Defect in Gauge Theory and Quantum Hall States

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Outline

Introduction

The study of low-energy physics of supersymmetric gauge theory and integrable model has been an active research for decades.

- SW curve of $\mathcal{N} = 2 \Longleftrightarrow$ Spectral curve of integrable model.

The correspondence is later extended to the quantum level by Nekrasov and Shatashivilli. [Nekrasov-Shatashivilli '09, '09]

Introduction

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The correspondence is later extended to the quantum level by Nekrasov and Shatashivilli. [Nekrasov-Shatashivilli '09, '09]
In this talk I will explore a corner of this correspondence between the $\mathcal{N}=2^*$ $U(N)$ gauge theory in 4 dimension and the fractional quantum Hall states (Laughlin [Nekrasov '19], Moore-Read, Read-Rezayi, etc).

Outline

Supersymmetric Gauge Theory

Let us consider $\mathcal{N}=2^* \; U(N)$ gague theory with adjoint mass m. The supersymmetric partition function

$$\mathcal{Z}(\mathbf{a},m,\tau,\boldsymbol{\varepsilon}) = \mathcal{Z}_{\text{tree}}(\mathbf{a},m,\tau,\boldsymbol{\varepsilon})\mathcal{Z}_{\text{1-loop}}(\mathbf{a},m,\boldsymbol{\varepsilon})\mathcal{Z}_{\text{inst}}(\mathbf{a},m,\mathfrak{q},\boldsymbol{\varepsilon}) \qquad (1)$$

depends on the Coulomb moduli parameters $\mathbf{a}=(a_1,\dots,a_N)$ and complex coupling

$$\tau = \frac{4\pi i}{g^2} + \frac{\vartheta}{2\pi}, \quad \mathfrak{q} = e^{2\pi i \tau}. \qquad (2)$$

The $\boldsymbol{\varepsilon}=(\varepsilon_1,\varepsilon_2)$ are the Ω-deformation parameters in $\mathbb{R}^4=\mathbb{C}^2$.
The instanton partition function is an infinite sum over number of instantons

$$\mathcal{Z}_{\text{inst}}(\mathbf{a},m,\mathfrak{q},\boldsymbol{\varepsilon}) = \sum_{k=0}^{\infty} \mathfrak{q}^k \mathcal{Z}_k(\mathbf{a},m,\boldsymbol{\varepsilon}). \qquad (3)$$

Let $\mathbf{N} = \mathbb{C}^N$, $\mathbf{K} = \mathbb{C}^k$ be two vector spaces, the k instanton of $U(N)$ is constructed by ADHM matrices $B_{1,2} \in \mathrm{Hom}(\mathbf{K}, \mathbf{K})$, $I \in \mathrm{Hom}(\mathbf{N}, \mathbf{K})$, $J \in \mathrm{Hom}(\mathbf{K}, \mathbf{N})$ satisfying

$$\mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0,$$
$$\mu_{\mathbb{C}} = [B_1, B_2] + IJ = 0 \tag{4}$$

modulo the $U(k)$ action of the form

$$(v) \cdot (B_{1,2}, I, J) = (vB_{1,2}v^{-1}, vI, Jv^{-1}) \tag{5}$$

for $v \in U(k)$.

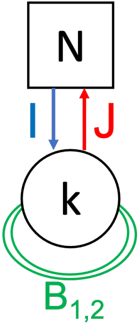

Figure: ADHM data

Instanton Partition Function

An anti-self-dual connection A can be constructed using ADHM data satisfying $\mu_{\mathbb{R}} = \mu_{\mathbb{C}} = 0$.

$$\mathcal{Z}_k(\mathbf{a}, m) = \int_{\mathcal{M}_{N,k}} 1 \tag{6}$$

where instanton moduli space is

$$\mathcal{M}_{N,k} = \{(B_1, B_2, I, J) \mid \mu_{\mathbb{R}} = 0,\ \mu_{\mathbb{C}} = 0\}. \tag{7}$$

The moduli space defined here has singularity. Regularization is needed.

Instanton Partition Funtion

Turn on the FI parameter for real moment map

$$\mu_{\mathbb{R}} = \zeta 1_k, \ \zeta > 0. \tag{8}$$

This immediately implies $J = 0$ and the stability condition $\mathbf{K} = \mathbb{C}[B_1, B_2]I(\mathbf{N})$. The regularized moduli space

$$\widetilde{\mathcal{M}}_{N,k} = \{(B_1, B_2, I, J)|\mu_{\mathbb{C}} = 0\}//\mathrm{GL}(k, \mathbb{C}) \tag{9}$$

which is equivalent to the non-commutative instanton moduli space $\mathcal{M}^{\zeta}_{N,k} = \{\mu_{\mathbb{R}} = \zeta, \ \mu_{\mathbb{C}} = 0\}/U(k)$.

