Off-diagonal Bethe ansatz approach to quantum integrable models

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- Introduction
- The Heisenberg Spin Chains
  - With $U(1)$-symmetry.
  - Without $U(1)$-symmetry.
- Thermodynamic limit of the Heisenberg spin chain
  - With periodic boundary condition.
  - With non-diagonal boundary terms.
- Conclusion and Comments

Quantum integrable systems have many applications in

- String/ Gauge theories: AdS/CFT, Super-symmetric Yang-Mills theories...
- Statistical mechanics: The Ising model, the six-vertex models...
- Condensed Matter Physics: The super-symmetric $t - J$ Model, the Hubbard model...
- Mathematics: Quantum group, Representation theory, Algebraic Topology, ...

There are many methods to solve quantum integrable systems (The case of $T = 0$):

- The Coordinate Bethe Ansatz method (H. Bethe 1931)
- The Baxter's $T - Q$ relation method (R. Baxter 1970s)
- The Quantum Inverse Scattering (or Algebraic Bethe Ansatz) method (L. Faddeev's School 1979s) and its generalizations
- The off-diagonal Bethe Ansaz method (2013s)

The Hamiltonian of the closed Heisenberg chain is

$$H = \sum_{k=1}^{N} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right),$$

where

$$\sigma_{N+1}^\alpha = \sigma_1^\alpha, \quad \alpha = x, y, z.$$

The system is **integrable**, i.e., there exist enough conserved charges

$$i\hbar \frac{\partial}{\partial t} h_i = [H, h_i] = 0, \qquad i = 1, \ldots.$$

and

$$[h_i, h_j] = 0.$$

It is convenient to introduce a generation function of these charges, the so-called transfer matrix

$$t(u) = \sum_{i=0} h_i u^i.$$

Then

$$[t(u),\ t(v)] = 0, \qquad H \propto \frac{\partial}{\partial u} \ln t(u)|_{u=0} + const,$$

or

$$H \propto h_0^{-1} h_1 + const,$$
$$h_0\, \sigma_i^{\alpha}\, h_0^{-1} = \sigma_{i+1}^{\alpha}.$$

The eigenstates and the corresponding eigenvalues can be obtained by Quantum Inverse Scattering Method (QISM). In the framework of QISM, the monodromy matrix $T(u)$

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix},$$

has played a central role. It is built from the six-vertex R-matrix of

$$T_0(u) = R_{0N}(u - \theta_N) \ldots R_{01}(u - \theta_1),$$

where the well-known six-vertex R-matrix is given by

$$R(u) = \begin{pmatrix} u+\eta & & & \\ & u & \eta & \\ & \eta & u & \\ & & & u+\eta \end{pmatrix}.$$

The transfer matrix is $t(u) = trT(u) = A(u) + D(u)$.

The R-matrix satisfies the Yang-Baxter equation (YBE)

$$R_{12}(u-v)\,R_{13}(u)\,R_{23}(v)=R_{23}(v)\,R_{13}(u)\,R_{12}(u-v). \tag{1}$$

The above fundamental relation leads to the following so-called RLL relation between the monodromy matrix

$$R_{0\,0'}(u-v)\;T_0(u)\;T_{0'}(v)=\;T_{0'}(v)\;T_0(u)\,R_{0\,0'}(u-v). \tag{2}$$

This leads to

$$[t(u),t(v)]=0, \tag{3}$$

which ensures the integrability of the Heisenberg chain with periodic boundary condition.

The eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ can be parameterized by some parameters $\{\lambda_1, \cdots, \lambda_M | M = 0, \cdots, N\}$ as follows (H. Bethe, Z. Phys. 71, 205 (1931)):

$$\Lambda(u) = a(u)\frac{Q(u-\eta)}{Q(u)} + d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{4}$$

where

$$a(u) = \prod_{l=1}^{N}(u - \theta_j + \eta) = d(u+\eta), \quad Q(u) = \prod_{j=1}^{M}(u - \lambda_j),$$

the parameters $\{\lambda_i\}$ should satisfy Bethe ansatz equations,

$$\prod_{k \neq j}^{M} \frac{\lambda_j - \lambda_k + \eta}{\lambda_j - \lambda_k - \eta} = \prod_{l=1}^{N} \frac{\lambda_j - \theta_l + \eta}{\lambda_j - \theta_l}, \qquad j = 1, \ldots, M.$$

