Shifted quantum groups and matter multiplets in SUSY gauge theories

Jean-Emile Bourgine

University of Melbourne (ACEMS)

Joint HEP-TH Seminar (SUDA?)

17-08-2022

[Based on [JEB 2107.10063, 2205.01309]**]**

AGT correspondence

6D $\mathcal{N} = (2,0)$ gauge theory on $\mathbb{C}^2 \times \Sigma$

4D $\mathcal{N} = 2$ gauge theory on $\mathbb{C}^2$

$G = \bigotimes U(2)$

$G = \bigotimes U(N)$

2D Conformal Field Theory on Σ

Liouville CFT

A_{N-1} Toda CFT

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Quantum Group ⟷ W-algebra

$\Rightarrow$ On the RHS, *all* physical quantities determined by the symmetry algebra!

How about gauge theory partition functions?

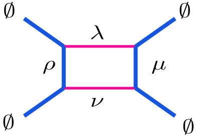
Can you reconstruct them using only representation theory?

- This point is better understood using the 5d uplift to $\mathcal{N}=1$ theories on $\mathbb{C}^2 \times S^1$.

⇝ AGT correspondence involves q-deformed W-algebras.

⇝ Non-perturbative (instanton) partition functions obtained as **topological strings amplitudes** on toric Calabi-Yau threefolds.

⇝ They can be computed using the (refined) topological vertex technique.



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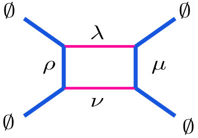

$$\mathcal{Z}_{\text{inst.}}\sim\sum_{\lambda,\mu,\nu,\rho}C_{\emptyset,\lambda,\rho}C_{\rho,\nu,\emptyset}C_{\nu,\mu,\emptyset}C_{\lambda,\emptyset,\mu}$$

- The topological vertex is the set of matrix elements of a **vertex operator**

$$C_{\lambda,\mu,\nu}=\langle\lambda|\,\Phi\left(|\mu\rangle\!\rangle\otimes|\nu\rangle\right)=\left(\langle\!\langle\lambda|\otimes\langle\mu|\right)\Phi^*\,|\nu\rangle\,.$$

$\rightsquigarrow$ Choice of basis actually irrelevant (equivalence between IKV and Awata-Kanno vertices).

[Awata, Feigin, Shiraishi 2011]

What is a vertex operator?

- The vertex operators of a 2d (chiral) CFT satisfy the primary field property

$$\rho_V$$

$$[L_n; V_\alpha(z)] = (z^{n+1}\partial_z + h_\alpha(n+1)z^n)V_\alpha(z)$$

$$\rho_H$$

$$\Leftrightarrow\ (\rho_V \otimes \rho_H\ \Delta(L_n))\, V_\alpha(z) = V_\alpha(z)\rho_H(L_n)$$

$$\Delta(L_n) = L_n \otimes 1 + 1 \otimes L_n$$

- We distinguish two types of representations:
  - **Vertical:** ρ_V conformal transformations acting on the coordinates (basis z^n)
  - **Horizontal:** ρ_H action of the Virasoro algebra on the Fock space (basis $J_{-\lambda_1}\cdots J_{-\lambda_\ell}\,|\emptyset\rangle$).

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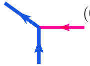
- The Virasoro algebra can be replaced by affine Lie algebras (WZW models), quantum groups (Integrable Systems), or even toroidal quantum groups (topological strings vertex).

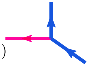
$\Rightarrow$ A vertex operator is an intertwiner between representations ρ_H and $\rho_V \otimes \rho_H$.

