Non-invertible Duality Defects: Lattice, Field Theory and Symmetry TFT

Yunqin Zheng
Kavli IPMU, ISSP @ U.Tokyo

Soochow University, HET seminar
Nov 2nd, 2022

Justin Kaidi, **Kantaro Ohmori**, Y.Z. 2111.01141
Justin Kaidi, **Kantaro Ohmori**, Y.Z. 2209.11062
Linhao Li, **Masaki Oshikawa**, Y.Z. To appear

Global Symmetry

- Global symmetries are powerful in constraining RG flows. ('t Hooft anomaly matching, Selection rule of correlation functions, checking dualities, ...)

- It is well known (before 2014)

  0-form global symmetry $\longrightarrow$ codimension-1 topological defect

- But the converse arrow $\longleftarrow$ is appreciated only until recently (since 2014). This leads to "generalized global symmetry"

  generalized global symmetry $\longleftrightarrow$ topological defect

Generalized global symmetry

- The topological operators, hence the generalized global symmetries, can be organized according to codimension and invertibility.

	defect codim = 1	defect codim > 1
Invertible	**Ordinary Sym**	**Higher form/group Sym**
Non-invertible	**Non-invertible Sym**	

- This talk will focus on the non-invertible symmetry.

Non-invertible symmetry is Unavoidable

- Start with an ordinary symmetry or higher form symmetry, one can get non-invertible symmetry upon various gauging.

- Gauging with a mixed anomaly ⇒ Duality defects;
  [Kaidi,Ohmori,**YZ**,2111.01141]...
- Gauging on half space ⇒ Duality defects;
  [Choi,Cordova,Hsin,Lam,Shao, 2111.01139]...
- Gauging a non-normal subgroup;
  [Bhardwaj,Bottini,Schafer-Nameki,Tiwari,2204.06564]...
  [Nguyen,Tanizaki,Unsal,2101.02227]...
- Gauging a multi-dimensional coupled system;
  [Bhardwaj,Schafer-Nameki,Wu,2208.05973]...
- Higher gauging ⇒ Condensation defects;
  [Roumpedakis,Seifnashri,Shao,2204.02407]...
- Higher gauging on half higher-codim space ⇒ Higher duality defects;
  [Kaidi,Ohmori,**YZ**,2209.11062]...

Non-invertible symmetry is Ubiquitous

- Various TQFTs.
- Free field theories: 2d compact boson, 4d Maxwell.
- G Yang-Mills theory with various G, $\mathcal{N}$, d.
- QED_4. Infinite number of non-invertible symmetries.
- QCD_4.
- Axion models.
- Class-S theories
- Any QFT with a non-anomalous higher form symmetry.
- ...

This Talk

- I will instead come back to (probably) the very first non-invertible symmetry discovered in history:

  Non-invertible Kramers-Wannier Duality Defect in $(1+1)$d

- Three ways to approach it:

  Lattice. [Li,Oshikawa,**YZ**,to appear]

  Field theory. [Choi,Cordova,Hsin,Lam,Shao,2111.01139]

  Symmetry TFT. [Kaidi,Ohmori,**YZ**,2209.11062]

KW on Lattice

[Li,Oshikawa,**YZ**, to appear]

Kramers-Wannier on lattice

- The KW transformation was first found in the 2d statistical Ising model, relating high temperature and low temperature. [Kramers,Wannier, 1941]

- Another version of KW found in $(1+1)$d quantum Ising model, relating large transverse field to small transverse field.

$$H = -\sum_i (\sigma_i^z \sigma_{i+1}^z + h\sigma_i^x)$$

  KW: $\sigma_i^x \to \sigma_i^z \sigma_{i+1}^z$, $\sigma_i^z \to ...\sigma_{i-2}^x \sigma_{i-1}^x \sigma_i^x$
  Then $\sigma_i^x \to \sigma_i^z \sigma_{i+1}^z$, $\sigma_{i-1}^z \sigma_i^z \to \sigma_i^x$, two terms are exchanged.
  Effectively: $h \to 1/h$.

- We will focus on the quantum Ising model below.

KW on lattice: Puzzles

$$\sigma_i^x \to \sigma_i^z \sigma_{i+1}^z, \qquad \sigma_i^z \to \ldots \sigma_{i-2}^x \sigma_{i-1}^x \sigma_i^x$$

1. On an infinite chain, fine. How about on a circle? This was discussed in the appendix of [Hsieh,Nakayama,Tachikawa,2002.12283], but we were confused by some formula there...