Instanton Partition Function

The equivariant action on the ADHM data are

$$(v, \eta) \cdot (B_1, B_2, I, J) = (vB_1v^{-1}, vB_2v^{-1}, vI\eta^{-1}, v^{-1}J\eta) \tag{10}$$

with $v \in \mathrm{GL}(k, \mathbb{C})$ and $\eta \in \mathrm{GL}(N, \mathbb{C})$. The Ω-deformation $(q_1, q_2) = (e^{\varepsilon_1}, e^{\varepsilon_2}) \in \mathbb{C}^\times \times \mathbb{C}^\times$ is the Cartan element of the $\mathrm{GL}(2, \mathbb{C})$ associated to the spacetime rotation. It acts on ADHM data by

$$(q_1, q_2) \cdot (B_1, B_2, I, J) = (q_1B_1, q_2B_2, I, q_1q_2J). \tag{11}$$

Instanton Partition Function

The fix point under the equivariant torus action by

$$q_1 B_1 = v B_1 v^{-1},\ q_2 B_2 = v B_2 v^{-1},\ I = v I \eta^{-1},\ q_1 q_2 J = v^{-1} J \eta. \quad (12)$$

Let $v = e^{\phi}$, $\eta = e^{\mathbf{a}}$ with $\phi \in \mathrm{Lie}(\mathrm{GL}(k))$, $\mathbf{a} = (a_1, \ldots, a_N) \in \mathrm{Lie}(\mathrm{GL}(N))$ with $I = \bigoplus_{\alpha=1}^{N} I_\alpha$, $J = \bigoplus_{\alpha=1}^{N} J_\alpha$. The infinitesimal transformation above is

$$[\phi, B_1] = \varepsilon_1 B_1,\ [\phi, B_2] = \varepsilon_2 B_2,\ \phi I_\alpha = a_\alpha I_\alpha,\ J_\alpha \phi = (a_\alpha - \varepsilon_+) J_\alpha \quad (13)$$

Instanton Partition Function

The $\mathcal{N}=2$ 4d instanton partition function obtained equivariant integral over the moduli space is evaluated over

$$\mathcal{Z}_k = \frac{1}{k!}\left(\frac{\varepsilon_+}{\varepsilon_1\varepsilon_2}\right)^k \oint \prod_{j=1}^{k} \frac{d\phi_j}{2\pi\imath}\mathbb{E}\left[(1-e^m)(NK^* + q_+N^*K - P_{12}KK^*)\right] \tag{14}$$

with $N=\sum_{\alpha=1}^{N} e^{a_\alpha}$ and $K=\sum_{j=1}^{k} e^{\phi_j}$ are the characters of the vector spaces $\mathbf{N}$ and $\mathbf{K}$, $q_i = e^{\varepsilon_i}$, $q_+ = q_1 q_2$, $P_i = 1-q_i$, $P_{12} = P_1 P_2$.
Given a virtual character $X = \sum_a \mathsf{n}_a e^{\mathsf{x}_a}$ we denote by $X^* = \sum_a \mathsf{n}_a e^{-\mathsf{x}_a}$ its dual virtual character.
the index functor $\mathbb{E}$ that converts the additive Chern class character to multiplicative class

$$\mathbb{E}\left[\sum_a \mathsf{n}_a e^{\mathsf{x}_a}\right] = \prod_a \mathsf{x}_a^{-\mathsf{n}_\alpha} \tag{15}$$

Instanton Partition Function

The contour integration pick up poles at $\phi_j = a_\alpha$ or $\phi_j = \phi_{j'} + \varepsilon_{1,2}$. The instanton configuration thus can be represented by a set of N Young diagrams $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(N)})$, $\lambda^{(\alpha)} = (\lambda_1^{(\alpha)}, \lambda_2^{(\alpha)}, \dots)$ satisfying

$$\lambda_{\mathrm{i}}^{(\alpha)} \geq \lambda_{\mathrm{i}+1}^{(\alpha)} \geq 0 \tag{16}$$

for $\alpha = 1, \dots, N$, $\mathrm{i} = 1, 2 \dots$. The character K is

$$K = \sum_{\alpha=1}^{N} \sum_{\square \in \lambda^{(\alpha)}} e^{c_\square}. \tag{17}$$

with $c_\square = a_\alpha + (\mathrm{i}-1)\varepsilon_1 + (\mathrm{j}-1)\varepsilon_2$.

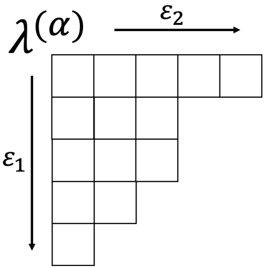

Figure: Young diagram $\lambda^{(\alpha)} = (5, 3, 3, 2, 1)$.

Instanton Partition Function

The instanton partition function is an ensemble over all instanton configurations labeled by N Young diagrams $\boldsymbol{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(N)})$.