- To be specific, the vertex operators of topological strings Φ and Φ^* are constructed as **intertwiners** between 3 representations of the **quantum toroidal $\mathfrak{gl}(1)$ algebra**:

$$\Phi\left(\rho^{(0,1)} \otimes \rho^{(1,n)}\ \Delta(e)\right) = \rho^{(1,n+1)}(e)\Phi$$



$$\Phi^* \rho^{(1,n+1)}(e) = \left(\rho^{(0,1)} \otimes \rho^{(1,n)}\ \Delta'(e)\right)\Phi^*$$



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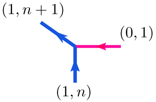

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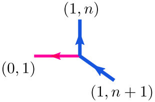
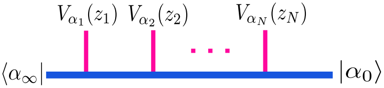


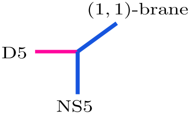
- Thus, **topological strings amplitudes are analogues of conformal blocks**



Geometric VS Algebraic Engineering

- This construction is in fact very general!

$\rightsquigarrow$ To undestand why, we need to recall the correspondence with **(p,q)-brane webs** in which the CY toric diagram is interpreted as a configuration of branes in type IIB string theory.



$\overbrace{\quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad}^{\mathbb{C}_{\varepsilon_1} \otimes \mathbb{C}_{\varepsilon_2} \otimes S^1}$

	0	1	2	3	4	5	6	7	8	9
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
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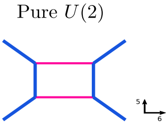

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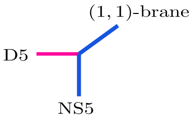

$\mathbb{C}_{\varepsilon_1} \otimes \mathbb{C}_{\varepsilon_2} \otimes S^1$ (over columns 0–4)

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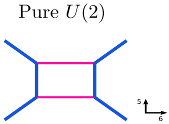

- The two types of representations are attached to different kind of branes:
  - **Vertical:** on D5-branes = COHA action on instanton moduli space $\rho_V = \rho^{(0,1)}$
  - **Horizontal:** on NS5-branes (+ n D5) = ‘auxiliary’ Fock space $\rho_H = \rho^{(1,n)}$

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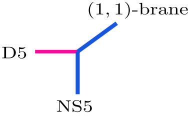

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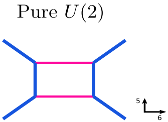

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This correspondence can be extended to many different brane systems and representations!!!

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iii. Construct an operator $\mathcal{T}$ acting on this network by **gluing vertex operators** Φ and Φ^*.

iv. Derive the gauge theory observables from **matrix elements** of this operator, e.g.

$$\mathcal{Z} = \langle \mathcal{T} \rangle$$

Partition Function

Operator acting on repr. network

$$\chi(z) = \langle X(z) \mathcal{T} \rangle$$

qq-character (BPS loop observable)

Algebra current

- Over the last few years, this program has been applied successfully to many SUSY QFT:

Theories	Quantum Group	Reference
5D $\mathcal{N} = 1$	quantum toroidal $\mathfrak{gl}(1)$	**[Awata, Feigin, Shiraishi 2011]**
5D $\mathcal{N} = 1$ on $\mathbb{Z}_p$-orbifold	quantum toroidal $\mathfrak{gl}(p)$	**[Awata, Kanno, et. al. 2017]**
5D $\mathcal{N} = 1$ on $\mathbb{Z}_p$-orbifold	new quantum toroidal algebras!	**[JEB, Jeong 2019]**
4D $\mathcal{N} = 2$	affine Yangian $\mathfrak{gl}(1)$	**[JEB, Zhang 2018]**
6D $\mathcal{N} = (1, 0)$	elliptic toroidal $\mathfrak{gl}(1)$	**[Zhu 2018] [Foda, Zhu 2018]**
3D $\mathcal{N} = 2^*$	quantum toroidal $\mathfrak{gl}(1)$	**[Zenkevich 2018]**
3D $\mathcal{N} = 2$	quantum affine $\mathfrak{sl}(2)$	**[JEB 2107.10063]**

- Other results include **D-type quiver gauge theories** **[JEB, Fukuda, Matsuo, Zhu 2017]**, **qq-characters** **[JEB, Fukuda, Harada, Matsuo, Zhu 2017]**, **R-matrices** and **KZ**-equations **[Awata, Kanno, et. al.]**...