2. It is now often said that the operator $\mathcal{N}$ implementing the KW transformation is non-invertible, with fusion rule

$$\mathcal{N} \times \mathcal{N} = 1 + U, \qquad \mathcal{N} \times U = U \times \mathcal{N} = \mathcal{N}, \qquad U \times U = 1$$

   How to see it on the lattice? It was discussed in [Aasen,Mong,Fendley,1601.07185], but they work on an infinite line (rather than a circle)...

3. It is also often said that KW is equivalent to gauging $\mathbb{Z}_2$. How to see the mapping between symmetry-twist sectors explicitly?

4. How about KW on open chains? Still non-invertible?

KW on a circle

Set-up:

- Spin-$\frac{1}{2}$ on each site i, $i = 1, \dots, L$.
- $s_{i+L} = s_i + t$
- Basis product state $|\{s_i\}\rangle$.

Instead of defining the KW by mapping between Pauli operators, we define the KW by an operation on the state:

$$\mathcal{N}\,|\{s_i\}\rangle = \frac{1}{2^{L/2}} \sum_{\{\widehat{s}_{i-\frac{1}{2}}\}} (-1)^{\sum_{j=1}^{L}(s_{j-1}+s_j)\widehat{s}_{j-\frac{1}{2}}+\widehat{t}s_L} \,|\{\widehat{s}_{i-\frac{1}{2}}\}\rangle\,.$$

This KW resolves all of the above puzzles!

KW on a circle

$$\mathcal{N}\left|\{s_i\}\right\rangle = \frac{1}{2^{L/2}} \sum_{\{\widehat{s}_{i-\frac{1}{2}}\}} (-1)^{\sum_{j=1}^{L}(s_{j-1}+s_j)\widehat{s}_{j-\frac{1}{2}}+\widehat{t}s_L} \left|\{\widehat{s}_{i-\frac{1}{2}}\}\right\rangle .$$

- On an infinite chain, it reproduces the standard KW transformation,

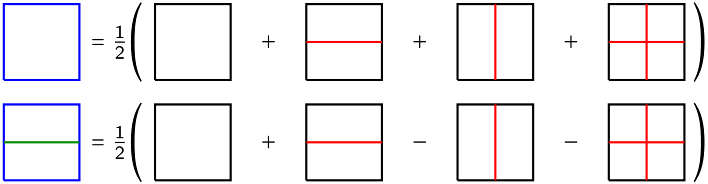
$$\widehat{\sigma}^x_{i-\frac{1}{2}}\mathcal{N} = \mathcal{N}\sigma^z_{i-1}\sigma^z_i , \qquad \widehat{\sigma}^z_{i-\frac{1}{2}}\mathcal{N} = \mathcal{N}(\ldots\sigma^x_{j-2}\sigma^x_{j-1})$$

- On a finite circle, it is consistent with gauging $\mathbb{Z}_2$ symmetry. Let us review what do we mean by gauging $\mathbb{Z}_2$ symmetry.

KW as gauging $\mathbb{Z}_2$

- Gauging $\mathbb{Z}_2$
  $=$ summing over $\mathbb{Z}_2$ connections
  $=$ summing over $\mathbb{Z}_2$ defects.

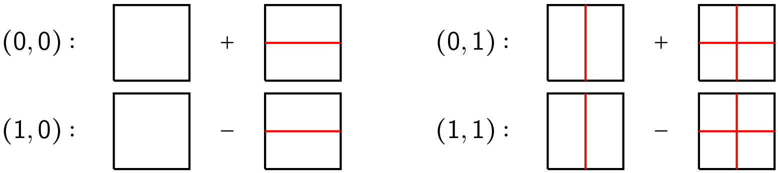
$$Z_{\mathcal{X}/\mathbb{Z}_2}[A] = \frac{1}{2} \sum_{a \in H^1(T^2, \mathbb{Z}_2)} Z_{\mathcal{X}}[a] (-1)^{\int a \cup A}$$



KW as gauging $\mathbb{Z}_2$

Use (u, t) to label the symmetry-twist sectors:

