The analytic, algebraic and geometric aspects of the Stokes phenomenon

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- Consider the linear system on z-plane

$$\frac{dF}{dz} = \left(\frac{A_k}{z^k} + \frac{A_{k-1}}{z^{k-1}} + \cdots + \frac{A_1}{z} + A_0 + B_1 z + B_2 z^2 + \cdots \right) F,$$

where $F(z) \in \mathrm{GL}_n$, A_i and B_i are constant matrices.

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where $F(z) \in \mathrm{GL}_n$, A_i and B_i are constant matrices.

- The asymptotics behavior of its solution $F(z)$ can differ in different sectors surrounding the essential pole $z = 0$.

Stokes matrices of ODEs with second order poles

- Consider the linear system on z-plane

$$\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z} \right) F,$$

$F(z) \in \mathrm{GL}_n$, $u = \mathrm{diag}(u_1, ..., u_n)$, and $A \in \mathfrak{gl}_n$.

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$F(z) \in \mathrm{GL}_n$, $u = \mathrm{diag}(u_1, ..., u_n)$, and $A \in \mathfrak{gl}_n$.

- Any fundamental solution $F(z) \in \mathrm{GL}_n$ has asymptotics

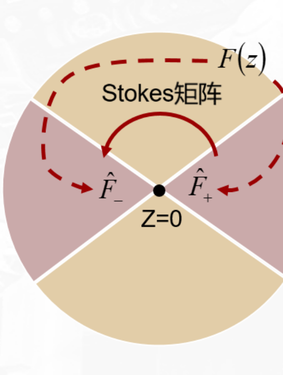
$$e^{\frac{u}{z}} z^{-[A]} \cdot F(z) \sim T_{\pm} \quad as\ z \to 0\ in\ left/right\ planes\ \mathbb{H}_{\pm},$$

for some invertible constant matrices $T_{\pm}$.

- The different asymptotics of $F(z)$ are measured by the ratio

$$S_{+}(A, u) = T_{+} \cdot T_{-}^{-1},$$

called Stokes matrix, similarly define $S_{-}(A, u)$.



Example: 2 by 2

- We consider

$$\frac{dF}{dz} = \left(\frac{1}{z^2}\begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} t_1 & a \\ b & t_2 \end{pmatrix}\right)F.$$

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Then

$$S_+(A,u) = \begin{pmatrix} e^{t_1} & \frac{a((u_2-u_1))^{t_1-t_2}}{\Gamma(1-\lambda_1+t_1)\Gamma(1-\lambda_2+t_1)} \\ 0 & e^{t_2} \end{pmatrix}$$

Here λ_1, λ_2 are eigenvalues of $\begin{pmatrix} t_1 & a \\ b & t_2 \end{pmatrix}$.

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I. the analysis aspect: an asymptotic solution to the Riemann-Hilbert problem/connection problem of the underlying nonlinear isomonodromy equations.

II. the algebraic aspect: a realization of quantum groups and crystals.

III. the geometric aspect: the Stokes matrices in the WKB approximation via the periods on spectral curves.

Part I

The analytic aspect of the Stokes phenomenon: Riemann-Hilbert problem and isomonodromy deformation equations

Asymptotic solution to Riemann-Hilbert problem

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Theorem (Xu, RH problem as $u_1 << u_2 << \cdots << u_n$)

Set $\{\lambda_i^{(k)}\}_{i=1,\dots,k}$ the eigenvalues of the k-th principal submatrix of $A \in \mathfrak{gl}_n$. Then as $u_k/u_{k+1} \to 0$ for all $k = 1, \dots, n-1$ in $u = \mathrm{diag}(u_1, \dots, u_n)$, the sub-diagonals of $S_+(u, A)$ are

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$$S_{k,k+1} = \sum_{i=1}^{k} \frac{\prod_{l=1,l\neq i}^{k} \Gamma(\lambda_l^{(k)} - \lambda_i^{(k)})}{\prod_{l=1}^{k+1} \Gamma(\lambda_l^{(k+1)} - \lambda_i^{(k)})} \frac{\prod_{l=1,l\neq i}^{k} \Gamma(\lambda_l^{(k)} - \lambda_i^{(k)})}{\prod_{l=1}^{k-1} \Gamma(\lambda_l^{(k-1)} - \lambda_i^{(k)})} \cdot m_i^{(k)} + O(\frac{\log(u_2 - u_1)}{u_2 - u_1}),$$

where $k = 1, ..., n-1$ and the other entries are given by explicit algebraic combinations of the sub-diagonal ones.

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- By the isomonodromy property, the leading term gives a model in the study of Stokes phenomenon.
- Solve a connection problem for the isomonodromy equation. Generalizing the result for Painlevé VI by Jimbo.

