The panorama of Spin Matrix theory

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- Decoupling limit

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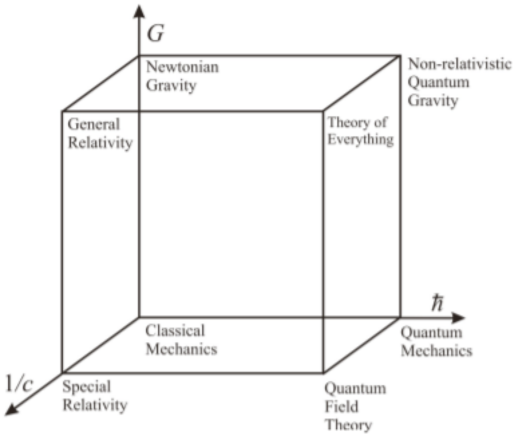
The theory cube



AdS/CFT motivation

$\mathcal{N}=4$ SYM in adjoint of SU(N) group $\leftrightarrow$ Type IIB strings in AdS$_5 \times S^5$:
Believed to be true for all couplings [Maldacena, 1997][Gubser et al, 1998][Witten, 1998]

1. Planar limit $N=\infty$ and the power of integrability [Minahan, Zarembo, 2002][Beisert et al, 2003]
2. Supersymmetric localization [Pestun, 2007]
3. Recent microstate counting of supersymmetric AdS$_5$ black hole [Kim, et al 2018] [Murthy et al, 2018] [Benini, Milan, 2018]

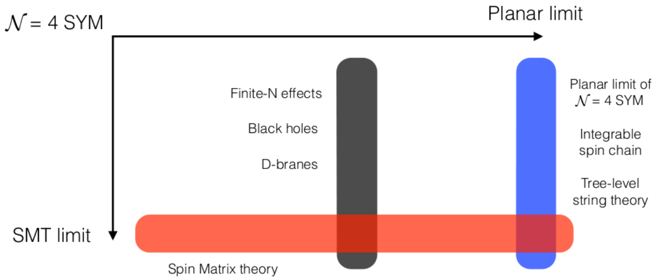
Problem

- Planar limit: gravity enters as $1/N$ perturbative corrections $\Rightarrow$ No access to black holes and D-branes dynamics
- Finite N but weak coupling: string theory is not geometrical

Spin Matrix Theory

Controlled finite N effects (strong coupled dynamics of gravity) and semiclassical geometry: Spin Matrix Theory limits [Harmark, Orselli, 2014].

- Decoupling limits of $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3 \Rightarrow$ the theory reduces to a subsector with only one-loop contributions of the dilatation operator [Harmark, Orselli, 2006][Harmark, Kristjansson, Orselli, 2006-07]
- Approach unitarity (BPS) bounds
- Understand how quantum gravity gets simplified in non-relativistic limit (as expansions of c^{-1})



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Contents of $\mathcal{N} = 4$ SYM

The set of letters of $\mathcal{N} = 4$ SYM

- 6 indepedent gauge components $F_{\pm,0}, \bar{F}_{\pm,0}$
- 6 complex scalars $Z, W, X, \bar{Z}, \bar{W}, \bar{X}$
- 16 complex fermions $\chi_i, \bar{\chi}_i$, $i = 1, .., 8$
- 4 components of covariant derivatives $d_{1,2}$ and $\bar{d}_{1,2}$

The letters are specified by dimension D_0, $SO(4)$ spin (S_1, S_2) and R-charges (Q_1, Q_2, Q_3). The BPS letters are those satisfying

$$D_0 = S_1 + S_2 + Q_1 + Q_2 + Q_3$$

BPS letters

Letter	$SO(4)[S_1, S_2]$	Name in 0510251	$Q = \frac{1}{2}(Q_1 + Q_2)$	Q_3	D_0
Z	$[0, 0]$	Z	$\frac{1}{2}$	0	1
X	$[0, 0]$	X	$\frac{1}{2}$	0	1
W	$[0, 0]$	Y	0	1	1
A	$[1, 1]$	F_{++}	0	0	2
χ_1	$[\frac{1}{2}, -\frac{1}{2}]$	$\psi_{0,+,+++}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
χ_2	$[-\frac{1}{2}, \frac{1}{2}]$	$\psi_{0,-,+++}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$\bar{\chi}_3$	$[\frac{1}{2}, \frac{1}{2}]$	$\psi_{+,0,-++}$	0	$\frac{1}{2}$	$\frac{3}{2}$
$\bar{\chi}_5$	$[\frac{1}{2}, \frac{1}{2}]$	$\psi_{+,0,+-+}$	0	$\frac{1}{2}$	$\frac{3}{2}$
$\bar{\chi}_7$	$[\frac{1}{2}, \frac{1}{2}]$	$\psi_{+,0,++-}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
d_1	$[1, 0]$	∂_{++}	0	0	1
d_2	$[0, 1]$	∂_{+-}	0	0	1