- $u = 0, 1$: $\mathbb{Z}_2$ even/odd symmetry sectors,
- $t = 0, 1$: $\mathbb{Z}_2$ untwisted/twisted sectors,



$\mathcal{X}$	$t = 0$	$t = 1$
$u = 0$	S	U
$u = 1$	T	V

$\mathcal{X}/\mathbb{Z}_2$	$\widehat{t} = 0$	$\widehat{t} = 1$
$\widehat{u} = 0$	S	T
$\widehat{u} = 1$	U	V

$$\widehat{u} = t, \qquad \widehat{t} = u$$

KW on a circle

$$\mathcal{N}\,|\{s_i\}\rangle = \frac{1}{2^{L/2}} \sum_{\{\widehat{s}_{i-\frac{1}{2}}\}} (-1)^{\sum_{j=1}^{L}(s_{j-1}+s_j)\widehat{s}_{j-\frac{1}{2}}+\widehat{t}s_L} \,|\{\widehat{s}_{i-\frac{1}{2}}\}\rangle\,.$$

- Indeed, this transformation exactly reproduces the mapping between symmetry and twist sectors,

$$\widehat{u} = t, \qquad \widehat{t} = u$$

- This formula is almost the same as in [Aasen,Mong,Fendley,1601.07185], except for the last term in the exponent, $\widehat{t}s_L$. This term is needed to match how the symmetry and twist sectors transform $(\widehat{u},\widehat{t}) = (t,u)$.

KW on a circle

$$\mathcal{N}\,|\{s_i\}\rangle = \frac{1}{2^{L/2}} \sum_{\{\widehat{s}_{i-\frac{1}{2}}\}} (-1)^{\sum_{j=1}^{L}(s_{j-1}+s_j)\widehat{s}_{j-\frac{1}{2}} + \widehat{t}s_L}\,|\{\widehat{s}_{i-\frac{1}{2}}\}\rangle\,.$$

- Non-invertiblity of $\mathcal{N}$ is now transparent: $\mathcal{N}^\dagger\mathcal{N}$ is not an identity operator.

$$\langle\{s_i\}|\mathcal{N}^\dagger\mathcal{N}\,|\{s'_i\}\rangle = \prod_{i=1}^{L}\delta_{s_i,s'_i} + (-1)^{\widehat{t}}\prod_{i=1}^{L}\delta_{s_i,s'_i+1}$$

- Symmetry and twist sectors are straightforward to check, $(\widehat{u},\widehat{t}) = (t,u)$.
- Fusion rule is also straightforward to find, but now interestingly depends on the twist parameter:

$$\mathcal{N}\times\mathcal{N} = 1 + (-1)^{\widehat{t}}U, \qquad \mathcal{N}\times U = (-1)^{\widehat{t}}\mathcal{N}$$

$\mathcal{N}$ is non-invertible. The standard fusion rule $\mathcal{N}\times\mathcal{N} = 1+U$ etc assumes PBC along the chain, i.e. no twist-defect U terminating on $\mathcal{N}$. This point was recently emphasized in [Choi,Cordova,Hsin,Lam,Shao,2204.09025].

KW on a circle

$$\mathcal{N}\left|\{s_i\}\right\rangle = \frac{1}{2^{L/2}} \sum_{\{\widehat{s}_{i-\frac{1}{2}}\}} (-1)^{\sum_{j=1}^{L}(s_{j-1}+s_j)\widehat{s}_{j-\frac{1}{2}}+\widehat{t}s_L} \left|\{\widehat{s}_{i-\frac{1}{2}}\}\right\rangle .$$

- Everything discussed so far is independent of the Hamiltonian. One can easily construct KW self-dual Hamiltonians by demanding $[H, \mathcal{N}] = 0$. For example, the Ising model at $h = 1$ is self-dual on a circle:

$$H = -\sum_{i=1}^{L}(\sigma_{i-1}^z \sigma_i^z + h\sigma_i^x)$$

Then KW becomes a non-invertible symmetry. By modular S transformation, $\mathcal{N}$ becomes a non-invertible duality defect.