Motivations

A few reasons for developing this technique:

- Emphasize the role of the non-perturbative symmetry
  - $\rightsquigarrow$ Correspondences (CFT, integrability), KZ equations, (q-)Painlevé,...
- Develop a diagrammatic technique that simplifies actual calculations
  - $\rightsquigarrow$ Generalize the topological vertex technique to other brane systems
- Study of non-Lagrangian theories
- Connect with string theory realizations of QFT and brane systems
  - $\rightsquigarrow$ Description of non-perturbative dualities, RG flows,...

- In this talk we...
  - Develop an algebraic description of fundamental hypermultiplets of 5D $\mathcal{N}=1$ $U(N)$ gauge theories.
  - Extend the algebraic construction to a class of 3D $\mathcal{N}=2$ theories corresponding to certain D3/D5/NS5-brane systems.
  - Relate the two through the Higgsing mechanism and a limiting procedure.

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  - Relate the two through the Higgsing mechanism and a limiting procedure.

To do this, we need the notion of **shifted** quantum groups!

Outline

I. Introduction

II. Shifted quantum groups

III. Higgsing and pit representations

IV. 3d SUSY gauge theories

V. Discussion

II. Shifted Quantum Groups

Main point

- Disclaimer: While this work is about 5d/3d theories realized in type IIB string theory, the illustrations are simpler in terms of 4d/2d theories in type IIA (no bending of branes). The latters are obtained by sending the radius $R \to 0$ of the extra compact dimension.

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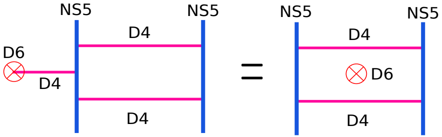
• Hanany-Witten transition for a 4d $\mathcal{N} = 2$ $SU(2)$ theory with one fundamental flavor:



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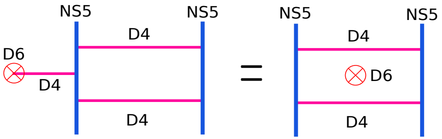

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Need to modify the representations acting on the branes modules!

Shifted Quantum Groups

- In the Drinfeld presentation of **quantum affine/toroidal algebras**, a set of currents $x_\omega^\pm(z)$ (or $e_\omega(z)$, $f_\omega(z)$) and $\psi_\omega^\pm(z)$ is attached to each node ω of the (./affine) Dynkin diagram.

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- We focus here on the quantum affine $\mathfrak{sl}(2)$ and quantum toroidal $\mathfrak{gl}(1)$ algebras.

$\rightsquigarrow$ No index ω. Shift parameters $\boldsymbol{\mu} = (\mu_+, \mu_-) \in \mathbb{Z} \times \mathbb{Z}$.

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$$\iota_P\rho(x^+(z)) = P(z)\rho(x^+(z)), \quad \iota_P\rho(x^-(z)) = \rho(x^-(z)), \quad \iota_P\rho(\psi^\pm(z)) = P(\hat{\gamma}^{\pm 1/2}z)\rho(\psi^\pm(z)),$$

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with $\boldsymbol{\mu}_P = (n-\mu_0, \mu_0)$ for $P(z) = z^{-\mu_0}\prod_{a=1}^n (1 - z/\nu_a)$.