Dirac equation

$$d_1\chi_2 - d_2\chi_1 = 0$$

Decoupling

Assume $(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (n_1\Omega, n_2\Omega, n_3\Omega, n_4\Omega, n_5\Omega)$ are the chemical potentials conjugate to the angular momenta and R-charges. The critical values are denoted as $(n_1, n_2, n_3, n_4, n_5)$, which is reached when $\Omega \to 1$. The critical overall charge is then $J \equiv n_1 S_1 + n_2 S_2 + n_3 Q_1 + n_4 Q_2 + n_5 Q_3$. We are interested in sectors $D_0 = J$. The partition function is

$$\begin{aligned} Z &= \mathrm{Tr}[e^{-\beta D + \beta\Omega J}] \\ &= \mathrm{Tr}[e^{-\beta(D_0 - J) + \beta(1-\Omega)J - \beta\lambda D_2 + \mathcal{O}(\lambda^{\frac{3}{2}})}] \end{aligned}$$

Take the following limit (decoupling limit)

$$\beta \to \infty, \quad \Omega \to 1, \quad \lambda \to 0, \quad \tilde{\beta} = \beta(1-\Omega),\ \beta\lambda \text{ fixed}$$

Then the effective partition function is just

$$Z = \mathrm{Tr}_{D_0 = J}[e^{-\tilde{\beta}(D_0 + \tilde{\lambda} D_2)}]$$

i.e. Only one-loop correction survives in the decoupling limit.

Non-relativistic

Recall that a known fact from special relativity is

$$E = \sqrt{m_0^2 c^4 + p^2 c^2} = m_0^2 c^2 + \frac{p^2}{2m_0} + \mathcal{O}(c^{-2})$$

The Newtonian mechanics is about the dynamics at the order c^0. A well-defined effective theory. The analogy in $\mathcal{N} = 4$ SYM is that we denote the classical conformal dimension by D_0. In the presence of weak interaction parametrized by 't Hooft coupling λ, the conformal dimension will receive the quantum correction

$$D = D_0 + \lambda D_2 + ...$$

where D_2 is the one loop correction. We can decouple the higher order of Feymann diagram corrections the same as we do for non-relativistic mechanics.

Spin Matrix theory [Harmark, Orselli, 2014]

Constructing D_2; as its letters carry both matrix indices from SU(N) and spin group indices from subgroup of PSU$(2, 2|4)$.

Magnon example

The dispersion of a single magnon in $\mathcal{N} = 4$ SYM is [Beisert, 2005]

$$E - Q = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\frac{p}{2}} - 1$$

Take the small momenta limit, we have

$$E - Q \approx \sqrt{1 + \frac{\lambda p^2}{4\pi^2}} - 1$$

In the SMT decoupling limit, $\lambda \to 0$, this becomes

$$E - Q = \frac{\lambda p^2}{8\pi^2}$$

Zoo of decoupling limits

Without going to the details, it has been explored by [Harmark et al, 2007] to find all the possible decoupling limits. Except the trivial $U(1)$ decoupling limit, there are 12 nontrivial decoupling limits

- There are letters $D_0 = J$
- All the letters satisfy $D_0 \geq J$

The compact subsectors are [Harmark, Orselli, 2014; Baiguera, Harmark, YL, 2021]

- $SU(2)$ limit, $\vec{n} = (0,0,1,1,0)$. Letter: Z, X.
- $SU(1|1)$ limit ($XX_{\frac{1}{2}}$ Heisenberg spin chain), $\vec{n} = (\frac{2}{3}, 0, 1, \frac{2}{3}, \frac{2}{3})$. Letter: Z, χ_1
- $SU(1|2)$ limit ($t-J$ model), $\vec{n} = (\frac{1}{2}, 0, 1, 1, \frac{1}{2})$. Letter: Z, X, χ_1 [Beisert, Staudacher, 2005]
- $SU(2|3)$ limit, $\vec{n} = (0,0,1,1,1)$. Letter: Z, X, W, χ_1, χ_2

Zoo of decoupling limits

Non-compact SU$(1,1)$ kind subsectors are [Baiguera, Harmark, YL, Wintergerst, 2020, 2021]

- Bosonic $SU(1,1)$ limit ($XXX_{-\frac{1}{2}}$ Heisenberg model), $\vec{n} = (1,0,1,0,0)$. Letter: $d_1^n Z$.
- Fermionic $SU(1,1)$ limit, $\vec{n} = (1,0,\frac{2}{3},\frac{2}{3},\frac{2}{3})$. Letter: $d_1^n \chi_1$.
- SU$(1,1|1)$ limit, $\vec{n} = (1,0,1,\frac{1}{2},\frac{1}{2})$, Letters $d_1^n Z, d_1^n \chi_1$
- PSU$(1,1|2)$ limit, $\vec{n} = (1,0,1,1,0)$. Letters: $d_1^n Z, d_1^n X, d_1^n \chi_1, d_1^n \bar{\chi}_7$

Non-compact SU$(1,2)$ kind subsectors are [Baiguera, Harmark, YL, Wintergerst, 2020, 2022]