[Nekrasov '02]

$$\mathcal{Z}(\mathbf{a}, m, \mathfrak{q}, \boldsymbol{\varepsilon}) = \sum_{\boldsymbol{\lambda}} \mathfrak{q}^{|\boldsymbol{\lambda}|} \mathcal{Z}[\boldsymbol{\lambda}]. \tag{18}$$

The pseudo-measure (combining with 1-loop contribution) is

$$\mathcal{Z}[\boldsymbol{\lambda}] = \prod_{(\alpha \mathrm{i}) \neq (\beta \mathrm{j})} \frac{\Gamma(\varepsilon_2^{-1}(x_{\alpha \mathrm{i}} - x_{\beta \mathrm{j}} - \varepsilon_1))}{\Gamma(\varepsilon_2^{-1}(x_{\alpha \mathrm{i}} - x_{\beta \mathrm{j}}))} \frac{\Gamma(\varepsilon_2^{-1}(x_{\alpha \mathrm{i}} - x_{\beta \mathrm{j}} - m))}{\Gamma(\varepsilon_2^{-1}(x_{\alpha \mathrm{i}} - x_{\beta \mathrm{j}} - m - \varepsilon_1))} \tag{19}$$

with $x_{\alpha \mathrm{i}} = a_\alpha + (\mathrm{i} - 1)\varepsilon_1 + \lambda_{\mathrm{i}}^{(\alpha)} \varepsilon_2$.

Defect in Gauge Theory

The first co-dimensional two surface defect is introduced in the form of a $\mathbb{Z}_N$ orbifold acting on $\mathbb{R}^4 = \mathbb{C}_1 \times \mathbb{C}_2$ by $(\mathbf{z}_1, \mathbf{z}_2) \to (\mathbf{z}_1, \xi \mathbf{z}_2)$ with $\xi^N = 1$. The Orbifold creates a chainsaw quiver structure: [Kanno Tachikawa '11, Nakajima '11]

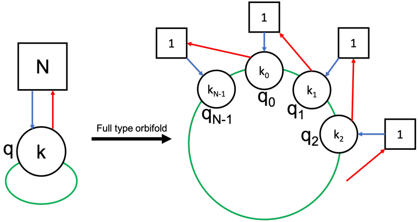

Figure: Orbifolding creates chainsaw quiver structure.

N Fractional instanton counting parameter $\mathfrak{q}_\omega$ satisfy

$$\mathfrak{q} = \prod_{\omega=0}^{N-1} \mathfrak{q}_\omega, \quad \mathfrak{q}_{\omega-1} = \frac{z_{\omega+1}}{z_\omega}. \tag{20}$$

Defect in Gauge Theory

An orbifold defect is characterized by a *coloring function* $c : [N] \to \mathbb{Z}_N$. The coloring functions assigns each Coulomb moduli parameter a_α to the $\mathcal{R}_{c(\alpha)}$ representation of $\mathbb{Z}_N$. The Young diagrams are also colored accordingly.

$$\mathcal{K}_\omega = \left\{ (\alpha, (\mathrm{i}, \mathrm{j})) \mid \alpha \in [N], \ (\mathrm{i}, \mathrm{j}) \in \lambda^{(\alpha)}, \right. \\ \left. \alpha + j - 1 = \omega \bmod N \right\} \tag{21}$$

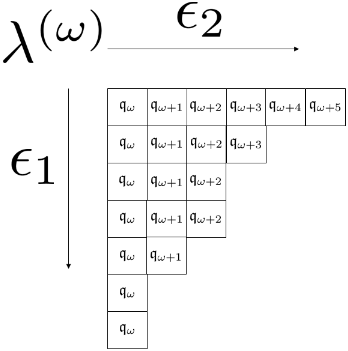

Figure:

Defect in Gauge Theory

The ADHM data is charges under the orbifold, characterized by a coloring function $c : \{1, \dots, N\} \to \mathbb{Z}_N$:

$$\begin{aligned} \hat{N} &= \sum_{\omega=0}^{N-1} N_\omega q_2^{\frac{\omega}{N}} \mathcal{R}_\omega, \quad N_\omega = \sum_{c(\alpha)=\omega} e^{\tilde{a}_\omega}, \; a_\alpha - \frac{\varepsilon_2}{N} c(\alpha) = \tilde{a}_\alpha; \\ \hat{K} &= \sum_{\omega=0}^{N-1} K_\omega q_2^{\frac{\omega}{N}} \mathcal{R}_\omega, \quad K_\omega = \sum_\alpha e^{\tilde{a}_\alpha} \sum_{J=0}^{\infty} \sum_{\substack{(i,j)\in\lambda^{(\alpha)} \\ c(\alpha)+j-1=\omega+NJ}} q_1^i q_2^J. \end{aligned} \tag{22}$$