KW on an interval

Define KW on an open chain with free open boundary condition:

$$\mathcal{N}\,|\{s_i\}\rangle = \frac{1}{2^{L/2}} \sum_{\{\widehat{s}_{i-\frac{1}{2}}\}} (-1)^{\sum_{j=2}^{L} s_{j-1}\widehat{s}_{j-\frac{1}{2}} + \sum_{j=1}^{L} s_j \widehat{s}_{j-\frac{1}{2}}}\, |\{\widehat{s}_{i-\frac{1}{2}}\}\rangle\,.$$

- Surprisingly, KW is now invertible and unitary!!! Check the overlap,

$$\langle\{s_i\}|\mathcal{N}^\dagger\mathcal{N}\,|\{s_i'\}\rangle = \prod_{i=1}^{L} \delta_{s_i,s_i'}$$

The non-invertible KW transformation on a closed chain becomes invertible on an open chain!!!

Bonus: This simple fact also clarifies a small puzzle in history.

Early history on SPT

Back in 1983, Duncan Haldane (Nobel laureate in 2016) conjectured that the spin-S Heisenberg AFM is gapless for half-integer S and gapped for integer S. He showed this in large S limit, but the conjecture is for any finite S.

Nonlinear Field Theory of Large-Spin Heisenberg Antiferromagnets: Semiclassically Quantized Solitons of the One-Dimensional Easy-Axis Néel State

F. D. M. Haldane
Phys. Rev. Lett. **50**, 1153 – Published 11 April 1983

Physics See Focus story: Nobel Prize—Topological Phases of Matter

Article | References | Citing Articles (2,880) | PDF | Export Citation

ABSTRACT

The continuum field theory describing the low-energy dynamics of the large-spin one-dimensional Heisenberg antiferromagnet is found to be the O(3) nonlinear sigma model. When weak easy-axis anisotropy is present, soliton solutions of the equations of motion are obtained and semiclassically quantized. Integer and half-integer spin systems are distinguished.

Received 31 January 1983

Early history on SPT

Four years later in 1987, Affleck, Lieb, Kennedy and Tasaki found an analytically tractable model for $S = 1$, and gapped exists! At that time, AKLT model was exotic, beyond Landau paradigm. Now, it is known as the first example of SPT.

Rigorous results on valence-bond ground states in antiferromagnets

Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki
Phys. Rev. Lett. **59**, 799 – Published 17 August 1987

Article | References | Citing Articles (1,628) | PDF | Export Citation

ABSTRACT

We present rigorous results on a phase in antiferromagnets in one dimension and more, which we call a valence-bond solid. The ground state is simply constructed out of valence bonds, is nondegenerate, and breaks no symmetries. There is an energy gap and an exponentially decaying correlation function. Physical applications are mentioned.

Received 26 May 1987

Kennedy-Tasaki transformation

Four years later in 1991, Kennedy and Tasaki wrote another influential paper, showed that the AKLT model is not that exotic: one can still understand it within Landau paradigm, i.e. via Hidden symmetry breaking. SSB phase and SPT phase on an open chain are related by a **non-local unitary transformation**, and local order parameter $\xrightarrow{KT}$ string order parameter.

Hidden $\mathbf{Z}_2 \times \mathbf{Z}_2$ symmetry breaking in Haldane-gap antiferromagnets

Tom Kennedy and Hal Tasaki
Phys. Rev. B **45**, 304 – Published 1 January 1992

Article | References | Citing Articles (316) | PDF | Export Citation

ABSTRACT

We show that the Haldane phase of the S=1 antiferromagnetic chain is closely related to the breaking of a hidden $Z_2 \times Z_2$ symmetry. When the chain is in the Haldane phase, this $Z_2 \times Z_2$ symmetry is fully broken, but when the chain is in a massive phase other than the Haldane phase, e.g., the Ising phase or the dimerized phase, this symmetry is broken only partially or not at all. The hidden $Z_2 \times Z_2$ symmetry is revealed by introducing a nonlocal unitary transformation of the chain. This unitary transformation also leads to a simple variational calculation which qualitatively reproduces the phase diagram of the S=1 chain.