- SU$(1,2)$ limit, $\vec{n} = (1,1,0,0,0)$. Letter: $d_1^n d_2^k A$
- SU$(1,2|1)$ limit, $\vec{n} = (1,1,\frac{1}{2},\frac{1}{2},0)$. Letter: $d_1^n d_2^k A, d_1^n d_2^k \bar{\chi}_7$
- SU$(1,2|2)$ limit, $\vec{n} = (1,1,1,0,0)$. Letter: $d_1^n d_2^k A,\ d_1^n d_2^k Z,\ d_1^n d_2^k \chi_1,\ d_1^n d_2^k \bar{\chi}_7$
- PSU$(1,2|3)$ limit, $\vec{n} = (1,1,1,1,1)$. Letter: $d_1^n d_2^k A,\ d_1^n d_2^k Z,\ d_1^n d_2^k W,\ d_1^n d_2^k X,\ d_1^n d_2^k \chi_{1,2},\ d_1^n d_2^k \bar{\chi}_{3,5,7}$

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Hamiltonian

Effectively, we want to derive the Hamiltonian as

$$H = \frac{1}{N} \sum_{i,j,m,n} U_{mn}^{ij} a_m^\dagger a_n^\dagger a_i a_j$$

Beissert proposed

$$H = -\frac{1}{N} C_{CD}^{AB} \ : \mathrm{Tr}[W_A, \check{W}^C][W_B, \check{W}^D] :$$

Sectors with effective (1+1)-dimensional theories

Focus on BPS bounds

$$H \geq S_1 + \sum_{i=1}^{3} \omega_i Q_i$$

- S_1, S_2 Cartan generators for rotations on S^3
- Q_i Cartan generators of $SU(4)$ R-symmetry group
- ω_i chemical potentials characterizing the bound

Spin Matrix Theory limit

$$\lambda \to 0\,, \qquad H_2 = \frac{H - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{\lambda} \text{ finite}\,, \qquad N \text{ fixed}$$

Sectors	Combination of $SU(4)$ Cartan charges $\sum_{i=1}^{3} \omega_i Q_i$
SU$(1,1)$ bosonic	Q_1
SU$(1,1)$ fermionic	$\frac{2}{3}(Q_1 + Q_2 + Q_3)$
SU$(1,1\|1)$	$Q_1 + \frac{1}{2}(Q_2 + Q_3)$
PSU$(1,1\|2)$	$Q_1 + Q_2$

SU$(1,1)$ subsector

General procedure:

- Isolate the propagating modes in a given near-BPS limit from the quadratic classical Hamiltonian
- Derive the form of the current which couple to the gauge field from the $\mathcal{N}=4$ SYM action – of order λ
- Integrate out additional non-dynamical modes giving rise to effective interactions in a given near-BPS limit
- Derive the interacting Hamiltonian from

$$H_{\text{int}} = \lim_{g\to 0} \frac{H - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{g^2 N}$$

$$H_{\mathsf{int}} = \frac{1}{2N} \sum_{l=1}^{\infty} \frac{1}{l} \mathsf{tr}\left(q_l^\dagger q_l\right)$$

where we defined scalar block as

$$q_l = \sum_{n=0}^{\infty} [\Phi_n^\dagger, \Phi_{n+l}]$$

SU$(1, 1|1)$

The full Hamiltonian of SU$(1, 1)$ bosonic sector is

$$H = L_0 + \frac{\tilde{g}^2}{2N} \sum_{l=0}^{\infty} \frac{1}{l} \mathrm{tr}\left(q_l^\dagger q_l\right)$$

For SU$(1, 1|1)$ subsector including bosons and fermions,

$$H_{\text{int}} = \frac{1}{2N} \sum_{l=0}^{\infty} \frac{1}{l} \mathrm{tr}\left(\hat{q}_l^\dagger \hat{q}_l\right) + \frac{1}{2N} \sum_{l=0}^{\infty} \mathrm{tr}\left(F_l^\dagger F_l\right)$$

where

$$\hat{q}_l = q_l + \tilde{q}_l$$

$$\tilde{q}_l = \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{n+l+1}} \{\psi_n^\dagger, \psi_{n+l}\}$$

$$F_l = \sum_{n=0}^{\infty} \frac{[\psi_{n+l}, \Phi_m^\dagger]}{\sqrt{n+l+1}}$$

SU(1,2) Hamiltonian

Letters: A, d_1, d_2

The block is

$$\mathfrak{q}_{l,\Delta\mu} \equiv \sum_{\mu_1,\mu_2} \sum_{s_2=0}^{\infty} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} \sqrt{\frac{(s_2+1)(s_2+2)}{(s_2+l+1)(s_2+l+2)}} [A^{\dagger}_{s_2\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

The final Hamiltonian is then

$$H_{\text{int}} = \sum_{l=1}^{\infty} \sum_{\Delta\mu=-l}^{l} \frac{1}{l} \text{tr}(\mathfrak{q}^{\dagger}_{l,\Delta\mu} \mathfrak{q}_{l,\Delta\mu})$$