The defect partition function is an integral over the $\mathbb{Z}_N$ invariant parts

$$\begin{aligned} \mathcal{Z}_{\text{defect}}(\mathbf{a}, m, \vec{\varepsilon}, \vec{z}; \mathfrak{q}) &= \sum_{\boldsymbol{\lambda}} \prod_{\omega} \left(\frac{z_{\omega+2}}{z_{\omega+1}} \right)^{|\mathcal{K}_\omega|} \mathcal{Z}_{\text{defect}}[\boldsymbol{\lambda}] \\ \mathcal{Z}_{\text{defect}}[\boldsymbol{\lambda}] &= \mathbb{E}\left[(1 - e^m)(\hat{N}\hat{K}^* + \hat{q}_{12}\hat{N}^*\hat{K} - \hat{P}_1\hat{P}_2\hat{K}\hat{K}^*) \right]^{\mathbb{Z}_N}. \end{aligned} \tag{23}$$

Defect in Gauge Theory

The defect partition function $\mathcal{Z}_{\text{defect}}$ in the NS-limit $\varepsilon_2 \to 0$ has the asymptotic

$$\mathcal{Z}_{\text{defect}} = e^{\frac{1}{\varepsilon_2}\mathcal{W}(\mathbf{a},m,\mathfrak{q},\vec{\varepsilon})} \left(\mathcal{Z}_{\text{surface}}(\mathbf{a}, m, \vec{\varepsilon}, \vec{\mathfrak{q}}) + \mathcal{O}(\varepsilon_2) \right) \tag{24}$$

with the singular part is identical to the bulk instanton partition function

$$\mathcal{W} = \lim_{\varepsilon_2 \to 0} \varepsilon_2 \log \mathcal{Z}_{\text{inst}}. \tag{25}$$

The leading order contribution $\mathcal{Z}_{\text{surface}}$ is the *surface partition function.*

Defect in Gauge Theory

By studying qq-character BPS observable, one can prove that the surface partition function with perturbative prefactor

$$\Psi(\mathbf{a},\vec{z},\varepsilon_1,\kappa,\mathfrak{q}) = \prod_{\omega=1}^{N} z_\omega^{-\frac{a_{c^{-1}(\omega-1)}}{\varepsilon_1}+\kappa\left(\omega-\frac{N-1}{2}\right)} \mathcal{Z}_{\text{surface}}(\mathbf{a},m,\varepsilon_1,\vec{z},\mathfrak{q}) \quad (26)$$

is the eigenfunction of the Hamiltonian of N interacting particles

[Nekrasov '17][Chen-Kimura-Lee '19]

$$\hat{\mathcal{H}} = \frac{1}{2}\sum_{\alpha=1}^{N}(\nabla_\alpha^z)^2 + \kappa\sum_{\alpha=1}^{N}\left(\nabla_\alpha^z \log\Theta_{A_{N-1}}(\vec{z};\mathfrak{q})\right)\nabla_\alpha^z$$
$$+\frac{\kappa^2}{2}\sum_{\alpha=1}^{N}\left[\left((\nabla_\alpha^z)^2\log\Theta_{A_{N-1}}(\vec{z};\mathfrak{q})\right)+\left(\nabla_\alpha^z\log\Theta_{A_{N-1}}(\vec{z};\mathfrak{q})\right)^2\right] \quad (27)$$

with $\varepsilon_1\kappa = m+\varepsilon_1$ and the eigenvalue

$$E = \frac{1}{\varepsilon_1}\mathfrak{q}\frac{\partial\mathcal{W}}{\partial\mathfrak{q}} + \sum_{\alpha=1}^{N}\frac{a_\alpha^2}{2\varepsilon_1^2} - \frac{\kappa^2}{2}\frac{N(N^2-1)}{12}. \quad (28)$$

Defect in Gauge Theory

Let us consider the weak coupling limit $\mathfrak{q} \to 0$, which translate to one of the fractional coupling $\mathfrak{q}_{N-1} \to 0$ in the quiver gauge theory. There are no instanton in the bulk in this limit, giving a vanishing superpotential

$$\mathcal{W} = 0. \tag{29}$$

Given an empty bulk instanton configuration, there can still be non-trivial surface instantons. The width of the coloring partition is limited

$$\lambda_i^{(\alpha)} \leq N - c(\alpha), \ \forall i = 1, 2, \ldots \implies \lambda^{T,(\alpha)}_{N-c(\alpha)} = 0 \tag{30}$$

For later convenience, we will take a transpose for all Young diagrams $\lambda^{(\alpha)} \to \lambda^{T,(\alpha)}$.

