complex value)}$$

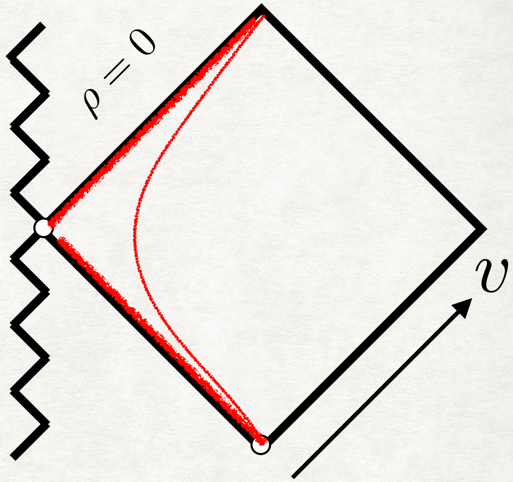
At least, for $\kappa_{\rm BF}$ so that ω is purely imaginary,
$\tilde{\kappa}_{\rm BF}$ is real, and $\tilde{\omega}$ is also purely imaginary

Application: scalars in near-horizon geometry

- We have derived conserved quantities along black hole horizons by exploiting mass ladder operators near the horizon [TK and Kimura 2021] [TK and Kimura 2022]

- Vicinity of black holes with zero Hawking temperature is highly-symmetric geometry called near-horizon geometry:

$$ds^2 = \underbrace{-\lambda_0 \rho^2 dv^2 + 2dvd\rho}_{\mathrm{AdS}_2} + \underbrace{\gamma_0 d\Omega^2_{n-2}}_{S^{n-2}}$$



- Reduction of scalars on near-horizon geometry to that on AdS2:

$$\Box \Phi = 0 \longrightarrow \left[\Box_{\mathrm{AdS}_2} - \frac{\ell(\ell + n - 3)}{\gamma_0}\right]\phi_\ell = 0$$

$$\Phi = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \phi_\ell(v, \rho) Y_{\ell m}(\theta, \varphi)$$

Mass ladder operators connect different multipole modes

- We obtain conservation laws along the horizon: $\partial_v \left[\partial_\rho D_1 D_2 \cdots D_\ell \phi_\ell\right]\big|_{\rho=0} = 0$