Twisted Elliptic Genera

Kaiwen Sun (KIAS)

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An overview

For the last 10 years, there has been huge progress on the following relation web in both classification and computation.

6d (1,0) SCFTs classified in (Heckman-Morrison-Rudelius-Vafa 15)

1. are engineered by F-theory compactified on local elliptic CY3
2. contain BPS strings with worldsheet theory as 2d (0,4) SCFTs
3. compactified on S^1 give 5d Kaluza-Klein (KK) theories

$$\mathbb{E}^{\mathrm{2d\ (0,4)\ SCFT}} = Z^{\mathrm{6d\ (1,0)\ SCFT}}_{\mathbb{R}^4\times T^2} = Z^{\mathrm{5d\ KK}}_{\mathbb{R}^4\times S^1} = Z^{\mathrm{ref.\ top.}}_{\mathrm{local\ elliptic\ CY3}}$$

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A simple example

1. 6d (1,0) E-string theory
2. 5d $SU(2)+8\mathbf{F}$ KK theory
3. local half-K3 Calabi-Yau threefold
4. a series of 2d (0,4) $O(k)$ gauge theories

An overview

When 6d (1,0) SCFT has a discrete global symmetry, we can do twisted circle compactification to a 5d KK theory (Bhardwaj-Jefferson-Kim-Tarazi-Vafa 19). There are two kinds

1. gauge algebra allows outer automorphism: folding vector multiplet
2. quiver structure has discrete symmetry: folding tensor multiplet

We focus on the first kind of twisted compactification. In such cases, the relations are generalized as

$$\mathbb{E}^{\text{2d (0,4) SCFT}}_{\text{twisted}} = Z^{\text{6d (1,0) SCFT}}_{\mathbb{R}^4\times S^1\times S^1,\text{twisted}} = Z^{\text{5d KK}}_{\mathbb{R}^4\times S^1} = Z^{\text{ref. top.}}_{\text{local genus-one fibered CY3}}$$

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Two simple examples

1. $\mathbb{Z}_2$ twist of 6d (1,0) $SU(3)$ SCFT $\to$ 5d $\mathcal{N}=1$ $SU(3)_9$
2. $\mathbb{Z}_3$ twist of 6d (1,0) $SO(8)$ SCFT $\to$ 5d $\mathcal{N}=1$ $SU(4)_8$

Outline

- 6d (1,0) SCFTs and twisted compactification
- Twisted elliptic genera
- Twisted elliptic blowup equations
- Modular bootstrape on $\Gamma(N)$
- Spectral flow symmetry
- Summary

6d SCFTs and elliptic non-compact Calabi-Yau

Six is the highest dimension for SCFT

6d $(2,0)$

- ADE classification

6d $(1,0)$

- Tons
- F-theory compactified on elliptic non-compact CY3
- Elliptic fibration over non-compact base surface B in which all curve classes can be simultaneously shrinkable to zero volume
- Atomic classification (Heckman-Morrison-Rudelius-Vafa 15)
- Generalized quiver from non-Higgsable clusters
- Tensor branch dimension called "rank" i.e. $H^{1,1}(B,\mathbb{Z})$

Rank One 6d $(1,0)$ SCFTs

Rank *one* 6d $(1,0)$ SCFTs are the natural elliptic lift of 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$ gauge theories.

- Associated to Calabi-Yau as elliptic fibration over $\mathcal{O}(-n) \to \mathbb{P}^1$
- Kodaira singularity type gives the gauge algebra G supported on the $-n$ curve
- Global flavor symmetry F on a non-compact curve
- matters in representation $\mathfrak{R}$ at intersection points between curves
- $(n, G, F, \mathfrak{R})$ are highly constrained by *Calabi-Yau condition*.

Full List of Rank One 6d $(1,0)$ SCFTs

e.g. $n=4$, $SO(r+8)+r\mathbf{F}$ theories

n	G	F	$2(R_G, R_F)$
12	E_8	$-$	$-$
8	E_7	$-$	$-$
7	E_7	$-$	$(\mathbf{56},\mathbf{1})$
6	E_6	$-$	$-$
6	E_7	$\mathfrak{so}(2)_{12}$	$(\mathbf{56},\mathbf{2})$
5	F_4	$-$	$-$
5	E_6	$\mathfrak{u}(1)_6$	$\mathbf{27}_{-1}\oplus c.c.$
5	E_7	$\mathfrak{so}(3)_{12}$	$(\mathbf{56},\mathbf{3})$
4	$\mathfrak{so}(8)$	$-$	$-$
4	$\mathfrak{so}(N\geq 9)$	$\mathfrak{sp}(N-8)_1$	$(\mathbf{N},\mathbf{2(N-8)})$
4	F_4	$\mathfrak{sp}(1)_3$	$(\mathbf{26},\mathbf{2})$
4	E_6	$\mathfrak{su}(2)_6\times\mathfrak{u}(1)_{12}$	$(\mathbf{27},\overline{\mathbf{2}})_{-1}\oplus c.c.$
4	E_7	$\mathfrak{so}(4)_{12}$	$(\mathbf{56},\mathbf{2}\oplus\mathbf{2})$
3	$\mathfrak{su}(3)$	$-$	$-$
3	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1$	$(\mathbf{8},\mathbf{4})$
3	$\mathfrak{so}(8)$	$\mathfrak{sp}(1)_1^a\times\mathfrak{sp}(1)_1^b\times\mathfrak{sp}(1)_1^c$	$(\mathbf{8}_v\oplus\mathbf{8}_c\oplus\mathbf{8}_s,\mathbf{2})$
3	$\mathfrak{so}(9)$	$\mathfrak{sp}(2)_1^a\times\mathfrak{sp}(1)_2^b$	$(\mathbf{9},\mathbf{4}^a)\oplus(\mathbf{16},\mathbf{2}^b)$
3	$\mathfrak{so}(10)$	$\mathfrak{sp}(3)_1^a\times(\mathfrak{su}(1)_4\times\mathfrak{u}(1)_4)^b$	$(\mathbf{10},\mathbf{6}^a)\oplus[(\mathbf{16}_s)_1^b\oplus c.c.]$
3	$\mathfrak{so}(11)$	$\mathfrak{sp}(4)_1^a\times\text{Ising}^b$	$(\mathbf{11},\mathbf{8}^a)\oplus(\mathbf{32},\mathbf{1}_s^b)$
3	$\mathfrak{so}(12)$	$\mathfrak{sp}(5)_1$	$(\mathbf{12},\mathbf{10})\oplus(\mathbf{32}_s,\mathbf{1})$
3	G_2	$\mathfrak{sp}(1)_1$	$(\mathbf{7},\mathbf{2})$
3	F_4	$\mathfrak{sp}(2)_3$	$(\mathbf{26},\mathbf{4})$
3	E_6	$\mathfrak{su}(3)_6\times\mathfrak{u}(1)_{18}$	$(\mathbf{27},\overline{\mathbf{3}})_{-1}\oplus c.c.$
3	E_7	$\mathfrak{so}(5)_{12}$	$(\mathbf{56},\mathbf{5})$