Overall Hamiltonian in $\mathrm{SU}(1,2|2)$

Summing all the interactions, we find

$$H_{\text{int}} = \frac{1}{2N}\sum_{l=1}^{\infty}\sum_{\Delta\mu=-l}^{l}\frac{1}{l}\mathrm{tr}\left(\mathbf{Q}^{\dagger}_{l,\Delta\mu}\mathbf{Q}_{l,\Delta\mu}\right)$$
$$+\frac{1}{2N}\sum_{a=2,3}\sum_{l=0}^{\infty}\sum_{\Delta\mu=-l}^{l}\mathrm{tr}\left((F^{\dagger}_{a}+K^{\dagger}_{a})_{l,\Delta\mu}(F^{a}+K^{a})_{l,\Delta\mu}\right)$$
$$+\frac{1}{2N}\sum_{l=0}^{\infty}\sum_{\Delta\mu=-l}^{l}\mathrm{tr}\left(W^{\dagger}_{l,\Delta\mu}W_{l,\Delta\mu}\right)$$

where

$$\mathbf{Q}_{l,\Delta\mu} \equiv q_{l,\Delta\mu} + \tilde{q}_{l,\Delta\mu} + \mathfrak{q}_{l,\Delta\mu}$$

Blocks

$$q_{l,\Delta\mu} \equiv \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} [\Phi^\dagger_{s_2\mu_2}, \Phi_{s_2+l,\mu_2+\Delta\mu}]$$

$$\tilde{q}_{l,\Delta\mu} \equiv \sum_{a=1,2} \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} \sqrt{\frac{s_2+1}{s_2+l+1}} \{(\zeta^\dagger_a)_{s_2\mu_2}, (\zeta^a)_{s_2+l,\mu_2+\Delta\mu}\}$$

$$\mathfrak{q}_{l,\Delta\mu} \equiv \sum_{\mu_1,\mu_2} \sum_{s_2=0}^{\infty} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} \sqrt{\frac{(s_2+1)(s_2+2)}{(s_2+l+1)(s_2+l+2)}} [A^\dagger_{s_2\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

$$(F^a)_{l,\Delta\mu} \equiv \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{l}{2},\frac{\Delta\mu}{2};\frac{s_2}{2},\frac{\mu_2}{2}} \epsilon^{ab} \frac{[(\zeta_b)_{s_2+l,\mu_2+\Delta\mu}, \Phi^\dagger_{s_2,\mu_2}]}{\sqrt{s_2+l+1}}$$

$$(K^a)_{l,\Delta\mu} \equiv \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} \sqrt{\frac{s_2+1}{(s_2+l+1)(s_2+l+2)}} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{l}{2},\frac{\Delta\mu}{2};\frac{s_2}{2},\frac{\mu_2}{2}} [(\zeta^\dagger_a)_{s_2,\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

$$W_{l,\Delta\mu} = \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} \sqrt{\frac{l+1}{(s_2+l+1)(s_2+l+2)}} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} [\Phi^\dagger_{s_2\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

Representations of SU$(1,2)$

Like the global symmetry of $\mathcal{N}=4$ SYM, the algebra can be represented by oscillators

$$[\mathbf{a}_\alpha, \mathbf{a}^\dagger_\beta] = \delta_{\alpha\beta}, \quad [\mathbf{b}_{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta_{\dot\alpha\dot\beta}, \quad \{\mathbf{c}^\dagger_a, \mathbf{c}_b\} = \delta_{ab}$$

Such that

$$\begin{aligned}
L_0 &= \frac{1}{2}(1+\mathbf{a}^\dagger_1\mathbf{a}_1+\mathbf{b}^\dagger_1\mathbf{b}_1), \quad L_1 = \mathbf{a}^\dagger_1\mathbf{b}^\dagger_1, \quad L_{-1} = \mathbf{a}_1\mathbf{b}_1 \\
\tilde{L}_0 &= \frac{1}{2}(1+\mathbf{a}^\dagger_2\mathbf{a}_2+\mathbf{b}^\dagger_1\mathbf{b}_1), \quad \tilde{L}_1 = \mathbf{a}^\dagger_2\mathbf{b}^\dagger_1, \quad \tilde{L}_{-1} = \mathbf{a}_2\mathbf{b}_1 \\
J_+ &= \mathbf{a}^\dagger_1\mathbf{a}_2, \qquad J_- = \mathbf{a}^\dagger_2\mathbf{a}_1
\end{aligned}$$

There are a few merits using this

- $L_1, \tilde{L}_1$ are d_1, d_2 operations respectively. Two spatial directions are treated equally.
- If $\tilde{L}_{0,\pm}$ is turned off, we can reacquire the algebra SU$(1,1)$ of subsectors. They are ghost like
- The descendants are labelled by (n,k) symmetrically.