Part II

The algebraic aspect of the Stokes phenomenon: quantum groups and crystals

Stokes matrices of ODEs in noncommutative rings

- $U(\mathfrak{gl}_n)$: generator $\{E_{ij}\}$, relation $[E_{ij}, E_{kl}] = \delta_{jk}E_{il} - \delta_{li}E_{kj}$.

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For any $u \in \mathfrak{h}_{\rm reg}(\mathbb{R})$ n by n diagonal matrices with distinct real eigenvalues, consider

$$\frac{dF}{dz} = \hbar\left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F,$$

for a $n \times n$ matrix function $F(z)$ with entries in $U(\mathfrak{gl}_n)[[\hbar]]$.

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- Denote the (quantum) Stokes matrices $S_{\hbar\pm}(u) = (s_{ij}^{(\pm)})$, with entries $s_{ij}^{(\pm)}$ in $U(\mathfrak{gl}_n)[[\hbar]]$.

Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{gl}_n)$

A unital associative algebra over $\mathbb{C}[[\hbar]]$ with generators $q^{\pm h_j}, e_i, f_i$, $1 \leq j \leq n, 1 \leq i \leq n-1$ and relations (where we set $q = e^{\hbar/2}$):

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- $$[e_i, f_j] = \delta_{ij} \frac{q^{h_i - h_{i+1}} - q^{-h_i + h_{i+1}}}{q - q^{-1}};$$
- for $|i-j| = 1$,
$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0,$$
$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0,$$
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As $\hbar \to 0$, it becomes the universal eveloping algebra $U(\mathfrak{gl}_n)$.

The algebraic aspect of (quantum) Stokes matrices

Theorem (Xu)

For any $u \in \mathfrak{h}_{\mathrm{reg}}(\mathbb{R})$, the map

$$\nu_\hbar(u) : U_\hbar(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)[[\hbar]] \ ; \quad e_i \mapsto s^{(+)}_{i,i+1}, \quad f_i \mapsto s^{(-)}_{i-1,i}$$

is an algebra isomorphism ($s^{(\pm)}_{ij}$ are the entries of $S_{\hbar\pm}(u)$).

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- In other word, the subdiagonal entries of Stokes matrices of

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• Realizations of quantum symmetric pairs, Yangians.

• Poisson structures on the space of (classical) Stokes matrices found by Boalch.

WKB analysis and crystals

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Conjecture (Xu, proved in a special case)

For any $u \in \mathfrak{h}_{\mathrm{reg}}$, there exists a canonical basis $\{v_I(u)\}$ of $L(\lambda)$, operators $\tilde{e}_k(u)$ and $\tilde{f}_k(u)$ for $i = 1, ..., n-1$ such that as $q = e^{\hbar} \to 0$

$$s_{k,k+1}^{(+)}(u) \cdot v_I(u) = q^c \tilde{e}_k(v_I(u)) + \textit{lower order terms},$$

$$s_{k+1,k}^{(-)}(u) \cdot v_I(u) = q^{c'} \tilde{f}_k(v_I(u)) + \textit{lower order terms}.$$

Furthermore, the datum $(\{v_I(u)\}, \tilde{e}_k(u), \tilde{f}_k(u))$ is a $\mathfrak{gl}_n$-crystal.

- Crystal limit $q \to 0$ in $U_q(\mathfrak{gl}_n)$, Kashiwara and Lusztig.

Part III

The geometric aspect of the Stokes phenomenon: WKB approximation and spectral curves

The geometry in the WKB approximation

• WKB analysis: $\frac{1}{\hbar^2}\frac{d^2\psi}{dz^2} - V(z)\psi = 0$. The $\hbar \to 0$ hehavior is related to the Stokes graphs on z-plane determined by $V(x)$.

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$$\varepsilon \frac{dF}{dz} = \Big(\frac{u}{z^2} + \frac{A}{z}\Big)F.$$

- Problem: between the asymptotics of $S(u, A; \varepsilon)$ as $\varepsilon \to 0$ and the geometry of the spectral curve

$$\det\left[\lambda - \left(\frac{u}{z^2} + \frac{A}{z}\right)\right] = 0.$$

Example (2 by 2 case)

$$S(A, u; \varepsilon) = \begin{pmatrix} e^{\frac{t_1}{\varepsilon}} & \frac{a\left(\frac{i(u_2-u_1)}{\varepsilon}\right)^{\frac{t_1-t_2}{2\pi i\varepsilon}}}{\Gamma(1-\frac{\lambda_1-t_1}{2\pi i\varepsilon})\Gamma(1-\frac{\lambda_2-t_1}{2\pi i})} \\ 0 & e^{\frac{t_2}{\varepsilon}} \end{pmatrix} \sim \begin{pmatrix} e^{\frac{t_1}{\varepsilon}} & e^{\frac{\max(\lambda_1,\lambda_2)}{\varepsilon}} \\ 0 & e^{\frac{t_2}{\varepsilon}} \end{pmatrix}$$