Received 29 July 1991

Kennedy-Tasaki transformation

In the same year, Masaki Oshikawa, in his first paper in graduate school, generalized the KT transformation to arbitrary integer spin, and found a much nicer and compact formula of the **non-local unitary transformation**

$$U_{\mathrm{KT}} = \prod_{i>j} \exp\left(i\pi S_i^z S_j^x\right).$$

Hidden Z2* Z2 symmetry in quantum spin chains with arbitrary integer spin

著者	Masaki Oshikawa
公開日	1992/9/7
論文誌	Journal of Physics: Condensed Matter
巻	4
号	36
ページ	7469
出版社	IOP Publishing
説明	The author studies integer S> 1 spin chains. He extends the Kennedy-Tasaki nonlocal unitary transformation for S= 1 to arbitrary integer S. He shows the main results of Kennedy and Tasaki (1992) are maintained for S> 1: Heisenberg-type Hamiltonians are transformed to Hamiltonians of nearest-neighbour interactions with Z 2* Z 2 symmetry, and the den Nijs-Rommelse string observables are transformed to the ferromagnetic correlation observables. He asserts that in general values of integer S there exist several phases with the hidden Z 2* Z 2 symmetry breaking. The den Nijs-Rommelse string order parameters, which measure the hidden Z 2* Z 2 symmetry breaking, are calculated explicitly for several variants of the VBS-type states. In the standard VBS state, the hidden Z 2* Z 2 symmetry breaks down when S is odd but remains unbroken when S is even. His results for partially dimerized VBS states suggest that …
総被引用数	引用元 220

Kennedy-Tasaki transformation: Puzzle

$$U_{\mathrm{KT}} = \prod_{i>j} \exp\left(i\pi S_i^z S_j^x\right).$$

However, there is a puzzle...

The **non-local unitary transformation** has been only defined on an open chain. How to define it on a circle? We haven't found a literature addressing this question...

Non-invertible KT transformation

	circle	interval
Kramers-Wannier	non-invertible non-unitary S	invertible, unitary
Kennedy-Tasaki	non-invertible non-unitary $STS = TST$	invertible, unitary

In an upcoming work with Linhao Li and Masaki Oshikawa, we showed that non-local unitary KT transformation can be lifted on a circle if we sacrifice unitarity!

It turns out that the non-unitary transformation is basically $TST = STS$, whose operator/defect obeys non-invertible fusion rule.

KW from Field theory: Half-space gauging

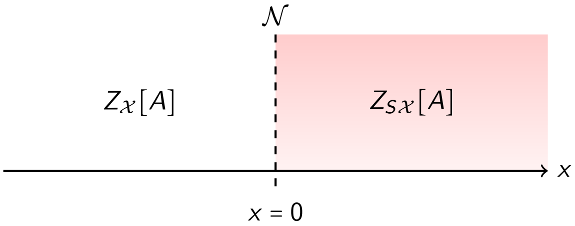
[Choi,Cordova,Hsin,Lam,Shao,2111.01139]

[Kaidi,Ohmori,**YZ**,2209.11062]

KW Defect from field theory

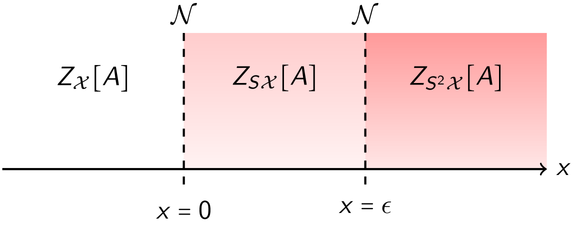
- For an arbitrary $(1+1)$d bosonic QFT $\mathcal{X}$ with an non-anomalous $\mathbb{Z}_N$ symmetry, one can gauge $\mathbb{Z}_N$ to generate a new theory $S\mathcal{X} := \mathcal{X}/\mathbb{Z}_N$ by a KW transformation. $\mathcal{X}$ is KW self-dual if $\mathcal{X} = \sigma\mathcal{X}$, which we assume below.

- Gauging $\mathbb{Z}_N$ on half-of the spacetime with Dir.b.c. defines a duality defect.

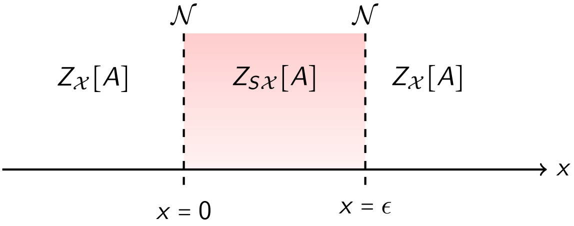


Fusion rule of Duality defects

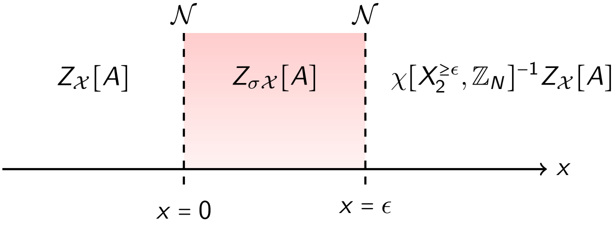
- To find the fusion of two duality defects, we place them parallel to each other:



- Gauging $\mathbb{Z}_N$ in a slab.