Classification of (p, q) representations

Like SU$(1,1)$ shown in the notes, we need to understand how the generators act on a given state and preserve the unitarity. Recall SU(N) has $N-1$ independent Casimir operators. Thus we need

$$C_2 = -1 - x_1x_2 - x_2x_3 - x_3x_1 = p + q + \frac{1}{3}(p^2 + pq + q^2)$$

$$C_3 = x_1x_2x_3 = \frac{1}{27}(p-q)(p+2q+3)(q+2p+3)$$

Remind the Casimir of SU$(1,1)$ is

$$C = -j(j-1)$$

- Principle representation: p, q can be complex (analogous to continuous)
- p-series: p is integer while q is not
- q-series: q is integer while p is not
- $(p+q)$-series: Neither of p, q is integer but $p+q$ is
- Integer series: $p, q \in \mathbb{Z}$
- Supplementary series

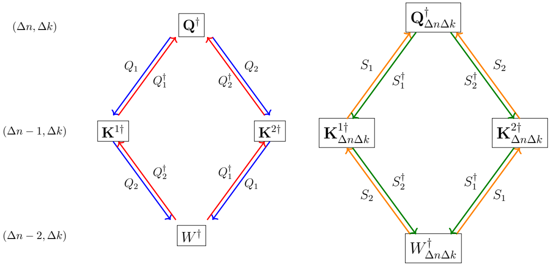
Representations

- **Gauge field**. The gauge field $\bar{F}_+$ is parametrized by $(p, q) = (0, 0)$ representation.
- **Fermion**: The fermions $\bar{\chi}_{5,7}$ are parametrized by $(p, q) = (0, -1)$ representation.
- **Scalar**: The scalar Z is parametrized by $(p, q) = (0, -2)$ representation.

Symmetry actions

What about blocks?

- The block $\mathbf{Q}^{\dagger}_{nk}$ is parametrized by $(p,q)=(0,-3)$ representation. This is a representation of fermion!
- The fermionic block $\mathbf{K}^{a\dagger}_{nk}=(K^{a\dagger}_{nk}+F^{a\dagger}_{nk})$ are parametrized by $(p,q)=(0,-2)$ representation.
- The scalar block $W^{\dagger}_{nk}$ is parametrized by $(p,q)=(0,-1)$ representation.



The blocks and letters in SU$(1,2|2)$ are forming $\mathcal{N}=2$ vector multiplets.

SU(2|3) subsector

The decoupling condition: $H_0 = Q_1 + Q_2 + Q_3$

There are three scalars $\Phi_{1,2,3}$ and two chiral fermions $\chi_{1,2}$ in this sector. The full SMT Hamiltonian by spherical reduction is

$$\begin{aligned} H_{\text{int}} =& \frac{1}{4N}\text{tr}\left([\Phi_b^\dagger, \Phi_a^\dagger][\Phi_a, \Phi_b]\right) + \frac{1}{4N}\text{tr}\left(\{\chi_\beta^\dagger, \chi_\alpha^\dagger\}\{\chi_\alpha, \chi_\beta\}\right) \\ &+ \frac{1}{2N}\text{tr}\left([\Phi_a^\dagger, \chi_\beta^\dagger][\chi_\beta, \Phi_a]\right) \end{aligned}$$

D/F term

D-term means

$$[W^\dagger, W][W^\dagger, W]$$

while F-term means

$$[W, W][W^\dagger, W^\dagger]$$

The Hamiltonian in this subsector are made by F-terms.

PSU(1, 1|2) subsector

The quantum version of the Hamiltonian was obtained in [Bellucci, Casteill, 06]. We have decoupling limit $H_0 = S_1 + Q_1 + Q_2$.

The letters are two scalars $\Phi_{1,2}$, a chiral fermion $\chi_1 = \psi_1$ and antichiral fermion $\bar{\chi}_7 = \psi_2$, including their descendants generated by d_1.

$$H_{\text{int}} = H_B + \frac{1}{N}\sum_{l=1}^{\infty}\frac{1}{l} : \text{tr}\left(\mathbf{Q}_l^\dagger\,\mathbf{Q}_l\right) : + \frac{1}{N}\sum_{l=0}^{\infty} : \text{tr}\Big((F_{ab})_l^\dagger (F_{ab})_l\Big) :$$
$$-\frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\frac{1}{m+n+l+1} : \text{tr}\left(\epsilon^{ac}\epsilon^{bd}[(\Phi_a^\dagger)_m, (\Phi_b)_{m+l}][(\Phi_c^\dagger)_{n+l}, (\Phi_d)_n]\right) :$$
$$+\frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\frac{\sqrt{(m+1)(n+1)}}{\sqrt{(m+l+1)(n+l+1)}}\frac{: \text{tr}\left(\epsilon^{ac}\epsilon^{bd}\{(\psi_a^\dagger)_m, (\psi_b)_{m+l}\}\{(\psi_c^\dagger)_{n+l}, (\psi_d)_n\}\right) :}{m+n+l+2}$$
$$+\frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\sqrt{\frac{m+1}{n+l+1}}\frac{\epsilon^{ac}\epsilon^{bd}}{m+n+l+2} : \text{tr}\left([(\psi_a^\dagger)_m, (\Phi_b)_{m+l+1}][(\psi_c^\dagger)_{n+l}, (\Phi_d)_n]\right) :$$
$$-\frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\sqrt{\frac{m+1}{n+l+1}}\frac{\epsilon^{ac}\epsilon^{bd}}{m+n+l+2} : \text{tr}\left([(\Phi_a^\dagger)_{m+l+1}, (\psi_b)_m][(\Phi_c^\dagger)_n, (\psi_d)_{n+l}]\right) : ,$$