The surface partition function becomes

$$\mathcal{Z}_{\text{surface}} = \sum_{\boldsymbol{\lambda}} \prod_{\omega=1}^{N-1} \mathfrak{q}_{\omega-1}^{\sum_{c(\alpha)<\omega} \lambda^{(\alpha)}_{\omega-c(\alpha)}} \prod_{c(\alpha),c(\beta)<\omega} \frac{\Gamma\left(\frac{y_{\omega,\alpha}-y_{\omega,\beta}-m}{\varepsilon_1}\right)}{\Gamma\left(\frac{y_{\omega,\alpha}-y_{\omega,\beta}}{\varepsilon_1}\right)} \times \prod_{c(\alpha)<\omega+1,\ c(\beta)<\omega} \frac{\Gamma\left(\frac{y_{\omega+1,\alpha}-y_{\omega,\beta}}{\varepsilon_1}\right)}{\Gamma\left(\frac{y_{\omega+1,\alpha}-y_{\omega,\beta}-m}{\varepsilon_1}\right)} \tag{31}$$

with $y_{\omega,\alpha} := a_\alpha + \lambda^{(\alpha)}_{\omega-c(\alpha)}\varepsilon_1$. The ensemble runs over all Young diagram with limit width $\lambda^{(\alpha)}_{N-c(\alpha)} = 0$.

Defect in Gauge Theory

The Hamiltonian $\hat{\mathcal{H}}$ in the weak coupling limit

$$\lim_{\mathfrak{q}\to 0} \hat{\mathcal{H}} = \frac{1}{2}\sum_{\alpha=1}^{N} (\nabla_\alpha^z)^2 + \frac{\kappa}{2}\sum_{\alpha>\beta} \frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \left(\nabla_\alpha^z - \nabla_\beta^z\right). \tag{32}$$

is identify $\hat{\mathcal{H}}$ as half of the Laplace-Beltrami operator:

$$\mathcal{H}_{\mathrm{LB}}(\kappa) = \sum_{\alpha=1}^{N} (\nabla_\alpha^z)^2 + \kappa \sum_{\alpha>\beta} \frac{z_\alpha + z_\beta}{z_\alpha - z_\beta} \left(\nabla_\alpha^z - \nabla_\beta^z\right). \tag{33}$$

The surface partition function in the weak coupling limit is eigenfunction of the Laplace-Beltrami operator with eigen value

$$E = \frac{\vec{a}^2}{\varepsilon_1^2} - \kappa^2 \frac{N(N^2-1)}{12} \tag{34}$$

Quantization

The D-brane construction of the $\mathcal{N} = 2^*$ $U(N)$ gauge theory is realized as follows:

- A single NS5-brane is introduced in the 012345 direction.
- N D4 branes in the 01236 direction. The adjoint mass is introduced by a twisted boundary condition on the $x^{4,5}$ plane. The two ends of the D4 brane on the NS5 brane no longer meets but separated by m.

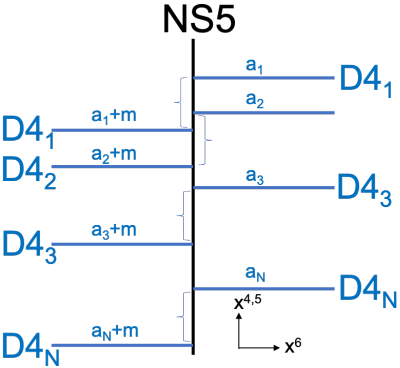

Figure: D-brane engineering of the $\mathcal{N} = 2^*$ gauge theory.

Quantization

We now introduce a second surface defect by Higgsing the Coulomb moduli parameters.

$$a_j - a_{j+1} + m + \varepsilon_1 = (n_{j+1} - n_j)\varepsilon_1, \tag{35}$$

with $j = 1, \ldots, N-1$, $n_j \geq n_{j+1}$. This condition imposes a locus on the Higgs branch where it meets the Coulomb branch, known as the *root of Higgs branch* for $\mathcal{N} = 2^*$ theory.

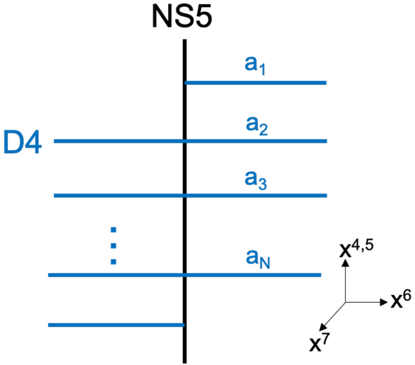

Figure: D4-branes join together to form a helical D4 in the absence of magnetic flux ($n_j = 0$ for all j).

The physical interpretation of n_j is turning on a magnetic flux in the 23-direction for the j-th $U(1)$ factor in the $U(N)$ gauge group

$$\frac{1}{2\pi}\int (F_{23})_j dx_2 \wedge dx_3 = n_j. \quad (36)$$

We denote the set of these $U(1)$ fluxes by $\mathbf{n} = (n_j)_{j=1,\dots,N}$.

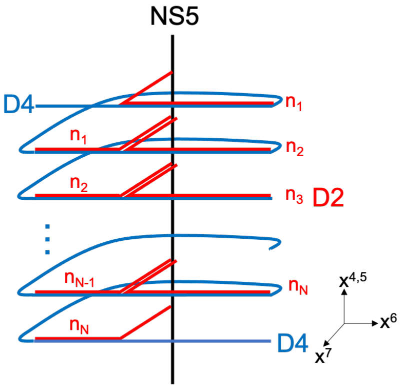

Figure: n_j D2-branes realizing the magnetic flux dissolving into j-th D4.