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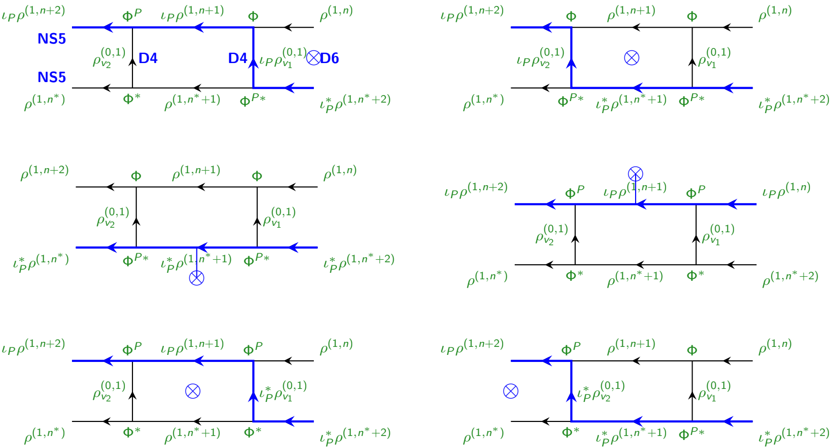
The presence of D6 branes induce a shift of the representations!

zeros of $P(z)$ = positions of D6 = masses hypermultiplets

- Example of (equivalent) realizations of $SU(2)$ **gauge theory with one fundamental flavor**:



$\rightsquigarrow$ The position of the transverse D6 is a conjecture.

- Main lessons:
  - The shifts propagate.

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  - Vertex operators at brane junctions have to be modified.

Representations	Algebras	Intertwiner
$\iota_P\rho_v^{(0,1)} \otimes \rho_u^{(1,n)} \to \iota_P\rho_{u'}^{(1,n+1)}$	$\mathcal{E}^{\mu_P} \otimes \mathcal{E} \to \mathcal{E}^{\mu_P}$	Φ
$\iota_P^*\rho_v^{(0,1)} \otimes \rho_u^{(1,n)} \to \iota_P\rho_{u'}^{(1,n+1)}$	$\mathcal{E}^{\mu_P} \otimes \mathcal{E} \to \mathcal{E}^{\mu_P}$	Φ^P
$\rho_v^{(0,1)} \otimes \iota_P^*\rho_u^{(1,n)} \to \iota_P^*\rho_{u'}^{(1,n+1)}$	$\mathcal{E} \otimes \mathcal{E}^{\mu_P} \to \mathcal{E}^{\mu_P}$	Φ
$\rho_v^{(0,1)} \otimes \iota_P\rho_u^{(1,n)} \to \iota_P\rho_{u'}^{(1,n+1)}$	$\mathcal{E} \otimes \mathcal{E}^{\mu_P} \to \mathcal{E}^{\mu_P}$	Φ^P
$\iota_P^*\rho_{u'}^{(1,n+1)} \to \iota_P^*\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}$	$\mathcal{E}^{\mu_P} \to \mathcal{E}^{\mu_P} \otimes \mathcal{E}$	Φ^*
$\iota_P^*\rho_{u'}^{(1,n+1)} \to \iota_P\rho_v^{(0,1)} \otimes \rho_u^{(1,n)}$	$\mathcal{E}^{\mu_P} \to \mathcal{E}^{\mu_P} \otimes \mathcal{E}$	Φ^{P*}
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- Main lessons:
  - The shifts propagate.
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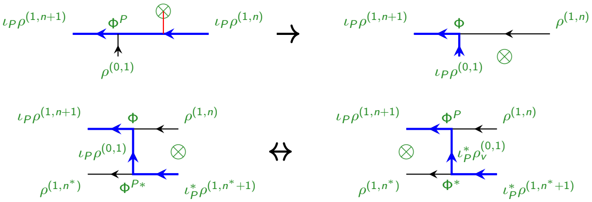
  - Algebraic realization of Hanany-Witten and *Vertical* transitions.



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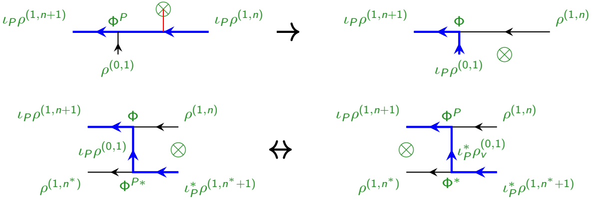


$\Rightarrow$ We can construct in this way the partition function and fundamental qq-character of 5d $\mathcal{N}=1$ $U(N)$ linear quiver gauge theories with a number of (anti)fundamental hypermultiplets.