Positiveness

Although the Hamiltonian is tedious, we can show

$$\hat{\mathcal{Q}}^\dagger=\sum_{m,n=0}^{\infty}\Big[\tfrac{1}{\sqrt{n+1}}\mathrm{tr}\Big([(\Phi_a^\dagger)_{m+n+1},(\Phi_a)_m](\psi_2)_n\Big)+\sqrt{\tfrac{m+1}{(n+1)(m+n+2)}}\mathrm{tr}\Big(\{(\psi_1^\dagger)_{m+n+1},(\psi_1)_m\}(\psi_2)_n\Big)$$
$$+\tfrac{1}{2}\sqrt{\tfrac{m+n+2}{(m+1)(n+1)}}\mathrm{tr}\Big(\{(\psi_2^\dagger)_{m+n+1},(\psi_2)_m\}(\psi_2)_n\Big)-\tfrac{1}{2\sqrt{m+n+1}}\,\epsilon^{ab}\mathrm{tr}\Big((\psi_1^\dagger)_{m+n}[(\Phi_a)_m,(\Phi_b)_n]\Big)\Big]$$

We can then show

$$\{\hat{\mathcal{Q}},\hat{\mathcal{Q}}^\dagger\}=H_{\mathrm{int}}$$

The cubic supercharges are from the extra PSU$(1|1)^2$ symmetry of this subsector [Beisert, Zwiebel, 2007]

Representation

How can the positiveness be manifest as square of blocks?

F-term problem

We define

$$J_L = \sum_{n=0}^{L} [\Phi^1_{L-n}, \Phi^2_n]$$

and $L = m + n + l$ We can show

$$-\frac{1}{2N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\frac{1}{m+n+l+1}\mathrm{tr}\left(\epsilon^{ac}\epsilon^{bd}[(\Phi^\dagger_a)_m, (\Phi_b)_{m+l}][(\Phi^\dagger_c)_{n+l}, (\Phi_d)_n]\right)$$

$$= \sum_{L=0}^{\infty}\frac{1}{L+1}\mathrm{tr}(J^\dagger_L J_L)$$

To derive this we need to use the Jacobi identity

$$\mathrm{tr}([\Phi_a, \Phi_b][\Phi^\dagger_b, \Phi^\dagger_a]) = \mathrm{tr}([\Phi_b, \Phi^\dagger_a][\Phi_a, \Phi^\dagger_b]) - \mathrm{tr}([\Phi_a, \Phi^\dagger_a][\Phi_b, \Phi^\dagger_b])$$

Then PSU$(1, 1|2)$ symmetry generator action:

$$(L_+)_D J^\dagger_L = (L+1)J^\dagger_{L+1}\,, \quad (L_+)_D J_L = -(L+1)J_{L-1}$$

Question

In the known case, we have achieve some manifest symmetry structure for two of the $1/8$-BPS subsector:

1. SU$(1,2|2)$ block as $\mathcal{N}=2$ vector multiplet
2. SU$(2|3)$ block as three $\mathcal{N}=1$ chiral multiplet

But we do not know how to organize the Hamiltonian of PSU$(1,1|2)$.

- Why PSU$(1,1|2)$ is not that manifestly positive definite?
- What is the PSU$(1,2|3)$ sector would be like?

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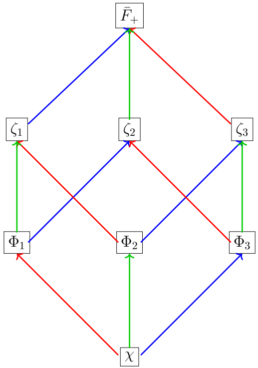
PSU(1, 2|3) letters

PSU(1, 2|3) limit, $\vec{n} = (1,1,1,1,1)$. Letter:
$d_1^n d_2^k A,\ d_1^n d_2^k Z,\ d_1^n d_2^k W,\ d_1^n d_2^k X,\ d_1^n d_2^k \chi_{1,2},\ d_1^n d_2^k \bar{\chi}_{3,5,7}$
Also $d_1\chi_2 - d_2\chi_1 = 0$ due to Dirac equation. We then define the ancestor fermion such that

$$\chi_1 = d_1\chi, \qquad \chi_2 = d_2\chi$$



Cubic supercharges

We want to construct a few cubic supercharges which could result in $H = \{\mathcal{Q}^\dagger, \mathcal{Q}\}$.

$$T_A = \sum_{n,k,n',k'=0}^{\infty} P^{(i,j)}_{n,k,n',k'} \mathrm{tr}([V^\dagger_{n,k} \tilde{V}^\dagger_{n',k'}\} \hat{V}_{n+n',k+k'}) \,,$$

with

$$P^{(i,j)}_{n,k,n',k'} = \sqrt{\frac{(k+n+i-1)!(k'+n'+j-1)!(n+n')!(k+k')!}{(k+k'+n+n'+i+j-1)!n!k!n'!k'!}} \,.$$