e.g. $SU(r) + 2r\mathbf{F}$ theories

n	G	F	$2(R_G, R_F)$
2	$\mathfrak{su}(1)$	$\mathfrak{su}(2)_1$	–
2	$\mathfrak{su}(2)$	$\mathfrak{so}(7)_1 \times \text{Ising}$	$(\mathbf{2}, \mathbf{8}_s \times \mathbf{1}_s)$
2	$\mathfrak{su}(N \geq 3)$	$\mathfrak{su}(2N)_1$	$(\mathbf{N}, \overline{\mathbf{2N}}) \oplus c.c.$
2	$\mathfrak{so}(7)$	$\mathfrak{sp}(1)_1^a \times \mathfrak{sp}(4)_1^b$	$(\mathbf{7}, \mathbf{2}^a) \oplus (\mathbf{8}, \mathbf{8}^b)$
2	$\mathfrak{so}(8)$	$\mathfrak{sp}(2)_1^a \times \mathfrak{sp}(2)_1^b \times \mathfrak{sp}(2)_1^c$	$(\mathbf{8}_v, \mathbf{4}^a) \oplus (\mathbf{8}_s, \mathbf{4}^b) \oplus (\mathbf{8}_c, \mathbf{4}^c)$
2	$\mathfrak{so}(9)$	$\mathfrak{sp}(3)_1^a \times \mathfrak{sp}(2)_2^b$	$(\mathbf{9}, \mathbf{6}^a) \oplus (\mathbf{16}, \mathbf{4}^b)$
2	$\mathfrak{so}(10)$	$\mathfrak{sp}(4)_1^a \times (\mathfrak{su}(2)_4 \times \mathfrak{u}(1)_8)^b$	$(\mathbf{10}, \mathbf{8}^a) \oplus [(\mathbf{16}_s, \mathbf{2}^b)_1 \oplus c.c.]$
2	$\mathfrak{so}(11)$	$\mathfrak{sp}(5)_1^a \times ?^b$	$(\mathbf{11}, \mathbf{10}^a) \oplus (\mathbf{32}, \mathbf{2}^b)$
2	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(6)_1^a \times \mathfrak{so}(2)_8$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32}_s, \mathbf{2}^b)$
2	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(6)_1^a \times \text{Ising}^b \times \text{Ising}^c$	$(\mathbf{12}, \mathbf{12}^a) \oplus (\mathbf{32}_s, \mathbf{1}_s^b) \oplus (\mathbf{32}_c, \mathbf{1}_s^c)$
2	$\mathfrak{so}(13)$	$\mathfrak{sp}(7)_1$	$(\mathbf{13}, \mathbf{14}) \oplus (\mathbf{64}, \mathbf{1})$
2	G_2	$\mathfrak{sp}(4)_1$	$(\mathbf{7}, \mathbf{8})$
2	F_4	$\mathfrak{sp}(3)_3$	$(\mathbf{26}, \mathbf{6})$
2	E_6	$\mathfrak{su}(4)_6 \times \mathfrak{u}(1)_{24}$	$(\mathbf{27}, \overline{\mathbf{4}})_{-1} \oplus c.c.$
2	E_7	$\mathfrak{so}(6)_{12}$	$(\mathbf{56}, \mathbf{6})$

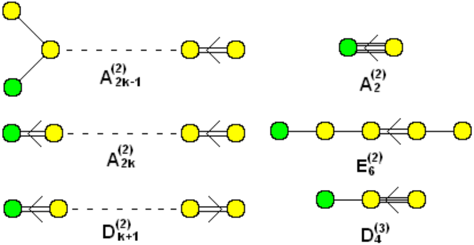
Full List of Rank One 6d $(1,0)$ SCFTs

e.g. $SU(r)+(r+8)\mathbf{F}+\mathbf{\Lambda}^2$ theories, $Sp(r)+(2r+8)\mathbf{F}$ theories