It can be proven that the transfer matrix $t(u)$ satisfies the relations:

$$t(\theta_j)\, t(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \cdots, N, \tag{5}$$

$$t(u) = 2u^N \times \mathrm{id} + \cdots, \quad u \to \infty. \tag{6}$$

This leads to that the eigenvalues $\Lambda(u)$ satisfy the corresponding relations:

$$\Lambda(\theta_j)\, \Lambda(\theta_j - \eta) = a(\theta_j) d(\theta_j - \eta), \quad j = 1, \cdots, N, \tag{7}$$

$$\Lambda(u) = 2u^N + \cdots, \quad u \to \infty. \tag{8}$$

The Hamiltonian of the Heisenberg chain with parallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{\eta}{p}\sigma_1^z + \frac{\eta}{q}\sigma_N^z + N.$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = tr(K^+(u)\,\mathcal{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \\ & q-u-\eta \end{pmatrix}.$$

The K-matrix $K^-(u)$ satisfies the reflection equation (RE)

$$\begin{aligned} &R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)K_2^-(u_2) \\ &\quad = K_2^-(u_2)R_{12}(u_1+u_2)K_1^-(u_1)R_{21}(u_1-u_2), \end{aligned}$$

while the dual K-matrix $K^+(u)$ satisfies the following dual RE

$$\begin{aligned} &R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-2)K_2^+(u_2) \\ &\quad = K_2^+(u_2)R_{12}(-u_1-u_2-2)K_1^+(u_1)R_{21}(u_2-u_1). \end{aligned}$$

The eigenvalues of the associated transfer matrix is also given in terms of a $T-Q$ relation (E.K. Sklyanin, J. Phys. A 21, 2375 (1988))

$$\Lambda(u)=a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)},$$

where

$$a(u)=\frac{2(u+\eta)}{2u+\eta}(u+p)(u+q)\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta),$$

$$d(u)=a(-u-\eta),\quad Q(u)=\prod_{j=1}^{M}(u-\lambda_j)(u+\lambda_j+\eta).$$

The Hamiltonian of the Heisenberg chain with unparallel boundary fields is

$$H = \sum_{k=1}^{N-1} \left(\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \sigma_k^z \sigma_{k+1}^z \right) + \frac{\eta}{p} \sigma_1^z + \frac{\eta}{q} (\sigma_N^z + \xi \sigma_N^x) + N. \qquad (9)$$

The system is **integrable**, i.e., the corresponding transfer matrix $t(u)$ can be constructed by the R-matrix and the associated K-matrices

$$t(u) = tr(K^+(u)\mathcal{T}(u)) = tr\left(K^+(u)T(u)K^-(u)T^{-1}(-u)\right),$$

where the K-matrices $K^{\pm}(u)$ are the diagonal K-matrices

$$K^-(u) = \begin{pmatrix} p+u & \\ & p-u \end{pmatrix}, \quad K^+(u) = \begin{pmatrix} q+u+\eta & \xi(u+\eta) \\ \xi(u+\eta) & q-u-\eta \end{pmatrix}.$$

$$H = \eta \frac{\partial}{\partial u} \ln t(u)|_{u=0,\{\theta_j\}=0}.$$

Without losing generality, we set $\eta = i$.

The eigenvalue $\Lambda(u)$ satisfies the properties (J. Cao et al, Nucl. Phys. B 875 (2013), 152-165):

$$\Lambda(-u-\eta) = \Lambda(u), \tag{10}$$

$$\Lambda(0) = 2pq\prod_{l=1}^{N}(1-\theta_l)(1+\theta_l), \tag{11}$$

$$\lim_{u\to\infty} \Lambda(u) = 2u^{2N+2} + \cdots, \tag{12}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\Delta_q(\theta_j)}{(2\theta_j+\eta)(2\theta_j-\eta)}, \quad j=1,\cdots,N, \tag{13}$$

where the quantum determinant $\Delta_q(u)$ is given by

$$\Delta_q(u) = 4(u^2-1)(p^2-u^2)((1+\xi^2)u^2-q^2)\prod_{l=1}^{N}((u-1)^2-\theta_j^2)((u+1)^2-\theta_j^2).$$