We want to have some constraints

- For example: each ingredient is invariant under bosonic symmetry.

$$\{L_+, T_A\}_D = \{L_-, T_A\}_D = \{J_+, T_A\}_D = 0$$

- $\{\mathcal{Q}, Q_a\} = 0.$

Explicit T_A charges

T_A invariant under bosonic symmetry.

$$T_1 = \frac{1}{2} \sum_{n,k,n',k'=0}^{\infty} P^{(0,0)}_{n,k,n',k'} \mathrm{tr}(\chi^\dagger_{n,k} \{\chi^\dagger_{n',k'}, \chi_{n+n',k+k'}\})$$

$$T_2 = \sum_{n,k,n',k'=0}^{\infty} P^{(0,1)}_{n,k,n',k'} \delta^{ab} \mathrm{tr}(\chi^\dagger_{n,k} [(\Phi^\dagger_a)_{n',k'}, (\Phi_b)_{n+n',k+k'}])$$

$$T_3 = \sum_{n,k,n',k'=0}^{\infty} P^{(0,2)}_{n,k,n',k'} \delta^{ab} \mathrm{tr}(\chi^\dagger_{n,k} \{(\zeta^\dagger_a)_{n',k'}, (\zeta_b)_{n+n',k+k'}\})$$

$$T_4 = \sum_{n,k,n',k'=0}^{\infty} P^{(0,3)}_{n,k,n',k'} \mathrm{tr}(\chi^\dagger_{n,k} [A^\dagger_{n',k'}, A_{n+n',k+k'}])$$

$$T_5 = \frac{1}{2} \sum_{n,k,n',k'=0}^{\infty} P^{(1,1)}_{n,k,n',k'} \epsilon^{abc} \mathrm{tr}\Big([(\Phi^\dagger_a)_{n,k}, (\Phi^\dagger_b)_{n',k'}](\zeta_c)_{n+n',k+k'}\Big)$$

$$T_6 = \sum_{n,k,n',k'=0}^{\infty} P^{(1,2)}_{n,k,n',k'} \delta^{ab} \mathrm{tr}\Big([(\Phi^\dagger_a)_{n,k}, (\zeta^\dagger_b)_{n',k'}] A_{n+n',k+k'}\Big)$$

$\mathcal{Q}$ invariant under supersymmetry:

$$\mathcal{Q} = T_1 + T_2 + T_3 + T_4 + T_5 - T_6$$

Let's compute

$$H = \{\mathcal{Q}^\dagger, \mathcal{Q}\}$$

We get

$$H_{\text{int}} = H_D + H_F \, ,$$

$$H_D = \sum_{n,k=0}^{\infty} \text{tr} \left[(\mathcal{B}_0^\dagger)_{n,k} (\mathcal{B}_0)_{n,k} + \sum_{a=1}^{3} \sum_{I=1,2} (\mathcal{B}_I^{a\dagger})_{n,k} (\mathcal{B}_I^a)_{n,k} + (\mathcal{B}_3^\dagger)_{n,k} (\mathcal{B}_3)_{n,k} \right] ,$$

$$H_F = \sum_{n,k=0}^{\infty} \text{tr} \left[(\mathcal{F}_0^\dagger)_{n,k} (\mathcal{F}_0)_{n,k} + \sum_{a=1}^{3} \sum_{I=1,2} (\mathcal{F}_I^{a\dagger})_{n,k} (\mathcal{F}_I^a)_{n,k} + (\mathcal{F}_3^\dagger)_{n,k} (\mathcal{F}_3)_{n,k} \right] ,$$

D-term blocks

$$
\begin{aligned}
(\mathcal{B}_0)_{n,k} = & \sum_{n',k'=0}^{\infty} P^{(0,0)}_{n,k;n',k'}\{\chi^{\dagger}_{n',k'},\chi_{n+n',k+k'}\} \\
& + \sum_{a=1}^{3} P^{(0,1)}_{n,k;n',k'}[(\Phi^{\dagger}_a)_{n',k'},(\Phi_a)_{n+n',k+k'}] \\
& + \sum_{a=1}^{3} P^{(0,2)}_{n,k;n',k'}\{(\zeta^{\dagger}_a)_{n',k'},(\zeta_a)_{n+n',k+k'}\} + P^{(0,3)}_{n,k;n',k'}[A^{\dagger}_{n',k'},A_{n+n',k+k'}] \ , \\
(\mathcal{B}^a_1)_{n,k} \equiv & \sum_{n',k'=0}^{\infty} P^{(1,1)}_{n,k:n',k'}\epsilon^{abc}[(\zeta_b)_{n+n',k+k'},(\Phi^{\dagger}_c)_{n',k'}] \\
& - P^{(1,2)}_{n,k;n',k'}[(\zeta^{\dagger}_a)_{n',k'},A_{n+n',k+k'}] + P^{(1,0)}_{n,k;n',k'}[(\Phi_a)_{n+n',k+k'},\chi^{\dagger}_{n',k'}] , \\
(\mathcal{B}^a_2)_{n,k} \equiv & \sum_{n',k'=0}^{\infty} P^{(2,1)}_{n,k;n',k'}[(\Phi^{\dagger}_a)_{n',k'},A_{n+n',k+k'}] + P^{(2,0)}_{n,k;n',k'}\{(\zeta_a)_{n+n',k+k'},\chi^{\dagger}_{n',k'}\} , \\
(\mathcal{B}_3)_{n,k} \equiv & \sum^{\infty} P^{(3,0)}_{n,k;n',k'}[A_{n+n',k+k'},\chi^{\dagger}_{n',k'}] .
\end{aligned}
$$