n	G	F	$2(R_G,R_F)$
1	$\mathfrak{sp}(0)$	$(E_8)_1$	$-$
1	$\mathfrak{sp}(N\geq 1)$	$\mathfrak{so}(4N+16)_1$	$(\mathbf{2N},\mathbf{4N+16})$
1	$\mathfrak{su}(3)$	$\mathfrak{su}(12)_1$	$(\mathbf{3},\overline{\mathbf{12}})_1\oplus c.c.$
1	$\mathfrak{su}(4)$	$\mathfrak{su}(12)_1^a\times\mathfrak{su}(2)_1^b$	$[(\mathbf{4},\overline{\mathbf{12}}_1^a)\oplus c.c.]\oplus(\mathbf{6},\mathbf{2}^b)$
1	$\mathfrak{su}(N\geq 5)$	$\mathfrak{su}(N{+}8)_1\times\mathfrak{u}(1)_{2N(N-1)(N+8)}$	$[(\mathbf{N},\overline{\mathbf{N+8}})_{-N+4}\oplus(\mathbf{\Lambda}^2,\mathbf{1})_{N+8}]\oplus c.c.$
1	$\mathfrak{su}(6)_*$	$\mathfrak{su}(15)_1$	$[(\mathbf{6},\overline{\mathbf{15}})\oplus c.c.]\oplus(\mathbf{20},\mathbf{1})$
1	$\mathfrak{so}(7)$	$\mathfrak{sp}(2)_1^a\times\mathfrak{sp}(6)_1^b$	$(\mathbf{7},\mathbf{4}^a)\oplus(\mathbf{8},\mathbf{12}^b)$
1	$\mathfrak{so}(8)$	$\mathfrak{sp}(3)_1^a\times\mathfrak{sp}(3)_1^b\times\mathfrak{sp}(3)_1^c$	$(\mathbf{8}_v,\mathbf{6}^a)\oplus(\mathbf{8}_s,\mathbf{6}^b)\oplus(\mathbf{8}_c,\mathbf{6}^c)$
1	$\mathfrak{so}(9)$	$\mathfrak{sp}(4)_1^a\times\mathfrak{sp}(3)_2^b$	$(\mathbf{9},\mathbf{8}^a)\oplus(\mathbf{16},\mathbf{6}^b)$
1	$\mathfrak{so}(10)$	$\mathfrak{sp}(5)_1^a\times(\mathfrak{su}(3)_4\times\mathfrak{u}(1)_{12})^b$	$(\mathbf{10},\mathbf{10}^a)\oplus[(\mathbf{16}_s,\mathbf{3}^b)_1\oplus c.c.]$
1	$\mathfrak{so}(11)$	$\mathfrak{sp}(6)_1^a\times ?^b$	$(\mathbf{11},\mathbf{12}^a)\oplus(\mathbf{32},\mathbf{3}^b)$
1	$\mathfrak{so}(12)_a$	$\mathfrak{sp}(7)_1^a\times\mathfrak{so}(3)_8^b$	$(\mathbf{12},\mathbf{14}^a)\oplus(\mathbf{32}_s,\mathbf{3}^b)$
1	$\mathfrak{so}(12)_b$	$\mathfrak{sp}(7)_1^a\times ?^b\times ?^c$	$(\mathbf{12},\mathbf{14}^a)\oplus(\mathbf{32}_s,\mathbf{2}^b)\oplus(\mathbf{32}_c,\mathbf{1}^c)$
1	G_2	$\mathfrak{sp}(7)_1$	$(\mathbf{7},\mathbf{14})$
1	F_4	$\mathfrak{sp}(4)_3$	$(\mathbf{26},\mathbf{8})$
1	E_6	$\mathfrak{su}(5)_6\times\mathfrak{u}(1)_{30}$	$(\mathbf{27},\overline{\mathbf{5}})_{-1}\oplus c.c.$
1	E_7	$\mathfrak{so}(7)_{12}$	$(\mathbf{56},\mathbf{7})$

Twisted 6d $(1,0)$ rank-one SCFTs

- When the gauge algebra allows outer automorphism, i.e. folding the Dynkin diagrams, we can construct twisted 6d theories
- Twisted circle compactification to 5d KK theories (Bhardwaj-Jefferson-Kim-Tarazi-Vafa 19,. . .).
- In rank one, one can fold both 6d vector and hyper multiplets.
- We denote the truncated gauge algebra, flavor algebra and matter representation as $\mathring{G}, \mathring{F}, \mathring{R}$.
- Upon twist, fractional KK charges naturally appear.



KK-momentum shifts of representations under twist

G	$\mathring{G}$	R		$\mathring{R}$
$A_{2r}^{(2)}$	C_r	$\mathbf{Adj}$	$\to$	$\mathbf{Adj}_0 \oplus \mathbf{F}_{1/4} \oplus \mathbf{\Lambda}^2_{1/2} \oplus \mathbf{1}_{1/2} \oplus \mathbf{F}_{3/4}$
		$\mathbf{F} \oplus \overline{\mathbf{F}}$	$\to$	$\mathbf{F}_0 \oplus \mathbf{1}_{1/4} \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_{3/4}$
$A_{2r-1}^{(2)}$	C_r	$\mathbf{Adj}$	$\to$	$\mathbf{Adj}_0 \oplus \mathbf{\Lambda}^2_{1/2}$
		$\mathbf{F} \oplus \overline{\mathbf{F}}$	$\to$	$\mathbf{F}_0 \oplus \mathbf{F}_{1/2}$
		$\mathbf{\Lambda}^2 \oplus \overline{\mathbf{\Lambda}^2}$	$\to$	$\mathbf{\Lambda}^2_0 \oplus \mathbf{\Lambda}^2_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2}$
$D_{r+1}^{(2)}$	B_r	$\mathbf{Adj}$	$\to$	$\mathbf{Adj}_0 \oplus \mathbf{F}_{1/2}$
		$\mathbf{V}$	$\to$	$\mathbf{V}_0 \oplus \mathbf{1}_{1/2}$
		$\mathbf{S} \oplus \mathbf{C}$	$\to$	$\mathbf{S}_0 \oplus \mathbf{S}_{1/2}$
$E_6^{(2)}$	F_4	$\mathbf{Adj}$	$\to$	$\mathbf{Adj}_0 \oplus \mathbf{F}_{1/2}$
		$\mathbf{F} \oplus \overline{\mathbf{F}}$	$\to$	$\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2}$
$D_4^{(3)}$	G_2	$\mathbf{Adj}$	$\to$	$\mathbf{Adj}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3}$
		$\mathbf{V} \oplus \mathbf{S} \oplus \mathbf{C}$	$\to$	$\mathbf{F}_0 \oplus \mathbf{1}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{1}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_{2/3}$

Some typical twisted 6d $(1,0)$ rank one theories

There exist four pure gauge twisted theories: $n=6$, $E_6^{(2)}$, $n=4$, $D_4^{(2)}$, $D_4^{(3)}$ and $n=3$, $A_2^{(2)}$.