A fake analysis

- solutions of $\varepsilon \frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z}\right) F$ have the WKB type expansion,

$$F(z, \varepsilon) \sim e^{\frac{\omega(z)}{\varepsilon}} (v(z) + \sum \phi_k \varepsilon^k), \quad as \quad \varepsilon \to 0, \tag{1}$$

where $\omega(z) = \mathrm{diag}(\omega_1, ..., \omega_n)$ is a diagonal matrix, and $v(z)$ is a $n \times n$ matrix with columns v_k.

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- Leading asymptotics: $d\omega_k/dz$ and $v_k(z)$ satisfy

$$\frac{d\omega_k}{dz} \cdot v_k(z) = \left(\frac{u}{z^2} + \frac{A}{z} \right) \cdot v_k(z),$$

i.e., $\omega(z) = \text{diag}(\int_{p_1}^{z} \lambda dt, ..., \int_{p_n}^{z} \lambda dt)$.

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i.e., $\omega(z) = \mathrm{diag}(\int_{p_1}^{z} \lambda dt, ..., \int_{p_n}^{z} \lambda dt)$.

- Stokes phenomenon takes place in the asymptotics $\varepsilon \to 0$ in a way that the approximation in (1) is not uniformly valid w.r.t z around 0. Then the asymptotics of Stokes matrices as $\varepsilon \to 0$ should be encoded by certain periods on the spectral curve.

Main conjecture

- A coordinate chart on the space of upper triangular matrices from cluster algebra theory: $\{\Delta_i^{(j)}\}_{1 \leq i \leq j \leq n}$ the minor formed by intersecting columns $i - j + 1$ to i and the first j rows.

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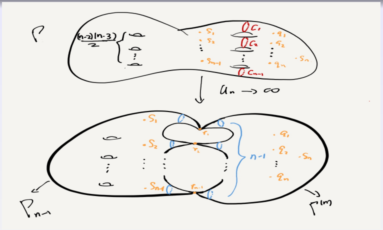
Conjecture (Alekseev-X-Zhou)

For generic u and A, there exists a canonical set of cycles $\{C_i^{(k)}\}_{1\leq i\leq k\leq n}$ on $\Gamma(u,A)$ such that

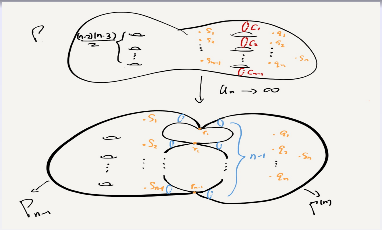
$$\lim_{\varepsilon\to 0}\left(\varepsilon\log\left|\Delta_i^{(k)}(S(A,u,\varepsilon))\right|\right)=\int_{C_i^{(k)}(u,A)}\omega.$$

- analytic difficult (left); • the discrete choice of cycles (right).

Proof of the conjecture near a limiting point



- Fixing $u_1, ..., u_{n-1}$, let $u_n \to \infty$,



- Fixing $u_1, ..., u_{n-1}$, let $u_n \to \infty$,
- As $u_1 << \cdots << u_n$, denote by $\{V_i^{(k)}\}_{1 \le i \le k \le n}$ the distinguish vanishing cycles.

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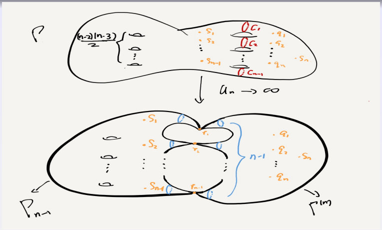

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Theorem (Alekseev-X-Zhou)

As u_k/u_{k+1} sufficiently small, the cycles $C_i^{(k)} = \sum_{j=1}^{i} V_{k-j}^{(k)}$ on $\Gamma(u, A)$ such that

$$\lim_{\varepsilon \to 0} \left(\varepsilon \lim_{u_1 \ll \cdots \ll u_n} (\log \left| \Delta_i^{(k)}(S(A, u; \varepsilon)) \right|) \right) = \lim_{u_1 \ll \cdots \ll u_n} \int_{C_i^{(k)}} \omega$$

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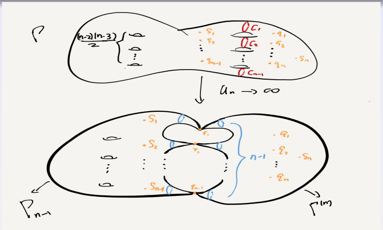

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Thank you very much!