The eigenvalue $\Lambda(u)$ of the corresponding transfer matrix is given in terms of an inhomogeneous $T-Q$ relation

$$\begin{aligned}\Lambda(u) &= a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)} \\ &\quad +2\left[1-(1+\xi^2)^{\frac{1}{2}}\right]u(u+\eta)\frac{a(u)d(u)}{Q(u)}, \end{aligned} \tag{14}$$

where

$$\begin{aligned} a(u) &= \frac{2(u+\eta)}{2u+\eta}(u+p)[(1+\xi^2)^{\frac{1}{2}}u+q]\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta), \\ d(u) &= a(-u-\eta), \quad Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+\eta). \end{aligned}$$

The roots of $Q(u)$ satisfy the BAEs

$$\frac{a(\lambda_j)}{d(\lambda_j)}+\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}=-2[1-(1+\xi^2)^{\frac{1}{2}}]\lambda_j(\lambda_j+\eta)\frac{a(\lambda_j)}{Q(\lambda_j-\eta)}, \quad j=1,\cdots,N. \tag{15}$$

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Let us introduce a gauge matrix U

$$U = \begin{pmatrix} \xi & \sqrt{1+\xi^2}-1 \\ \xi & -\sqrt{1+\xi^2}-1 \end{pmatrix},$$

which diagonalizes the K-matrix $K^+(u)$, namely,

$$\begin{aligned} K^+(u) &= \begin{pmatrix} q+u+1 & \xi(u+1) \\ \xi(u+1) & q-u-1 \end{pmatrix} \\ &= U^{-1} \begin{pmatrix} q+\sqrt{1+\xi^2}(u+1) & 0 \\ 0 & q-\sqrt{1+\xi^2}(u+1) \end{pmatrix} U. \end{aligned}$$

Let us introduce a gauged double-row monodromy matrix $\bar{\mathcal{T}}(u)$

$$\bar{\mathcal{T}}(u) = U\,\mathcal{T}(u)\,U^{-1} = U \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix} U^{-1} = \begin{pmatrix} \bar{\mathcal{A}}(u) & \bar{\mathcal{B}}(u) \\ \bar{\mathcal{C}}(u) & \bar{\mathcal{D}}(u) \end{pmatrix}$$

For an example,

$$\bar{\mathcal{B}}(u) = -\frac{1}{2\xi\sqrt{1+\xi^2}}\Big\{-\xi(\sqrt{1+\xi^2}-1)\mathcal{A}(u) - (\sqrt{1+\xi^2}-1)^2\mathcal{C}(u) + \xi^2\mathcal{B}(u) + \xi(\sqrt{1+\xi^2}-1)\mathcal{D}(u)\Big\}.$$

The associated Bethe state is given by

$$|\lambda_1,\ldots,\lambda_N\rangle = \bar{\mathcal{B}}(\lambda_1)\ldots\bar{\mathcal{B}}(\lambda_N)\,|\Omega\rangle.$$

The eigenvalue can be given in terms of a homogeneous $T-Q$ relation

$$\Lambda(u)=a(u)\frac{Q(u-\eta)}{Q(u)}+d(u)\frac{Q(u+\eta)}{Q(u)}, \tag{16}$$

where the roots of $Q(u)$ satisfy the Bethe ansatz equations (BAEs)

$$\frac{a(\lambda_j)}{d(\lambda_j)}=-\frac{Q(\lambda_j+\eta)}{Q(\lambda_j-\eta)}, \quad j=1,\cdots,M. \tag{17}$$

BAEs $\Rightarrow$ TBA

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The eigenvalue $\Lambda(u)$ of the corresponding transfer matrix is given in terms of an inhomogeneous $T-Q$ relation

$$\begin{aligned}\Lambda(u) &= \frac{2(u+1)^{2N+1}}{2u+1}(u+p)\left[(1+\xi^2)^{\frac{1}{2}}u+q\right]\frac{Q(u-1)}{Q(u)} \\ &+\frac{2u^{2N+1}}{2u+1}(u-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(u+1)-q\right]\frac{Q(u+1)}{Q(u)} \\ &+2\left[1-(1+\xi^2)^{\frac{1}{2}}\right]\frac{[u(u+1)]^{2N+1}}{Q(u)}, \end{aligned} \tag{18}$$

where the function $Q(u)$ can be parameterized as $Q(u)=\prod_{j=1}^{N}(u-\lambda_j)(u+\lambda_j+1)$.