n	G	$\mathring{G}$	$\mathring{R}$	$\mathring{F}$
6	$E_6^{(2)}$	F_4	$-$	$-$
4	$D_4^{(3)}$	G_2	$-$	$-$
4	$D_4^{(2)}$	B_3	$-$	$-$
4	$D_{r+4}^{(2)}$	B_{r+3}	$2r(\mathbf{V}_0 \oplus \mathbf{1}_{1/2})$	$\mathfrak{sp}(2r)$
4	$E_6^{(2)}$	F_4	$\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2}$	$\mathfrak{sp}(1)$
3	$A_2^{(2)}$	C_1	$-$	$-$
3	$D_4^{(2)}$	B_3	$\mathbf{V}_0 \oplus \mathbf{1}_{1/2} \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2}$	$\mathfrak{sp}(1) \times \mathfrak{sp}(1)$
3	$D_4^{(3)}$	G_2	$\mathbf{F}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/3} \oplus \mathbf{1}_{2/3}$	$\mathfrak{sp}(1)$
2	A_{2r}	C_r	$(2r+1)(\mathbf{F}_{1/4} \oplus \mathbf{F}_{3/4} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2})$	$\mathfrak{so}(4r+2)$
2	A_{2r-1}	C_r	$2r(\mathbf{F}_0 \oplus \mathbf{F}_{1/2})$	$\mathfrak{so}(4r)$
2	$D_4^{(2)}$	B_3	$2(\mathbf{V}_0 \oplus \mathbf{1}_{1/2} \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2})$	$\mathfrak{sp}(2) \times \mathfrak{sp}(2)$
2	$D_4^{(3)}$	G_2	$2(\mathbf{F}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/3} \oplus \mathbf{1}_{2/3})$	$\mathfrak{sp}(2)$
2	$D_5^{(2)}$	B_4	$4(\mathbf{V}_0 \oplus \mathbf{1}_{1/2}) \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2}$	$\mathfrak{sp}(4) \times \mathfrak{sp}(1)$
2	$D_6^{(2)}$	B_5	$6(\mathbf{V}_0 \oplus \mathbf{1}_{1/2}) \oplus \frac{1}{2}\mathbf{S}_0 \oplus \frac{1}{2}\mathbf{S}_{1/2}$	$\mathfrak{sp}(6)$
2	$E_6^{(2)}$	F_4	$2(\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/2})$	$\mathfrak{sp}(2)$
1	$A_2^{(2)}$	C_1	$6(\mathbf{F}_0 \oplus \mathbf{F}_{1/2} \oplus \mathbf{1}_{1/4} \oplus \mathbf{1}_{3/4})$	$\mathfrak{so}(12)$
1	$A_3^{(2)}$	C_2	$6(\mathbf{F}_0 \oplus \mathbf{F}_{1/2}) \oplus \mathbf{1}_0 \oplus \mathbf{\Lambda}^2_{1/2}$	$\mathfrak{so}(12) \times \mathfrak{sp}(1)$
1	$D_4^{(2)}$	B_3	$3(\mathbf{V}_0 \oplus \mathbf{1}_{1/2} \oplus \mathbf{S}_0 \oplus \mathbf{S}_{1/2})$	$\mathfrak{sp}(3) \times \mathfrak{sp}(3)$
1	$D_4^{(3)}$	G_2	$3(\mathbf{F}_0 \oplus \mathbf{F}_{1/3} \oplus \mathbf{F}_{2/3} \oplus \mathbf{1}_0 \oplus \mathbf{1}_{1/3} \oplus \mathbf{1}_{2/3})$	$\mathfrak{sp}(3)$

Elliptic Genera of 6d SCFT

- Consider a elliptic non-compact Calabi-Yau X
- Compactify F-theory on $\mathbb{C}^2_{\epsilon_1,\epsilon_2} \times T^2 \times X$
- The full partition function contains three parts

$$Z_{6d} = Z_{\mathrm{poly}} Z_{1-\mathrm{loop}} Z_{\mathrm{ell}},$$
$$Z_{\mathrm{ell}} = 1 + \mathbb{E}_1 Q_{\mathrm{ell}} + \mathbb{E}_2 Q_{\mathrm{ell}}^2 + \dots$$

- Z_{poly} comes from the perturbative part
- $Z_{1-\mathrm{loop}}$ contains the contributions from tensor, vector and hyper multiplets
- One main goal to study 6d SCFT is to compute the d-string elliptic genus $\mathbb{E}_d(\tau, m_G, m_F, \epsilon_1, \epsilon_2)$ of the associated 2d $(0,4)$ SCFTs on the BPS strings
- $\mathbb{E}_d$ is the natural elliptic lift the d-instanton Nekrasov partition function, also certain Jacobi form with $m_G, m_F, \epsilon_1, \epsilon_2$ as elliptic parameters

Twisted elliptic genera

In the twisted cases, similarly one can define twisted elliptic genera

$$Z_{6d}^{tw} = Z_{\text{poly}}^{\text{tw}} Z_{1-\text{loop}}^{\text{tw}} Z_{\text{ell}}^{\text{tw}}, \qquad Z_{\text{ell}}^{\text{tw}} = 1 + \mathbb{E}_1^{\text{tw}} Q_{\text{ell}} + \mathbb{E}_2^{\text{tw}} Q_{\text{ell}}^2 + \dots$$

Now $\mathbb{E}_d^{tw}$ contains fractional q_τ powers.

- In twisted circle compactification, $Z_{6d}^{tw} = Z_{5d}^{\text{KK}}$.
- Twisted elliptic genera for a few theories have been studied in (Kim[3]-Lee 21, Kim[3] 21)
- Now we would like go through all twistable theories
- We want to keep everything elliptic and as 6d gauge theory, rather than geometries or 5d KK theories.

Approaches to twisted elliptic genera

In the following, I mainly talk about the last three approaches.

- 2d quiver gauge theories
  - $n = 4, D_4^{(2)}$, (Kim^3 21)
- Twisting from Higgsing
  - many (Kim^2-Lee 19, Kim^3 21, Hayashi-Kim-Ohmori 21)
- topological vertex and brane-webs
  - (Hayashi-Zhu 20, Kim^3 21)
- Twisted elliptic blowup equations
  - $n = 3, A_2^{(2)}$, (Kim^3-Lee 21)
- Modular bootstrape
- Spectral flow symmetry

Background for blowup equations

- Blowup equations are functional equations for Nekrasov partition functions originally for 4d $\mathcal{N} = 2$ $SU(N)$ gauge theories (Nakajima-Yoshioka 03)
- K-theoretic version – 5d $\mathcal{N} = 1$ (Nakajima-Yoshioka 05, 09, Göttsche-Nakajima-Yoshioka 06, Keller-Song 12, Kim^2-Lee^2-Song 19)
- further generalize to refined topological strings on local Calabi-Yau threefolds (Huang-KS-Wang 17)
- elliptic version for 6d $(1, 0)$ SCFTs (Gu-Haghighat-Klemm-KS-Wang 18-20)
- 6d $(2, 0)$ SCFTs, 5d $\mathcal{N} = 1^{\star}$ (Duan-Lee-Nahmgoong-Wang 21)
- 6d twisted SCFTs and 5d KK theories (Kim^3-Lee 21)