Quantization

The Higgsing restricts the number of squares in the Young daigram each column to obey

$$n_\beta - n_{\beta+1} + \lambda^{(\beta+1)}_{\omega-\beta} \geq \lambda^{(\beta)}_{\omega-\beta}. \tag{37}$$

The partition function $\Psi_{\mathbf{n}}(\vec{z})$ consists only finite number of terms in the ensemble. Its eigenvalue of the Laplace-Beltrami operator is

$$E(\mathbf{n}) = \sum_{\alpha=1}^{N} \left(n_\alpha - \kappa \left(\alpha - \frac{N+1}{2} \right) \right)^2 - \kappa^2 \frac{N(N^2-1)}{12} \tag{38}$$

Quantization

Example: $N = 3$, $\mathbf{n} = (2,1,0)$. There are 8 instanton configurations.

$\boldsymbol{\lambda}$	counting	measure	$\boldsymbol{\lambda}$	counting	measure
$(\varnothing, \varnothing, \varnothing)$	$z_1^2 z_2$	1	$(\square, \square, \varnothing)$	$z_1 z_2 z_3$	$\frac{2\kappa}{\kappa+1}$
$(\square, \varnothing, \varnothing)$	$z_1 z_2^2$	1	$(\yng(1,1), \square, \varnothing)$	$z_1 z_3^2$	1
$(\yng(1,1), \varnothing, \varnothing)$	$z_1 z_2 z_3$	$\frac{2\kappa}{\kappa+1} \frac{\kappa+2}{2\kappa+1}$	$(\yng(2), \square, \varnothing)$	$z_2^2 z_3$	1
$(\varnothing, \square, \varnothing)$	$z_1^2 z_3$	1	$(\yng(2,1), \square, \varnothing)$	$z_2 z_3^2$	1

This gives

$$\begin{aligned} \Psi_{(2,1,0)}(\vec{z}) = {} & z_1^2 z_2 + z_1^2 z_3 + z_1 z_2^2 + z_1 z_3^2 \\ & + z_2^2 z_3 + z_2 z_3^2 + \frac{6\kappa}{2\kappa+1} z_1 z_2 z_3. \end{aligned} \tag{39}$$

Jack Polynomial

We now introduce an alternative to denote the instanton configuration using a single Young tableaux.
We consider a semi-standard Young tableaux $\mathbf{T}_\mathbf{n}[\boldsymbol{\lambda} = \varnothing]$ of shape $\mathbf{n} = (n_1, \dots, n_N)$. Given a Young diagram $\boldsymbol{\lambda}$, the reading on the j-th square at α-row of the Young Tableaux $\mathbf{T}_\mathbf{n}[\boldsymbol{\lambda}]$ by

$$T_{\alpha,j} = \lambda^{T,(\alpha)}_{n_\alpha+1-j} + \alpha \tag{40}$$

Jack Polynomial

Example: $N = 3$, $\mathbf{n} = \ydiagram{2,1} = (2,1,0)$. The eight instanton configurations:

$$
\begin{aligned}
&\mathbf{T}_{\mathbf{n}}[(\varnothing,\varnothing,\varnothing)] = \ytableaushort{11,2}, &\quad& \mathbf{T}_{\mathbf{n}}[(\square,\varnothing,\varnothing)] = \ytableaushort{12,2}, \\
&\mathbf{T}_{\mathbf{n}}[(\ydiagram{1,1},\varnothing,\varnothing)] = \ytableaushort{13,2}, && \mathbf{T}_{\mathbf{n}}[(\varnothing,\square,\varnothing)] = \ytableaushort{11,3}, \\
&\mathbf{T}_{\mathbf{n}}[(\square,\square,\varnothing)] = \ytableaushort{12,3}, && \mathbf{T}_{\mathbf{n}}[(\ydiagram{1,1},\square,\varnothing)] = \ytableaushort{13,3}, \\
&\mathbf{T}_{\mathbf{n}}[(\ydiagram{2},\square,\varnothing)] = \ytableaushort{22,3}, && \mathbf{T}_{\mathbf{n}}[(\ydiagram{2,1},\square,\varnothing)] = \ytableaushort{23,3}.
\end{aligned}
\tag{41}
$$

Jack Polynomial

Given a Young Tableaux $\mathbf{T_n}[\boldsymbol{\lambda}]$ whose largest reading is less or equal to N (not necessary equal to N). We can define a series of sub Young Tableaux

$$\varnothing = \mathbf{T}_{\mathbf{n}}^{(0)}[\boldsymbol{\lambda}] \subset \mathbf{T}_{\mathbf{n}}^{(1)}[\boldsymbol{\lambda}] \subset \cdots \subset \mathbf{T}_{\mathbf{n}}^{(N)}[\boldsymbol{\lambda}] = \mathbf{T_n}[\boldsymbol{\lambda}] \tag{42}$$