- Bethe ansatz equations

$$\left(\frac{\lambda_j+1}{\lambda_j}\right)^{2N+1}\frac{(\lambda_j+p)\left[(1+\xi^2)^{\frac{1}{2}}\lambda_j+q\right]}{(\lambda_j-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(\lambda_j+1)-q\right]}=$$

$$-\frac{\left[1-(1+\xi^2)^{\frac{1}{2}}\right](2\lambda_j+1)(\lambda_j+1)^{2N+1}}{(\lambda_j-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(\lambda_j+1)-q\right]\prod_{l=1}^{N}(\lambda_j-\lambda_l-1)(\lambda_j+\lambda_l)}$$

$$-\prod_{l=1}^{N}\frac{(\lambda_j-\lambda_l+1)(\lambda_j+\lambda_l+2)}{(\lambda_j-\lambda_l-1)(\lambda_j+\lambda_l)},\quad j=1,\cdots,N. \tag{19}$$

- The eigenvalue of the Hamiltonian

$$E=\sum_{j=1}^{N}\frac{2}{\lambda_j(\lambda_j+1)}+N-1+\frac{1}{p}+\frac{(1+\xi^2)^{\frac{1}{2}}}{q}. \tag{20}$$

- Contribution of the inhomogeneous term

We define the contribution of the inhomogeneous term to the ground state energy as

$$E_{inh} = E_{hom} - E_{true}. \tag{21}$$

Here E_{hom} is the ground state energy of the Heisenberg chain calculated by the homogeneous $T - Q$ relation

$$\begin{aligned} \Lambda_{hom}(u) \quad = \quad & \frac{2(u+1)^{2N+1}}{2u+1}(u+p)\left[(1+\xi^2)^{\frac{1}{2}}u+q\right]\frac{Q(u-1)}{Q(u)} \\ & +\frac{2u^{2N+1}}{2u+1}(u-p+1)\left[(1+\xi^2)^{\frac{1}{2}}(u+1)-q\right]\frac{Q(u+1)}{Q(u)}. \end{aligned} \tag{22}$$

The singular property of the $T-Q$ relation (22) gives the following BAEs

$$\left(\frac{\mu_j-\frac{i}{2}}{\mu_j+\frac{i}{2}}\right)^{2N}\frac{(\mu_j-i\bar{p})\,(\mu_j-i\bar{q})}{(\mu_j+i\bar{p})\,(\mu_j+i\bar{q})}=\prod_{l\neq j}^{M}\frac{(\mu_j-\mu_l-i)(\mu_j+\mu_l-i)}{(\mu_j-\mu_l+i)(\mu_j+\mu_l+i)}, \tag{23}$$

where we have put $\lambda=i\mu-\frac{1}{2}$, $\bar{p}=p-\frac{1}{2}$ and $\bar{q}=q(1+\xi^2)^{-\frac{1}{2}}-\frac{1}{2}$. Note, E_{hom} is given by equation (20) with the constraint (23).

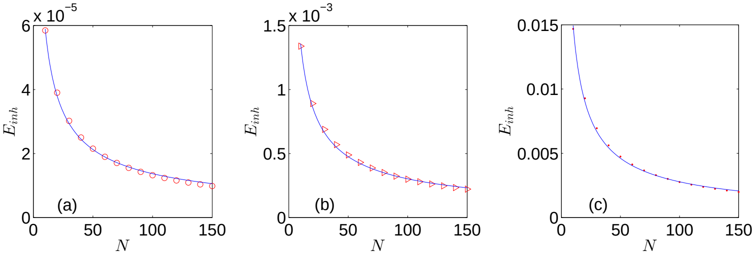

Figure 1: The contribution of the inhomogeneous term to the ground state energy E_{inh} versus the system size N. The data can be fitted as $E_{inh} = \gamma_1 N^{\beta_1}$. **Due to the fact $\beta_1 < 0$, when the N tends to infinity, the contribution of the inhomogeneous term tends to zero.** Here $p = 8$, $q = 4$, (a) $\xi = \frac{1}{8}$, $\gamma_1 = 0.000253$ and $\beta_1 = -0.6334$; (b) $\xi = \frac{5}{8}$, $\gamma_1 = 0.006096$ and $\beta_1 = -0.6521$; (c) $\xi = \frac{25}{8}$, $\gamma_1 = 0.080180$ and $\beta_1 = -0.7297$.