Twisted elliptic blowup equations

Let λ_0 be a coweight of G invariant upon the twist. If (λ_0, λ_F) is admissible for the original 6d SCFT, and let $\lambda_{\mathring{F}}$ be the reduction of λ_F, then the twisted elliptic genera $\mathbb{E}_d(\tau, m_{\mathring{G}}, m_{\mathring{F}}, \varepsilon_{1,2})$ satisfy the following

Twisted elliptic blowup equations

$$\sum_{\lambda_{\mathring{G}} \in \phi_{\lambda_0}(Q^\vee(\mathring{G}))}^{\frac{1}{2}||\lambda_{\mathring{G}}||^2 + d' + d'' = d+\delta} (-1)^{|\lambda_{\mathring{G}}|} \theta_i^{[a]}\left(n\tau, \begin{array}{c} -n\lambda_{\mathring{G}} \cdot m_{\mathring{G}} + k_F \lambda_{\mathring{F}} \cdot m_{\mathring{F}} + \\ (y - \frac{n}{2}||\lambda_{\mathring{G}}||^2)(\epsilon_1+\epsilon_2) - nd'\epsilon_1 - nd''\epsilon_2 \end{array}\right)$$
$$\times A_V^{\mathring{G}}(\tau, m_{\mathring{G}}, \lambda_{\mathring{G}}) A_V^{frac}(\tau, m_{\mathring{G}}, \lambda_{\mathring{G}}) A_H^{\mathring{R}}(\tau, m_{\mathring{G}}, m_{\mathring{F}}, \lambda_{\mathring{G}}, \lambda_{\mathring{F}})$$
$$\times \mathbb{E}^{\mathrm{tw}}_{d'}(\tau, m_{\mathring{G}} + \epsilon_1 \lambda_{\mathring{G}}, m_{\mathring{F}} + \epsilon_1 \lambda_{\mathring{F}}, \epsilon_1, \epsilon_2 - \epsilon_1)$$
$$\times \mathbb{E}^{\mathrm{tw}}_{d''}(\tau, m_{\mathring{G}} + \epsilon_2 \lambda_{\mathring{G}}, m_{\mathring{F}} + \epsilon_2 \lambda_{\mathring{F}}, \epsilon_1 - \epsilon_2, \epsilon_2)$$
$$= \Lambda(\delta)\, \theta_i^{[a]}(n\tau, k_F \lambda_{\mathring{F}} \cdot m_{\mathring{F}} + ny(\epsilon_1+\epsilon_2))\, \mathbb{E}^{\mathrm{tw}}_d(\tau, m_{\mathring{G}}, m_{\mathring{F}}, \epsilon_1, \epsilon_2).$$

For $\delta = 0$, $\Lambda(\delta) = 1$, we call it unity blowup equations. For $\delta > 0$, $\Lambda(\delta) = 0$, we call it vanishing blowup equations.

Twisted elliptic blowup equations

The parameters $y, \lambda_{\mathring{F}}$ of unity twisted elliptic blowup equations for rank one models. $\#$ is the number of equations with fixed characteristic a.

n	G	$\mathring{G}$	$\mathring{F}$	$\#$	y	$\lambda_{\mathring{F}}$
6	E_6	F_4	$-$	1	4	$\emptyset$
4	$D_4^{(3)}$	G_2	$-$	1	2	$\emptyset$
4	$D_{r+4}^{(2)}$	B_{r+3}	$\mathfrak{sp}(2r)$	2^{2r}	$r+2$	$(0\dots01)$
4	$E_6^{(2)}$	F_4	$\mathfrak{sp}(1)$	2	5	(1)
3	$A_2^{(2)}$	C_1	$-$	1	1	$\emptyset$
3	$D_4^{(2)}$	B_3	$\mathfrak{sp}(1)\times\mathfrak{sp}(1)$	4	$5/2$	(1),(1)
3	$D_4^{(3)}$	G_2	$\mathfrak{sp}(1)$	2	$5/2$	(1)
2	A_{2r}	C_r	$\mathfrak{so}(4r+2)$	2^{2r+1}	$r+\frac{1}{2}$	$(0\dots01)$
2	A_{2r-1}	C_r	$\mathfrak{so}(4r)$	2^{2r}	r	$(0\dots01)$
2	$D_4^{(2)}$	B_3	$\mathfrak{sp}(2)\times\mathfrak{sp}(2)$	16	3	(01),(01)
2	$D_4^{(3)}$	G_2	$\mathfrak{sp}(2)$	4	3	(01)
2	$D_5^{(2)}$	B_4	$\mathfrak{sp}(4)\times\mathfrak{sp}(1)$	32	4	(0001), (1)
2	$E_6^{(2)}$	F_4	$\mathfrak{sp}(2)$	4	6	(01)
1	$A_1^{(2)}$	C_1	$\mathfrak{so}(10)$	64	$3/2$	$(0\dots01)$
1	$A_2^{(2)}$	C_1	$\mathfrak{so}(12)$	64	2	$(0\dots01)$
1	$A_3^{(2)}$	C_2	$\mathfrak{so}(12)\times\mathfrak{sp}(1)$	128	$5/2$	$(0\dots01)$, (1)
1	$D_4^{(2)}$	B_3	$\mathfrak{sp}(3)\times\mathfrak{sp}(3)$	64	$7/2$	(001),(001)
1	$D_4^{(3)}$	G_2	$\mathfrak{sp}(3)$	8	$7/2$	(001)