Fusion rule of Duality defects



- Shrinking the slab by taking $\epsilon \to 0$, it simply means $\mathcal{N} \times \mathcal{N}$ is gauging $\mathbb{Z}_N$ on a co-dimension 1 submanifold M_1 → 1-gauging on M_1, leading to a condensation defect $\mathcal{C} = \sum_{n=0}^{N-1} \eta^n$.

- The full set of fusion rule (up to Euler counter terms) is

$$\mathcal{N} \times \mathcal{N} = \sum_{n=0}^{N-1} \eta^n, \qquad \eta \times \mathcal{N} = \mathcal{N} \times \eta = \mathcal{N}, \qquad \eta^N = 1$$

  $\mathcal{N}$ is non-invertible .

- The same discussion can be generalized to higher dimensions.

Constraints on RG

- When a QFT $\mathcal{X}$ is KW self-dual, KW is a non-invertible global symmetry of $\mathcal{X}$. As ordinary symmetries, the non-invertible KW duality symmetry also has non-trivial impact on the RG flow.

If a (1+1)d $\mathbb{Z}_N$ symmetric QFT $\mathcal{X}$ is self dual under KW, $\mathcal{X} = \mathcal{X}/\mathbb{Z}_N$, then $\mathcal{X}$ can not be gapped with a single ground state.

[Choi,Cordova,Hsin,Lam,Shao,2111.01139]

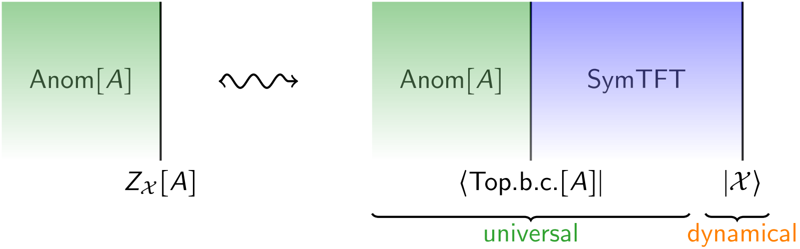
KW from Symmetry TFT

[Kaidi,Ohmori,**YZ**,2209.11062]

Global Symmetry from Symmetry TFT

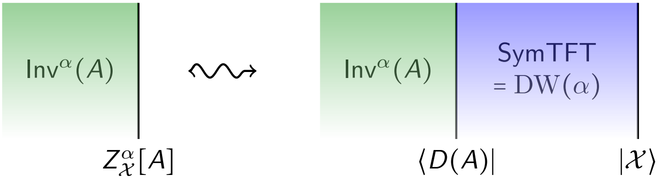
- For QFTs with finite symmetries, SymTFT nicely decouples the universal quantities (symmetries and 't Hooft anomalies) from the non-universal dynamics.

[Gaiotto,Kulp,2008.05960],[Freed,PI 22'],[Freed,Moore,Teleman,2209.07471]
[Apruzzi,Bonetti,García-Etxebarria,S.Hosseini,Schafer-Nameki,2112.02092]



SymTFT for Invertible Symmetry

- The SymTFT of a finite invertible symmetry G with a 't Hooft anomaly α is well-known: Dijkgraaf-Witten theory.



Symmetry TFT for $\mathbb{Z}_N$

- The KW is defined by gauging $\mathbb{Z}_N$. We will show that the properties of KW can be reproduced from the symmetry TFT of $\mathbb{Z}_N$.

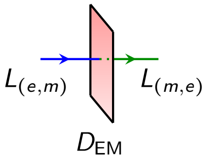
- For a non-anomalous $\mathbb{Z}_N$, the SymTFT is a $\mathbb{Z}_N$ DW without twist – $\mathbb{Z}_N$ gauge theory in $(2+1)$d. The action is

$$\frac{2\pi}{N}\int \widehat{a}\delta a$$

Topological Operators in $\mathbb{Z}_N$ Gauge Theory

- $\mathbb{Z}_N$ gauge theory has N^2 topological lines $L_{(e,m)}(\gamma)$, where $L_{(1,0)}$ and $L_{(0,1)}$ generate $\mathbb{Z}_N \times \mathbb{Z}_N$ 1-form symmetry.