- Boundary energy

$$
\begin{aligned}
E_b(p,q,\xi) &= \lim_{N\to\infty}\left[E_0(N;p,q,\xi) - 2E_0^{periodic}(N)\right] \\
&= -2\int_0^{\infty}\frac{e^{-p\omega}}{1+e^{-\omega}}d\omega - 2\int_0^{\infty}\frac{e^{-\frac{q}{\sqrt{1+\xi^2}}\omega}}{1+e^{-\omega}}d\omega \\
&\quad +\pi - 2\ln 2 - 1 + \frac{1}{p} + \frac{(1+\xi^2)^{\frac{1}{2}}}{q}.
\end{aligned} \tag{24}
$$

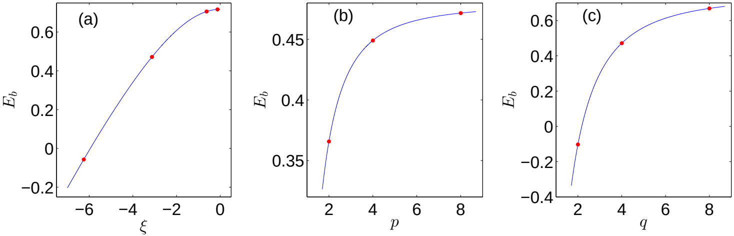

Figure 2: The boundary energies versus the boundary parameters. The blue curves are the ones calculated from equation (24), while the red points are the ones obtained from the Hamiltonian (9) with the BST algorithms. Here (a) $p = 8$ and $q = 4$; (b) $q = 4$ and $\xi = -\frac{25}{8}$; (c) $p = 8$ and $\xi = -\frac{25}{8}$.

When ξ is small, we can expand the boundary energy (24) with respect to ξ as

$$
\begin{aligned}
E_b(p,q,\xi) \quad \simeq \quad & \frac{1}{p} + \psi^{(0)}\left(\frac{p}{2}\right) - \psi^{(0)}\left(\frac{p+1}{2}\right) + \frac{1}{q} + \psi^{(0)}\left(\frac{q}{2}\right) - \psi^{(0)}\left(\frac{q+1}{2}\right) \\
& + \pi - 1 - 2\ln(2) + \xi^2\left[\frac{1}{2q} - \frac{1}{4}q\psi^{(1)}\left(\frac{q}{2}\right) + \frac{1}{4}q\psi^{(1)}\left(\frac{q+1}{2}\right)\right] \\
& + \xi^4\,\frac{\left[q^3\psi^{(2)}\left(\frac{q}{2}\right) - q^3\psi^{(2)}\left(\frac{q+1}{2}\right) + 6q^2\psi^{(1)}\left(\frac{q}{2}\right) - 6q^2\psi^{(1)}\left(\frac{q+1}{2}\right) - 4\right]}{32q} \\
& + O\left(\xi^6\right), \qquad (25)
\end{aligned}
$$

where $\psi^{(m)}(x)$ is the m-order derivative of digamma function. Up to the order ξ^2, our result coincides with that of R. Nepomechie, J. Phys. A 46 (2013), 442002.