III. Higgsing and pit representations

Vertical 'pit' representations

- The vertical representation acts on the instanton moduli space.

$\rightsquigarrow$ Instanton configurations are labelled by Young diagrams λ, and the action is

$$x^+(z)\,|\lambda\rangle = \sum_{\square\in A(\lambda)} \delta(z/\chi_\square) C_\lambda^+(\square)\,|\lambda+\square\rangle\,,$$

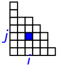
$$x^-(z)\,|\lambda\rangle = \sum_{\square\in R(\lambda)} \delta(z/\chi_\square) C_\lambda^-(\square)\,|\lambda-\square\rangle\,,$$

$$\psi^\pm(z)\,|\lambda\rangle = q_3^{-1/2}\left[\frac{\prod_{\square\in A(\lambda)}(1-q_3\chi_\square/z)\prod_{\square\in R(\lambda)}(1-\chi_\square/(q_3 z))}{\prod_{\square\in A(\lambda)}(1-\chi_\square/z)\prod_{\square\in R(\lambda)}(1-\chi_\square/z)}\right]_\pm |\lambda\rangle$$

where $C_\lambda^\pm(\square)$ are known coefficients (but not enlightening).

$\rightsquigarrow$ $x^+(z)$ add boxes ($A(\lambda)$ is the set of addable boxes), while $x^-(z)$ removes boxes ($R(\lambda)$ is the set of removable boxes) and $\psi^\pm(z)$ are diagonal.

$\rightsquigarrow$ For a box $\square$ of coordinates (i,j), we set $\chi_\square = vq_1^{i-1}q_2^{j-1}$ with the weight $v\in\mathbb{C}^\times$.



- The shifted vertical representation $\iota_P \rho_V$ of $\mathcal{E}^{(d,0)}$ reads

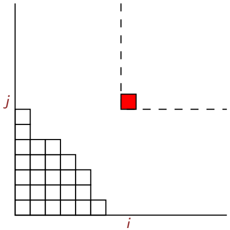
$$x^+(z)\,|\lambda\rangle = P(z) \sum_{\square \in A(\lambda)} \delta(z/\chi_\square) A_\lambda^+(\square)\,|\lambda + \square\rangle\,,$$

$$x^-(z)\,|\lambda\rangle = \sum_{\square \in R(\lambda)} \delta(z/\chi_\square) A_\lambda^-(\square)\,|\lambda - \square\rangle\,,$$

$$\psi^\pm(z)\,|\lambda\rangle = q_3^{-1/2} \left[P(z) \frac{\prod_{\square \in A(\lambda)}(1 - q_3\chi_\square/z) \prod_{\square \in R(\lambda)}(1 - \chi_\square/(q_3 z))}{\prod_{\square \in A(\lambda)}(1 - \chi_\square/z) \prod_{\square \in R(\lambda)}(1 - \chi_\square/z)} \right]_\pm |\lambda\rangle$$

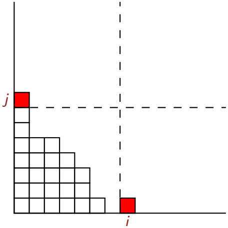
$\Rightarrow$ If $P(z)$ has a zero at $z = v q_1^{i-1} q_2^{j-1}$, the representation is reducible! The subrepresentation acts on a module spanned by Young diagrams **NOT** containing the box (i,j).

$\rightsquigarrow$ Young diagrams are restricted to a fat hook



The subrepresentation is called a **pit representation**!

- With two pits, we can even obtain **finite dimensional representations**:



Here $P(z)$ has two zeros at $z = vq_1^{i-1}$ and $z = vq_2^{j-1}$, Young diagrams are restricted to the rectangle $i \times j$: there are $\binom{i+j}{i}$ states.