The sub Young Tableaux $\mathbf{T}_{\mathbf{n}}^{(i)}[\boldsymbol{\lambda}] = \mathbf{n}^{(i)} = (n_1^{(i)}, n_2^{(i)}, \dots, n_N^{(i)})$ has its reading less or equal to i.
The weight t_α of the Young Tableaux $\mathbf{T_n}[\boldsymbol{\lambda}]$ denotes the number of squares in the Young tableaux which has reading α:

$$t_\alpha = |\mathbf{n}^{(\alpha)}| - |\mathbf{n}^{(\alpha-1)}|. \tag{43}$$

where $|\mathbf{n}^{(\alpha)}| = \sum_{j=1}^{\alpha} n_j^{(\alpha)}$.

Surface partition Ψ is now an ensemble over the Young Tableaux. Using the combinatorial formula of Jack polynomial, we find

$$\Psi(\mathbf{n}, \varepsilon_1, m, \vec{z}) = \sum_{\mathbf{T_n}} z^{\mathbf{T_n}} \psi_{\mathbf{T_n}} = J_{\mathbf{n}}^{\frac{1}{\kappa}}(\vec{z}) \tag{44}$$

where $z^{\mathbf{T_n}} = \prod_{j=1}^{N} z_i^{t_i}$.

Outline

Quantum Hall States

Let us consider N non-relativistic electrons gas moving on a two dimensional surface Σ in the presence of external magnetic field $B = dA$. A is the connection of some principle $U(1)$ bundle P over Σ. Let $\mathbf{x}_1, \ldots, \mathbf{x}_N \in \Sigma$ be the positions of electrons. The Hamiltonian governing the electrons is

$$\hat{\mathrm{H}} = \frac{1}{2m} \sum_{j=1}^{N} \nabla_j^2 + \sum_{j \neq k} U(\mathbf{x}_j, \mathbf{x}_k). \tag{45}$$

with $\nabla_j = -i\hbar d_j + A$, $\nabla^2 = \nabla \star \nabla$. The Hamiltonian acts on N-particle states $\Psi \in \mathcal{H} = \Lambda^N H$. H is the space of single particle solution.

Quantum Hall States

Let's take $\Sigma = \mathbb{C}$ and choose the gauge $A = \frac{B_0}{2}(-xdy + ydx)$. The single particle Hamiltonian is

$$\begin{aligned} \hat{\mathrm{H}}_{\mathrm{sp}} &= \frac{1}{2m}\left[\left(-2\hbar\partial_z + \frac{B_0}{2}\bar{z}\right)\left(2\hbar\partial_{\bar{z}} + \frac{B_0}{2}z\right) + \hbar B_0\right] \\ &= \hbar\omega_c\left(\hat{\mathrm{a}}^\dagger\hat{\mathrm{a}} + \frac{1}{2}\right),\ \hat{\mathrm{a}} = \frac{1}{\sqrt{\hbar B_0}}\left(2\hbar\partial_{\bar{z}} - \frac{B_0}{2}z\right) \end{aligned} \tag{46}$$

where $\omega_c = \frac{\hbar B_0}{m}$. By scaling z and $\bar{z}$, the eigenstate

$$|m,n\rangle = \frac{1}{\sqrt{2\pi m!n!}} e^{\frac{|z|^2}{4}} \frac{\partial^m}{\partial \bar{z}^m}\frac{\partial^n}{\partial z^n} e^{-\frac{|z|^2}{2}} \tag{47}$$

is degenerate on each Landau level

$$\hat{\mathrm{H}}_{\mathrm{sp}}|m,n\rangle = \hbar\omega_c\left(n + \frac{1}{2}\right)|m,n\rangle \tag{48}$$

Quantum Hall States

The many body Hamiltonian is

$$\hat{\mathrm{H}} = \sum_{j=1}^{N} \nabla_j^2 + V(z_j) + \sum_{j>k} \frac{1}{|z_j - z_k|} \tag{49}$$

where V is a potential generated by uniform neutralizing background. Laughlin pointed out the $\frac{1}{3}$ filling factor state can be realized with solely the lowest Landau level. The Laughlin wave function

$$\psi_L^{(m)} = \prod_{j>k} (z_j - z_k)^m e^{-\frac{|z|^2}{4}} \tag{50}$$

m is an odd integer to ensure the wave function is anti-symmetric.

[Laughlin '89]

Quantum Hall States

The Laughlin wavefunction not only accurately models the simplest abelian fractional quantum Hall states. It serves as the building block of more general cases both abelian and non-abelian including Moore-Read state and Read-Rezayi state.

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Apart form the Gaussian factor (which we will drop from now on), wavefunction of these models are conformally-invariant multivariable polynomials.