Universal formula for twisted elliptic genera $\mathbb{E}_1$

From the unity equations, we can solve twisted elliptic genera recursively. In particular, the twisted one-string elliptic genera for the pure gauge cases have the following universal formula:

$$\mathbb{E}_1^{\mathrm{tw}} = \sum_{\alpha\in\Delta_l^\vee}(-1)^{|\alpha^\vee|}\frac{D^{\alpha^\vee}}{D}\frac{\eta^4}{\theta_1(m_\alpha)\theta_1(m_\alpha-\varepsilon_{1,2})\theta_1(m_\alpha-2\varepsilon_+)}\prod_{\substack{\beta\in\Delta\\ \alpha^\vee\cdot\beta=1}}\frac{\eta}{\theta_1(m_\beta)}\prod_{\substack{\gamma\in\Delta_s\\ \alpha^\vee\cdot\gamma=1}}\frac{\eta}{\theta_1^{[k]}(m_\gamma)}.$$

Here $D^{\alpha^\vee}=Det^{\alpha^\vee}_{\{1,0,0\}}$ and $D=Det^{\alpha^\vee}_{\{0,0,0\}}$, and we define

$$\theta^{[a]}_{i,\{d_0,d_1,d_2\}}=\theta^{[a]}_i(n\tau,-nm_{\alpha^\vee}+(n-2)(\epsilon_1+\epsilon_2)-n((d_0+d_1)\epsilon_1+(d_0+d_2)\epsilon_2)),$$

$$Det^{\alpha^\vee}_{\{d_0,d_1,d_2\}}=\det\begin{pmatrix}\theta^{[a_1]}_{i,\{0,d,0\}} & \theta^{[a_1]}_{i,\{0,0,d\}} & \theta^{[a_1]}_{i,\{d_0,d_1,d_2\}}\\ \theta^{[a_2]}_{i,\{0,d,0\}} & \theta^{[a_2]}_{i,\{0,0,d\}} & \theta^{[a_2]}_{i,\{d_0,d_1,d_2\}}\\ \theta^{[a_3]}_{i,\{0,d,0\}} & \theta^{[a_3]}_{i,\{0,0,d\}} & \theta^{[a_3]}_{i,\{d_0,d_1,d_2\}}\end{pmatrix}.$$

Besides, k are the fractional KK charges for the short roots.

Structure of twisted elliptic genera

From the universal $\mathbb{E}_1^{\rm tw}$ formula, we find the following universal expansion for twisted pure gauge theories ($v = e^{\epsilon_+}$)

$$\mathbb{E}_1^{\rm tw} = v^{h^\vee_{\mathring{G}}-1}\Big(\sum_{n=0}^{\infty}\chi^{\mathring{G}}_{k\theta}v^{2n} + q_\tau^{\frac{1}{N}}(1+v^2)\sum_{n=0}^{\infty}\chi^{\mathring{G}}_{k\theta+f}v^{2n} + \dots\Big).$$

- q_τ^0 order gives 5d one $\mathring{G}$-instanton Hilbert series (Benvenuti-Hanany-Mekareeya 10)
- Fractional orders $q_\tau^{i/N}, i = 1, \dots, N-1$ all have exact v expansion formulas
- For example, for $D_4^{(3)}$ theory the $q_\tau^{2/3}$ order of $\mathbb{E}_1^{\rm tw}$ is

$$\sum_{n=0}^{\infty}(\chi^{G_2}_{2f+n\theta}(1+v^2+v^4) + \chi^{G_2}_{f+n\theta}(1+v^2) + \chi^{G_2}_{n\theta})v^{2n+3}.$$

Modular bootstrap on $\Gamma(N)$

We turn off all gauge and flavor fugacities to study the modular property of $\mathbb{E}_1(q,v)$. In many cases, this can fully determine $\mathbb{E}_1(q,v)$ with small amount of initial data.

- For untwisted theories, the modular ansatz was proposed in (Del Zotto-Lockhart 18) on $SL(2,\mathbb{Z})$
- For twisted cases, we propose the modular ansatz on congruence subgroup $\Gamma(N)$. Here N is the twist coefficient:

$$N = \begin{cases} 2, & A^{(2)}_{2r-1}, D^{(2)}_r, E^{(2)}_6, \\ 3, & D^{(3)}_4, \\ 4, & A^{(2)}_{2r}. \end{cases}$$

- Inspired by the modular ansatz for genus-one fibered CY3 (Cota-Klemm-Schimannek 19) (Knapp-Scheidegger-Schimannek 21)

Modular bootstrap on $\Gamma(N)$

For a theory on $(-n)$-curve with twist coefficient $N=2,3,4$, we propose modular ansatz for the reduced one-string elliptic genus:

$$\boxed{\mathbb{E}_1^{\mathrm{tw}}(\tau,\epsilon_+)=\frac{\mathcal{N}(\tau,\epsilon_+)}{\eta^{12(n-2)-4+24\delta_{n,1}}\Delta_{2N}(\frac{\tau}{N})^{\frac{s}{N}}\phi_{-2,1}(\tau,2\epsilon_+)^{h^\vee_{\mathring{G}}-1}}.}$$

Here $s=\frac{N}{N-1}(c-\frac{n-2}{2}-\delta_{n,1})$. Numerator $\mathcal{N}(\tau,\epsilon_+)$ is of weight $6(n-2)+2Ns+12\delta_{n,1}-2h^\vee_{\mathring{G}}$ and index $4(h^\vee_{\mathring{G}}-1)+n-h^\vee_G$ and

$$\boxed{\mathcal{N}(N\tau,\epsilon_+)\in M_\star(N)[\phi_{-2,1}(N\tau,\epsilon_+),\phi_{0,1}(N\tau,\epsilon_+)].}$$

Δ_{2N} are certain cusp forms on $\Gamma(N)$:

$$\Delta_4(\tau)=\frac{\eta(2\tau)^{16}}{\eta(\tau)^8},\quad \Delta_6(\tau)=\frac{\eta(3\tau)^{18}}{\eta(\tau)^6},\quad \Delta_8(\tau)=\frac{\eta(2\tau)^8\eta(4\tau)^{16}}{\eta(\tau)^8},$$

and $\phi_{-2,1},\phi_{0,1}$ are Eichler-Zagier's generators for weak Jacobi forms.