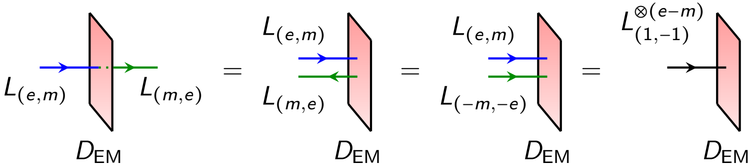
- More interestingly, there is a $\mathbb{Z}_2^{\text{EM}}$ 0-form symmetry, $a \leftrightarrow \widehat{a}$, i.e. $L_{(e,m)}(\gamma) \leftrightarrow L_{(m,e)}(\gamma)$. $\mathbb{Z}_2^{\text{EM}}$ comes with a surface defect D_{EM}.



- The $\mathbb{Z}_2^{\text{EM}}$ defect is a condensation defect.

[Roumpedakis,Seifnashri,Shao,2204.02407]

$\mathbb{Z}_2^{\text{EM}}$ Defect as Condensation defect



- Any power of $L_{(1,-1)}$ can be absorbed into the $\mathbb{Z}_2^{\text{EM}}$ defect. $\Rightarrow$ D_{EM} is a condensation defect of $L_{(1,-1)}$.

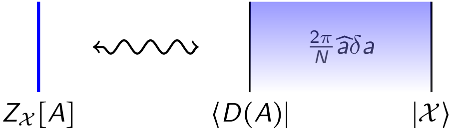
$$D_{\text{EM}}(M_2) = \frac{1}{|H^0(M_2,\mathbb{Z}_N)|} \sum_{\gamma\in H_1(M_2,\mathbb{Z}_N)} L_{(1,-1)}(\gamma)$$

- Indeed, it satisfies the desired properties:

$$L_{(e,m)}(\gamma) D_{\text{EM}}(M_2) = D_{\text{EM}}(M_2) L_{(m,e)}(\gamma), \qquad D_{\text{EM}}(M_2)^2 = \chi^{-1}$$

Dirichlet Boundary Conditions

- As a SymTFT for $\mathbb{Z}_N$, let us place the $\mathbb{Z}_N$ gauge theory on a slab



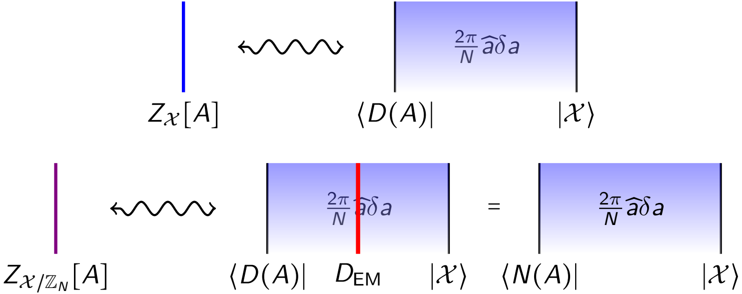
- Dirichlet b.c.: $L_{(1,0)}$ parallel to the bdy becomes trivial; $L_{(1,0)}$ orthogonal to the bdy can end; $L_{(0,1)}$ parallel to the boundary is still non-trivial; $L_{(0,1)}$ orthogonal to the bdy can not end;

- Upon shrinking the slab, $L_{(0,1)}$ survives as the symmetry operator of $\mathbb{Z}_N$. $L_{(1,0)}$ survives as a non-topological order parameter for $\mathbb{Z}_N$.

$$L_{(0,1)} \to \eta, \qquad L_{(1,0)} \to \mathcal{O}$$

Neumann Boundary Conditions

- The Neuman b.c. can be obtained by composing the Dirichlet b.c. with D_{EM}.