$\Rightarrow$ This is a new feature for quantum toroidal algebras, it could be very interesting in the context of quantum integrable systems!

Extra remarks

- Final dimensional representations of qu. tor. $\mathfrak{gl}(1)$ requires $\mu_+ + \mu_- \geq 2$. The most general finite dimensional representations are obtained as $\iota_P \rho_V$ with

$$P(z) = \prod_{\blacksquare \in A(\mu)} (1 - \chi_\blacksquare / z) \quad \Rightarrow \quad \mathcal{M}_\mu = \text{span}\ \{|\lambda\rangle\rangle \diagup \lambda \subseteq \mu\}.$$

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$$P(z) = \prod_{\blacksquare \in A(\mu)} (1 - \chi_\blacksquare / z) \quad \Rightarrow \quad \mathcal{M}_\mu = \text{span}\ \{|\lambda\rangle\rangle / \lambda \subseteq \mu\}.$$

• In the same way, "bulge representations" in which a number of boxes are **always present** are obtained as $\iota_P^* \rho_V$ with

$$P(z) = \prod_{\blacksquare \in R(\mu)} (1 - \chi_\blacksquare / z),$$

The submodule is spanned by states $|\lambda\rangle\rangle$ where λ is restricted to contain μ as a sub-Young diagram and $|\mu\rangle\rangle$ defines a new vacuum ($x^-(z)\,|\mu\rangle\rangle = 0$).

Higgsing

- In the gauge theory, the quantum group parameters q_1, q_2 are identified with the (exponentiated) omega-background parameters $(e^{R\epsilon_1}, e^{R\epsilon_2})$ (R radius of S^1).

$\rightsquigarrow$ The vertical weights $v_\ell = e^{Ra_\ell}$ are (exponentiated) Coulomb branch vevs.

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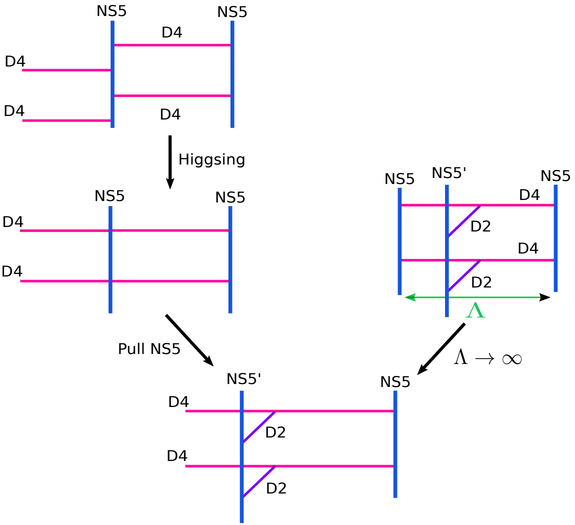
We can propose an algebraic description of the Higgsing phenomenon!

IV. 3d SUSY gauge theories

A cartoon for 2d $\mathcal{N} = (2, 2)$ gauge theories



3d $\mathcal{N} = 2$ gauge theories

- Consider a class of 3d $\mathcal{N} = 2$ gauge theories on $\mathbb{C}_{\epsilon_2} \times S^1$ obtained from 5d $\mathcal{N} = 1$ gauge theories on $\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2} \times S^1$ following a 2-steps procedure:

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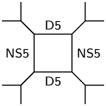
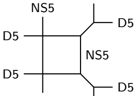
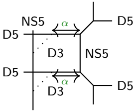
i. Higgsing at $\nu_a=v_aq_2$ which creates a single D3-brane per D5.