Examples on modular bootstrap

1. For pure $D_4^{(2)}$ theory on (-4) curve, twist coefficient $N=2$,

$$\mathbb{E}_1^{\mathrm{tw}}(\tau,\epsilon_+)=\frac{\Delta_4(\tau/2)^{\frac{1}{2}}\mathcal{N}(\tau,\epsilon_+)}{\eta(\tau)^{20}\phi_{-2,1}(\tau,2\epsilon_+)^4}.$$

   Here $\mathcal{N}(\tau,\epsilon_+)$ is of weight 0 and index 14. There are 64 coefficients to fix $\mathcal{N}(2\tau,\epsilon_+)$.

2. For pure $D_4^{(3)}$ theory on (-4) curve, twist coefficient $N=3$,

$$\mathbb{E}_1^{\mathrm{tw}}(\tau,\epsilon_+)=\frac{\Delta_6(\tau/3)^{\frac{2}{3}}\mathcal{N}(\tau,v)}{\eta(\tau)^{20}\phi_{-2,1}(\tau,2\epsilon_+)^3}.$$

   Here $\mathcal{N}(\tau,\epsilon_+)$ is of weight 0 and index 10. There are 67 coefficients to fix $\mathcal{N}(3\tau,\epsilon_+)$.

3. For pure $A_2^{(2)}$ theory on (-3) curve, twist coefficient $N=4$,

$$\mathbb{E}_1^{\mathrm{tw}}(\tau,\epsilon_+)=\frac{\Delta_8(\tau/4)^{\frac{1}{4}}\mathcal{N}(\tau,\epsilon_+)}{\eta(\tau)^{8}\phi_{-2,1}(\tau,2\epsilon_+)}.$$

   Here $\mathcal{N}(\tau,\epsilon_+)$ is of weight 0 and index 4. There are 15 coefficients to fix $\mathcal{N}(4\tau,\epsilon_+)$

Spectral flow symmetry

For twisted one-string elliptic genus, we find the spectral flow from R-R sector to NS-R sector is induced by transformation:

$$\boxed{\mathbb{E}^{\mathring{R}_{\mathrm{KK}}}_{\mathrm{NS-R}}\left(v,q\right) = \pm\left(\frac{q^{1/4}}{v}\right)^{n-h^\vee_G} \mathbb{E}^{\mathring{R}_{\mathrm{KK}}}_{\mathrm{R-R}}\left(\frac{q^{1/2}}{v},q\right).}$$

On the other hand, 2d $(0,4)$ analysis shows spectral flow shifts the KK charges of hypermultiplets by $1/2$:

$$\boxed{\mathbb{E}^{\mathring{R}_{\mathrm{KK}}}_{\mathrm{NS-R}} = \mathbb{E}^{\mathring{R}_{\mathrm{KK}+1/2}}_{\mathrm{R-R}}.}$$

These two properties together suggest interesting symmetry for the twisted elliptic genera.

Example: spectral flow symmetry for pure gauge theories

For 6d $(1,0)$ pure gauge $G^{(k)}$ theories on $(-n)$-curve, we find the following spectral flow symmetry for the reduced twisted one-string elliptic genera:

$$\boxed{\mathbb{E}_1^{G^{(k)}}\Big(q, \frac{q^{1/2}}{v}\Big) = \Big(-\frac{q^{1/2}}{v^2}\Big)^{n-3}\,\mathbb{E}_1^{G^{(k)}}(q,v).}$$

We explicitly checked this symmetry for $A_2^{(2)}, D_4^{(2)}, D_4^{(3)}, E_6^{(2)}$.

- generalization of (Del Zotto-Lockhart 16, 18) for the untwisted cases $A_2, D_4, F_4, E_{6,7,8}$.
- relate twisted R-R elliptic genus to twisted NS-R

Spectral flow symmetry for $E_6^{(2)}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\mathrm{tw}}(q_\tau, v, m_{F_4}=0) = v^{-3}\sum_{i,j=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

Then we find the following table for coefficients b_{ij}:

$q\backslash v$	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	1	0	52	0
1/2	0	0	0	0	0	0	26	0	1079
1	0	0	0	0	0	0	0	378	0
3/2	0	0	0	0	0	0	0	0	4056
2	0	0	0	0	0	−1	0	0	0
5/2	−1	0	0	0	1	0	−26	0	0
3	0	−26	0	0	0	26	0	−378	0
7/2	−52	0	−378	0	0	0	378	0	−4004
4	0	−1079	0	−4056	0	0	0	4004	0

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\mathrm{tw}}(q_\tau, v, m_{G_2}=0) = v^{-1}\sum_{i,j=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

Then we find the following table for coefficients b_{ij}:

$q\backslash v$	0			2			4			6
0	0	0	0	1	0	0	14	0	0	77
1/3	0	0	0	0	7	0	0	71	0	0
2/3	0	0	0	0	0	35	0	0	301	0
1	−1	0	0	0	0	0	141	0	0	1127
4/3	0	−7	0	0	0	0	0	497	0	0
5/3	0	0	−35	0	0	0	0	0	1582	0
2	−14	0	0	−141	0	0	0	0	0	4650
7/3	0	−71	0	0	−497	0	0	0	0	0
8/3	0	0	−301	0	0	−1582	0	0	0	0
3	−77	0	0	−1127	0	0	−4650	0	0	0

Spectral flow symmetry for $A_2^{(2)}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\mathrm{tw}}(q, v, m_{A_1} = 0) = \sum_{i=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

$q\backslash v$	0		1		2		3		4		5
0	0	0	1	0	0	0	3	0	0	0	5
1/4	0	0	0	2	0	0	0	6	0	0	0
1/2	1	0	0	0	4	0	0	0	9	0	0
3/4	0	2	0	0	0	8	0	0	0	18	0
1	0	0	4	0	0	0	17	0	0	0	39
5/4	0	0	0	8	0	0	0	30	0	0	0
3/2	3	0	0	0	17	0	0	0	51	0	0
7/4	0	6	0	0	0	30	0	0	0	86	0
2	0	0	9	0	0	0	51	0	0	0	147
9/4	0	0	0	18	0	0	0	86	0	0	0
5/2	5	0	0	0	39	0	0	0	147	0	0

A possible $Sp(1)_\pi$ theory from $A_2^{(2)}$

Interestingly, we find the twisted elliptic blowup equations of $A_2^{(2)}$ and modular ansatz allow another solution of twisted one-string elliptic genus. The spectral flow symmetry also holds, and the 5d limit is $Sp(1)_\pi$ theory.