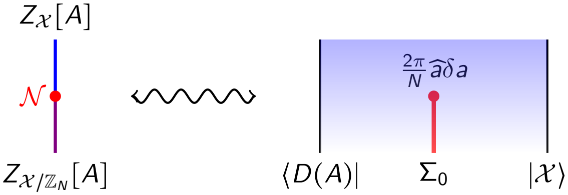
- Changing the b.c. from Dirichlet to Neuman exchanges the role of $L_{(0,1)} \leftrightarrow L_{(1,0)}$. Upon shrinking, $L_{(1,0)}$ survives as a symmetry operator of the quantum $\widehat{\mathbb{Z}}_N$ symmetry, and $L_{(0,1)}$ survives as a non-topological order parameter.

$$L_{(1,0)} \to \widehat{\eta}, \qquad L_{(0,1)} \to \widehat{\mathcal{O}}$$

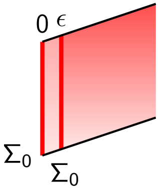
Twist Defects induces KW-like interfaces in (1+1)d

- Twist defect Σ: $\mathbb{Z}_2^{\text{EM}}$ exchanging defect on half-surface with Dirichlet boundary → higher KW interface.

- Colliding the twist defect with Dirichlet boundary condition yields boundary-changing line operator. Further shrinking the slab yields the KW duality defect.



Fusion rule of Twist defects



$\Longrightarrow$

- Fusing two twist defects Σ_0's is 2-gauging of $L_{(1,-1)}$ on a line.
  $\Sigma_0 \times \Sigma_0 = \sum_{n=0}^{N-1} L_{(n,-n)}$. $\Rightarrow$ twist defect is non-invertible!

- Due to Dirichlet boundary condition of the condensate,
  $\Sigma_0 \times L_{(e+n,-n)} = \Sigma_e$.

Invertibility / Non-invertibility transmutation

The invertibility of a defect on an open and closed manifold can transmute.

	Closed M	Open M
Kramers-Wannier	non-invertible	invertible
$\mathbb{Z}_2^{\mathsf{EM}}$ (condensation) defect	invertible	non-invertible

Fusion rule of KW duality defects from Symmetry TFT

- We already see

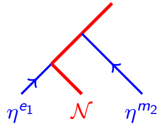
$$\text{Twist defects } \Sigma_e \longleftrightarrow \text{Duality defects } \mathcal{N}$$
$$\text{Magnetic line } L_{(0,1)} \longleftrightarrow \mathbb{Z}_N^{(0)} \text{ symmetry defect } \eta$$
$$\text{Electric line } L_{(1,0)} \longleftrightarrow \mathbb{Z}_N^{(0)} \text{ order parameter } \mathcal{O}$$

- Then the fusion rule of twist defects immediately implies the fusion rule of duality defects.

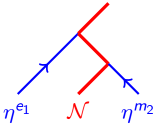
$$\begin{array}{ccc} L_{(e,m)}^N = 1 & & \eta^N = 1 \\ \Sigma_e \times L_{(0,1)} = \Sigma_{e+1} & \Rightarrow & \mathcal{N} \times \eta = \mathcal{N} \\ \Sigma_e \times \Sigma_{e'} = \sum_{n=0}^{N-1} L_{(n+e+e',-n)} & & \mathcal{N} \times \mathcal{N} = \sum_{n=0}^{N-1} \eta^n \end{array}$$

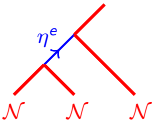
F-symbols for the duality defects

- As a bonus, the symmetry TFT also enables us to obtain the F-symbols of KW duality defects with less efforts. They can be inferred from the F-symbols of the twist defects in the $\mathbb{Z}_N$ gauge theory!

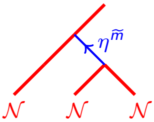


$= \quad e^{\frac{2\pi i}{N} e_1 m_2}$





$\simeq \quad \frac{1}{\sqrt{N}} \sum_{\widetilde{m}=0}^{N-1} e^{-\frac{2\pi i}{N} e \widetilde{m}}$



Summary

- We revisited the non-invertible KW duality defects from three different perspectives

  Lattice. [Li,Oshikawa,**YZ**, to appear]

  Field theoretical. [Choi,Cordova,Hsin,Lam,Shao,2111.01139]

  Symmetry TFT [Kaidi,Ohmori,**YZ**,2209.11062]

  Although KW duality defect has been discovered for 80 years, there are still interesting aspects of it to explore.

- More to explore:

  Other symmetries, including subsystem
  Higher dimensions
  Classification
  Dynamical application
  ...

Thank you for your attention!