F-term blocks

$$(\mathcal{F}_0)_{n,k} \equiv \frac{1}{2} \sum_{n'=0}^{n} \sum_{k'=0}^{k} P^{(0,0)}_{n',k';n-n',k-k'} \{\chi_{n-n',k-k'}, \chi_{n',k'}\} ,$$

$$(\mathcal{F}_1^a)_{n,k} \equiv \sum_{n'=0}^{n} \sum_{k'=0}^{k} P^{(0,1)}_{n',k',n-n',k-k'} [(\Phi_a)_{n-n',k-k'}, \chi_{n',k'}] ,$$

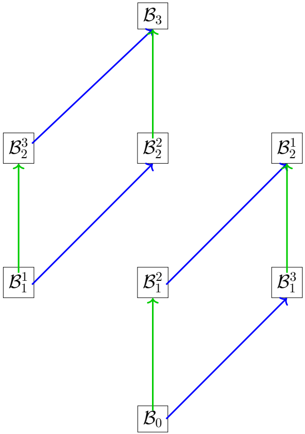
$$(\mathcal{F}_2^a)_{n,k} \equiv \frac{1}{2} \sum_{n'=0}^{n} \sum_{k'=0}^{k} P^{(1,1)}_{n',k',n-n',k-k'} \epsilon^{abc} [(\Phi_c)_{n',k'}, (\Phi_b)_{n-n',k-k'}]$$

$$+ P^{(0,2)}_{n',k',n-n',k-k'} \{(\zeta_a)_{n-n',k-k'}, \chi_{n',k'}\} ,$$

$$(\mathcal{F}_3)_{n,k} \equiv \sum_{n'=0}^{n} \sum_{k'=0}^{k} P^{(1,2)}_{n',k',n-n',k-k'} [(\zeta_a)_{n-n',k-k'}, (\Phi_a)_{n',k'}]$$

$$- P^{(0,3)}_{n',k';n-n'k-k'} [A_{n-n',k-k'}, \chi_{n',k'}] .$$

Symmetry structure



All the block $\mathcal{B}_I$, $\mathcal{F}_I$ and letters W_I are transforming in the $(p, q) = (0, I - 3)$ representations of SU$(1, 2)$ algebra, $I = 0, 1, 2, 3$.

Puzzles in PSU$(1,1|2)$

We will have two $\mathcal{N}=2$ hypermultiplets in PSU$(1,1|2)$ subsector. One in terms of D-term blocks while the other one is like F-term blocks.

Summary: New physics in each subsector

- SU$(2|3)$: $\chi_{1,2}, Z, W, X$, the fermionic doublet block $\{\chi_1, \chi_2\}$ and scalar block, pure F-term
- SU$(1,1|1)$, telescopic sum, pure D-term tr$(q_l^\dagger q_l)$, infinite modes
- PSU$(1,1|2) \times SU(2)_F$: $\chi_1, \bar{\chi}_7, Z, X, d_1$, infinite modes$+$ F-term, fermionic doublet block? SU$(2)_F$ automorphism; Simultaneous presence of D-term and F-term supermultiplets
- SU$(1,2|2)$: $F, \Phi, \bar{\chi}_{5,7}, d_{1,2}$, new $[A, \Phi]$ blocks
- PSU$(1,2|3)$: $F, Z, W, X, \chi_{1,2}, \bar{\chi}_{3,5,7}, d_{1,2}$,
  1. Dirac equation leads to ancestor fermion
  2. Simultaneous presence of D-term and F-term supermultiplets. Enhanced supersymmetry!

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5 Future work

Some future work

- Local field theory in $\mathrm{SU}(1,2)$ subsector [Baiguera, Harmark, Lei, 2023] and relation to chiral algebra?
- Relation to the work $\mathrm{SU}(1,D)$ field theory: [Lambert, Mouland, Orchard, 2022]
- $\frac{1}{16}-$BPS black hole interpretation
- Understanding how Kerr/CFT appears holographically based on $\mathrm{AdS}_5/\mathrm{CFT}_4$ (Based on [Goldstein, Jejjala, Lei, Leuven, Li, 2019]).

$$\mathrm{PSU}(1,2|3) \to \mathrm{PSU}(1,1|2)$$

- Factorization of partition function with finite N?
- Relation to strings in TNC gravity [Harmark, Hartong, Obers, Yan, 2018, 2021]