$q\backslash v$	0		1		2		3		4		5		6
0	0	0	0	0	2	0	0	0	4	0	0	0	6
1/4	0	0	0	0	0	4	0	0	0	8	0	0	0
1/2	0	0	0	0	0	0	6	0	0	0	12	0	0
3/4	0	0	0	0	0	0	0	12	0	0	0	24	0
1	2	0	0	0	4	0	0	0	28	0	0	0	52
5/4	0	4	0	0	0	8	0	0	0	48	0	0	0
3/2	0	0	6	0	0	0	12	0	0	0	74	0	0
7/4	0	0	0	12	0	0	0	24	0	0	0	124	0
2	4	0	0	0	28	0	0	0	52	0	0	0	220
9/4	0	8	0	0	0	48	0	0	0	88	0	0	0
5/2	0	0	12	0	0	0	74	0	0	0	136	0	0
11/4	0	0	0	24	0	0	0	124	0	0	0	224	0
3	6	0	0	0	52	0	0	0	220	0	0	0	388

Example with matters

Consider the $\mathbb{Z}_2$ twist of $E_6 + 2\mathbf{F}$ theory on (-4) curve. Upon twist

$$\begin{aligned} \text{vector multiplet}: &\quad \mathbf{78} \to \mathbf{52}_0 + \mathbf{26}_{1/2}, \\ \text{hyper multiplet}: &\quad 2 \cdot \mathbf{27} \to \mathbf{26}_0 + \mathbf{26}_{1/2} + \mathbf{1}_0 + \mathbf{1}_{1/2} \end{aligned}$$

Thus the twisted theory can be written as $E_6^{(2)} + \mathbf{F}_0 + \mathbf{F}_{1/2}$.

On the other hand, the $\mathbb{Z}_2$ twist of $E_6 + \mathbf{F}$ theory on (-5) curve have two possibilities: $E_6^{(2)} + \mathbf{F}_0$ and $E_6^{(2)} + \mathbf{F}_{1/2}$.

The twisted elliptic genera of all these theories can be exactly computed by the recursion formula from twisted elliptic blowup equations.

Example with matters: $E_6^{(2)} + \mathbf{F}_0 + \mathbf{F}_{1/2}$

Let us expand the twisted one-string elliptic genus as

$$\mathbb{E}_1^{\mathrm{tw}}(q_\tau, v, m_{F_4} = m_{\mathbf{F}} = 0) = \sum_{i,j=0}^{\infty} b_{i,j} q^i v^{2i+j}.$$

Then we find the following table for coefficients b_{ij}:

$q\backslash v$	4	5	6	7	8	9	10
0	0	0	0	−1	−4	78	−754
1/2	0	0	0	2	−27	−108	2509
1	0	0	0	3	54	−512	−1736
3/2	1	−2	−3	0	86	874	−7436
2	4	27	−54	−86	0	1437	10756
5/2	−78	108	512	−874	−1437	0	16930
3	754	−2509	1736	7436	−10756	−16930	0

Orange numbers give 5d one-instanton partition function $Z_1^{F_4+\mathbf{F}}(\mathrm{v})$.

Example with matters: $E_6^{(2)} + \mathbf{F}_{1/2}$ and $E_6^{(2)} + \mathbf{F}_0$

Let us expand the twisted one-string elliptic genus of $E_6^{(2)} + \mathbf{F}_{1/2}$ as

$$\mathbb{E}_1^{\mathrm{tw}}(q_\tau, v) = q_\tau^{-4/3} \sum_{i,j=0}^{\infty} c_{ij} q_\tau^i v^j = q_\tau^{-\frac{4}{3}} \sum_{i,j=0}^{\infty} b_{ij} q_\tau^i v^{2i+j}.$$

We obtain the following coefficients b_{ij}:

$q \backslash v$	5	6	7	8	9	10	11	12
0	0	0	0	1	0	52	0	1053
1/2	0	0	0	0	26	−52	1079	−2106
1	0	0	0	0	−2	381	−1300	15209
3/2	0	0	0	0	−3	−46	4235	−18670
2	0	0	0	0	−2	−73	−610	38768
5/2	1	−2	1	0	−1	8	−1100	−6134
3	4	23	−50	22	2	4	578	−12778
7/2	−78	154	307	−766	441	−112	−167	10866
4	754	−2197	2610	3113	−8788	6143	−2582	−4314

Orange numbers give 5d one-instanton partition function $Z_1^{F_4}(v)$.

Example with matters: $E_6^{(2)} + \mathbf{F}_0$ and $E_6^{(2)} + \mathbf{F}_{1/2}$

Let us expand the twisted one-string elliptic genus of $E_6^{(2)} + \mathbf{F}_0$ as

$$\mathbb{E}_1^{\rm tw}(q_\tau, v) = q_\tau^{-\frac{7}{12}} \sum_{i,j=0}^{\infty} c_{ij} q_\tau^i v^j = q_\tau^{-\frac{7}{12}} \sum_{i,j=0}^{\infty} b_{ij} q_\tau^i v^{2i+j}.$$

We obtain the following coefficients b_{ij}:

$q\backslash v$	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	1	4	−78	754
1/2	0	0	0	0	0	−2	23	154	−2197
1	0	0	0	0	0	1	−50	307	2610
3/2	1	0	0	0	0	0	22	−766	3113
2	0	26	−2	−3	−2	−1	2	441	−8788
5/2	52	−52	381	−46	−73	8	4	−112	6143
3	0	1079	−1300	4235	−610	−1100	578	−167	−2582

The twisted one-string elliptic genera of $E_6^{(2)} + \mathbf{F}_0$ and $E_6^{(2)} + \mathbf{F}_{1/2}$ theories are exactly spectral dual to each other!

Summary

- We find all twisted elliptic blowup equations for all twisted rank one 6d $(1,0)$ SCFTs
- We compute the twisted elliptic genera for most of them
- We determine the modular ansatz for most of the twisted elliptic genera
- We find study the spectral flow symmetry for the twisted elliptic genera
- In the cases with 2d quiver gauge description, we can directly prove the spectral flow symmetry

Thank you for listening!