The simplest model of FHS considers anti-symmetric polynomials. However it is also useful to study symmetric polynomial describing a bosonic FHS from which they are obtained by multiplying an odd power of Vandermonde determinant.

Quantum Hall States

Bernevig and Haldane established a framework to describe bosonic fractional quantum Hall effect states using Jack symmetric polynomial.

[Bernevig-Haldane '07]

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The Jack polynomial $J^{\alpha}_{\mathbf{n}}(\mathbf{z})$ is a symmetric polynomial in the variables $\mathbf{z} = \{z_1, \ldots, z_N\}$ labeled by the partition $\mathbf{n} = (n_1, \ldots, n_N)$, $n_i \geq n_{i+1}$, and a parameter α.

Example:

$$\begin{aligned} J^{\alpha}_{(1)}(\mathbf{z}) &= p_1; \\ J^{\alpha}_{(2)}(\mathbf{z}) &= \frac{\alpha}{1+\alpha} p_2 + \frac{1}{1+\alpha} p_1^2 \end{aligned} \tag{51}$$

where $p_n = \sum_{j=1}^{N} z_j^n$. General case are defined recursively or by combinatorial formula.

Quantum Hall States

A partition $\mathbf{n}$ is called (k,r)-*admissible* if

$$n_{i+k} - n_i \geq r, \quad i = 1, \ldots, N-k. \tag{52}$$

Feigin et. al. found that Jack polynomial defined on (k,r)-admissible partition naturally implements a *generalized Pauli principle*:

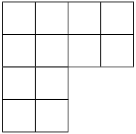
$$(k,r)\text{-admissible Jack } J_{\mathbf{n}}^{\alpha=-\frac{k+1}{r-1}}(\mathbf{z}) = 0 \text{ if } k+1 \text{ variables } z_i \text{ coincide.}$$

[Feigin-Jimbo-Miwa-Mukhin '02]

We denote $\mathbf{n}^{\circ}_{k,r}$ the (k,r)-admissible partition $\mathbf{n}$ that minimize $|\mathbf{n}|$.

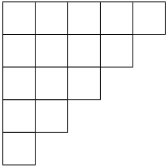
Example: Consider $N = 6$:

$$\mathbf{n}^{\circ}_{(2,2)} = (4,4,2,2,0,0) = \tag{53}$$



is $(k = 2, r = 2)$ admissible.

$$\mathbf{n}^{\circ}_{(1,1)} = (5,4,3,2,1,0) = \tag{54}$$



is both $(k = 2, r = 2)$-admissible and $(k = 1, r = 1)$-admissible.

Quantum Hall State

Occupation number $l_{\mathtt{m}}$ of each of the lowest Landau level orbits with angular momentum $L = \mathtt{m}\hbar$ is given by the multiplicity $l_{\mathtt{m}}$ of $\mathtt{m}$ in $\mathbf{n}$.

$$(n_1, \ldots, n_N) \leftrightarrow [0^{l_0} 1^{l_1} 2^{l_2} \cdots] \tag{55}$$

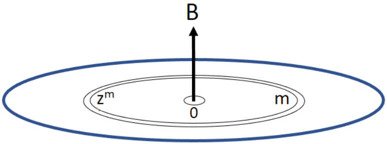


Figure: Orbital occupation in the Landau problem on a disk.

Quantum Hall State

The (bosonic) fractional quantum Hall state wavefunctions of filling fraction $\nu = \frac{k}{r}$ can be explicitly written as a single (k, r)-admissible Jack symmetric polynomial with $\alpha_{k,r} = -\frac{k+1}{r-1}$ (multiply a Gaussian factor). [Bernevig-Haldane '07]

- Laughlin: $\psi_{\mathrm{L}}^{(r)}(\mathbf{z}) = \prod_{i<j}(z_i - z_j)^r = J^{\alpha_{1,r}}_{\mathbf{n}^\circ_{1,r}}(\mathbf{z})$;
- Moore-Read: $\psi_{\mathrm{MR}}(\mathbf{z}) = \prod_{i<j}(z_i - z_j)\mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) = J^{\alpha_{2,2}}_{\mathbf{n}^\circ_{2,2}}(\mathbf{z})$;
- Read-Rezayi: $\psi_{\mathrm{RR}}(\mathbf{z}) = J^{\alpha_{k,2}}_{\mathbf{n}^\circ_{k,2}}(\mathbf{z})$;

Outline

Summary

- By introducing two types of surface defects, orbifold and Higgsing, we find the partition function of $\mathcal{N} = 2^*$ four dimensional $U(N)$ gauge theory in the weak coupling limit yields symmetric Jack polynomials.
- The Quantum Hall state wavefunctions of filling fractions $\nu = \frac{k}{r}$ can be explicitly written in a single (k, r)-admissible Jack polynomial with $\alpha = -\frac{k+1}{r-1}$.
- Remark: The gauge theory computation can be extended to 5d. One obtains MacDonald polynomial.

Thank you for your attention.