	0	1	2	3	4	5	6	7	8	9
Ω-bg		ϵ_2	ϵ_2	ϵ_1	ϵ_1				ϵ_3	ϵ_3
(p,q)	x	x	x	x	x	θ	θ			
NS5	x	x	x	x	x	x				
D5	x	x	x	x	x		x			
D3	x	x	x					x		

$\Rightarrow$ 3d $\mathcal{N}=4$ vector multiplet broken to $\mathcal{N}=2$ vector $\oplus$ chiral by Ω-background $m_{\text{adj}}=-\epsilon_1$.

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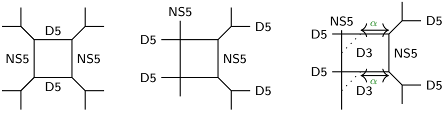

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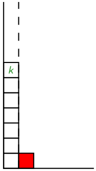
$\Rightarrow$ 3d $\mathcal{N} = 4$ vector multiplet broken to $\mathcal{N} = 2$ vector $\oplus$ chiral by Ω-background $m_{\text{adj}} = -\epsilon_1$.

ii. Send $\epsilon_1 \to \infty$ to decouple the adjoint chiral multiplet.

$\Rightarrow$ 3d $\mathcal{N} = 2$ vector multiplet $\oplus$ N_f fund. chiral $\oplus$ $\bar{N}_f$ antifund. chiral.

- Algebraic description of this procedure:

i. Higgsing with $\nu = vq_2$: pit subrepresentation on the vortex moduli space.



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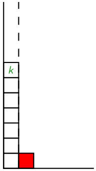

ii. Limit $q_1 \to \infty$ (q_2 fixed): quantum toroidal $\mathfrak{gl}(1)$ reduces to (shifted) quantum affine $\mathfrak{sl}(2)$:

$$G(z) = \frac{(z-q_1)(z-q_2)(z-q_3)}{(z-q_1^{-1})(z-q_2^{-1})(z-q_3^{-1})} \to q^{-2}\frac{z-q^2}{z-q^{-2}}, \quad q_2 \to q^2.$$

⚠ It is a **formal** limit: it holds for the currents, not their modes!

$\rightsquigarrow$ Mathematical part of the second paper tries to make it rigorous for some representations.

Engineering 3d $\mathcal{N} = 2$ gauge theories

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i. Vertical representation (D3): prefundamental representation of **[Hernandez-Jimbo 2012]**

$\rightsquigarrow$ Acts on the vortex moduli space, states $|k\rangle\rangle$ with $k \in \mathbb{Z}^{\geq 0}$

$$x^+(z)\,|k\rangle\rangle = \delta(\nu q^{2k}/z)\,|k+1\rangle\rangle,$$

$$x^-(z)\,|k\rangle\rangle = -\delta(\nu q^{2k-2}/z)(1-q^{2k})\,|k-1\rangle\rangle,$$

$$\psi^\pm(z)\,|k\rangle\rangle = q^{2k}\left[\frac{z(z-\nu q^{-2})}{(z-\nu q^{2k})(z-\nu q^{2k-2})}\right]_\pm |k\rangle\rangle.$$

Engineering 3d $\mathcal{N} = 2$ gauge theories

- We need the following ingredients:

i. Vertical representation (D3): prefundamental representation of **[Hernandez-Jimbo 2012]**

ii. Horizontal representation (NS5): new twisted Fock representation with $\psi^+(z) = 0$!!!

$\rightsquigarrow$ Currents are represented by the vertex operators

$$x^+(z) = e^Q e^{\sum_{k>0} \frac{z^k}{k}(1-q^{2k})J_{-k}} e^{-\sum_{k>0} \frac{z^{-k}}{k} J_k} q^{-2J_0},$$

$$x^-(z) = e^{-Q} e^{-\sum_{k>0} \frac{z^k}{k}(1-q^{-2k})J_{-k}} e^{\sum_{k>0} \frac{z^{-k}}{k} J_k},$$

$$\psi^+(z) = 0, \quad \psi^-(z) = e^{-\sum_{k>0} \frac{z^k}{k}(1+q^{2k})(1-q^{-2k})J_{-k}} q^{-2J_0},$$