The eigenvalue $\Lambda(u)$ satisfies the properties (J. Cao et al, Nucl. Phys. B 875 (2013), 152-165):

$$\Lambda(-u-\eta) = \Lambda(u), \tag{26}$$

$$\Lambda(0) = 2pq\prod_{l=1}^{N}(\eta-\theta_l)(\eta+\theta_l), \tag{27}$$

$$\lim_{u\to\infty} \Lambda(u) = 2u^{2N+2} + \cdots, \tag{28}$$

$$\Lambda(\theta_j)\Lambda(\theta_j-\eta) = -\frac{\Delta_q(\theta_j)}{(2\theta_j+\eta)(2\theta_j-\eta)}, \quad j = 1, \cdots, N. \tag{29}$$

In order to make the Hamiltonian (9) hermitian the boundary parameters have to be taken as follows:

$$p* = -p, \quad q^* = -q, \quad \xi^* = \xi,$$

which leads to

$$(t(u))^\dagger = t(-u^*), \quad \Lambda^*(u) = \Lambda(-u^*).$$

This fact allows us to give the decomposition of $\Lambda(u)$ for an eigenvalue of the transfer matrix

$$\begin{aligned}\Lambda(u) \quad = \quad & 2\prod_{j=1}^{M_1}(u-\mu_j+\frac{\eta}{2})(u+\mu_j+\frac{\eta}{2}) \\ & \times\prod_{j=1}^{M_2}(u-z_j+\frac{\eta}{2})(u+z_j+\frac{\eta}{2})(u-z_j^*+\frac{\eta}{2})(u+z_j^*+\frac{\eta}{2}) \\ & \times\prod_{j=1}^{M_b}(u-\eta\nu_j+\frac{\eta}{2})(u+\eta\nu_j+\frac{\eta}{2}),\end{aligned} \tag{30}$$

where μ_j and ν_j are real numbers, $z_j^* \neq (\pm)z_k$, and $M_b + M_1 + 2M_2 = N + 1$.

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For the ground state for a large even N, the corresponding $\Lambda(u)$ takes the decomposition

$$\begin{aligned}\Lambda(u) &= 2\,(u-\eta\nu+\frac{\eta}{2})(u+\eta\nu+\frac{\eta}{2}) \\ &\times\prod_{j=1}^{\frac{N}{2}}(u-x_j-\frac{\eta}{2})(u+x_j+\frac{3\eta}{2})(u-x_j+\frac{3\eta}{2})(u+x_j-\frac{\eta}{2}).\end{aligned}$$

This implies that $z_j \simeq x_j \pm \eta$ for a large even N where x_j is a real number which corresponds to the position of the 2-string, and that ν is a real number which is the position of a boundary bound state.

The corresponding ground state energy $E_0(N; P, Q, \xi)$ is expressed in terms of roots as

$$\begin{aligned}\lim_{N\to\infty} E_0(N; P, Q, \xi) &= \frac{1}{\frac{1}{4} - v^2} + \lim_{N\to\infty}\left\{\sum_{j=1}^{\frac{N}{2}}\left(\frac{3}{x_j^2 + \frac{9}{4}} - \frac{1}{x_j^2 + \frac{1}{4}}\right)\right\} \\ &= \frac{1}{\frac{1}{4} - v^2} + \int_{-\infty}^{+\infty}\left(\frac{3}{x^2 + \frac{9}{4}} - \frac{1}{x^2 + \frac{1}{4}}\right)\rho(x)\, dx,\end{aligned}$$

where $\rho(x)$ is the density of the distribution of roots.
Numerical study shows that inhomogeneous real parameters $\{\theta_j\}$ almost does not affect the imaginary parts of the roots $\{z_j\}$. Namely, $z_j \simeq x_j \pm \eta$. This fact allows us to derive a linear integral equation of the density of the roots with an auxiliary density $\sigma(\theta)$ of the inhomogeneities. Taking $\sigma(\theta) = \delta(0)$ or the homogeneous limit and the value of $\Lambda(0)$ allow us to determine $\rho(x)$ and the real value ν. Finally, we obtain the same expression (24) of the boundary energy $E_b(p, q, \xi)$.

So far, many typical $U(1)$-symmetry-broken models have been solved by the method:

- The spin-$\frac{1}{2}$ Heisenberg chain with arbitrary boundary fields.
- The open spin chains with general boundary condition associated with the $A_n^{(1)}$ algebra.
- The t-J model with unparallel boundary fields.
- The Hubbard model with unparallel boundary fields.
- The open spin chains associated with the $B_n^{(1)}$, $C_n^{(1)}$ and $D_n^{(1)}$ algebras.
- The open spin chains associated with the $A_n^{(2)}$ and $D_n^{(2)}$ twisted algebras.

Thank for your attentions