$\rightsquigarrow$ $x^-(z)$ coincides with Jing's t-fermion $\Rightarrow$ **Hall-Littlewood polynomials!!!**

Engineering 3d $\mathcal{N} = 2$ gauge theories

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i. Vertical representation (D3): prefundamental representation of **[Hernandez-Jimbo 2012]**

ii. Horizontal representation (NS5): new twisted Fock representation with $\psi^+(z) = 0$!!!

iii. Vertex operators Φ and Φ^* solving the intertwining relations

$$\varrho_{u,n}^{(LT)}(e)\Phi = \Phi\left(\varrho_\nu \otimes \iota_{P_\nu}^* \varrho_{u,n}^{(LT)}\Delta(e)\right), \quad \left(\varrho_\nu \otimes \varrho_{u,n}^{(LT)}\Delta'(e)\right)\Phi^* = \Phi^* \iota_{P_\nu^*} \varrho_{-u\nu,n+1}^{(LT)}(e).$$

$\Rightarrow$ The vortex partition function and fundamental qq-characters of 3d $\mathcal{N} = 2$ gauge theories constructed using the representation theory of the shifted quantum affine $\mathfrak{sl}(2)$ algebra.

V. Discussion

Epilogue

To summarize...

- Fundamental hypermultiplets of 5d $\mathcal{N} = 1$ gauge theories are realized in the algebraic formalism using shifted representations.

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- Fundamental hypermultiplets of 5d $\mathcal{N} = 1$ gauge theories are realized in the algebraic formalism using shifted representations.

- Higgsing is associated to 'pit' representations.

- A class of 3d $\mathcal{N} = 2$ gauge theories can be 'engineered' using representations of the shifted quantum affine $\mathfrak{sl}(2)$ algebra.

- These representations are the limits of the vertical/horizontal representations of the quantum toroidal $\mathfrak{gl}(1)$ algebra as $q_1 \to \infty$ with q_2 fixed.

Director's cuts

Some important results have been omitted in this talk, including:

- Relations between various representations (incl. vector repr.) of the toroidal algebra:

$$
\begin{tikzcd}
\rho_v^{(0,1)}(\mathcal{E}) \curvearrowright \mathcal{F} \arrow[dr, "P(z)\propto 1-vq_1/z"] & & \rho_v^{(0,0)}(\mathcal{E}) \curvearrowright \mathcal{V} \arrow[dl, "P'(z)\propto 1-vq_2^{-1}/z"'] \arrow[dr, "\mathrm{LII}"] & \\
& \begin{matrix} \iota_P \rho_v^{(0,1)}(\mathcal{E}^{(0,1)}) \curvearrowright \mathcal{L} \\ \simeq \iota_{P'}^* \rho_v^{(0,0)}(\mathcal{E}^{(0,1)}) \curvearrowright \mathcal{L} \end{matrix} \arrow[dr, "\mathrm{LII}"] & & \varrho_v^{(0)}(\mathcal{U}^{(-1,-1)}) \curvearrowright \mathcal{V} \arrow[dl, "P''(z)\propto 1-zq^2/v"'] \\
& & \begin{matrix} \varrho_v(\mathcal{U}^{(0,-1)}) \curvearrowright \mathcal{L} \\ \simeq \iota_{P''}^* \varrho_v^{(0)}(\mathcal{U}^{(0,-1)}) \curvearrowright \mathcal{L} \end{matrix} &
\end{tikzcd}
$$

- Limit of the vertical representation without Higgsing
- Definition of twisted Fock representations ($\psi^+(z) = 0$) for quantum toroidal $\mathfrak{gl}(1)$.
- ...

Sequels?

- Application of the 3D algebraic construction?
  - $\rightsquigarrow$ Integrability: **[Gadde, Gukov, Putrov 2014]**? Nekrasov-Shatashvili limit $\epsilon_2 \to 0$?
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Thank you!