CELESTIAL RECURSION

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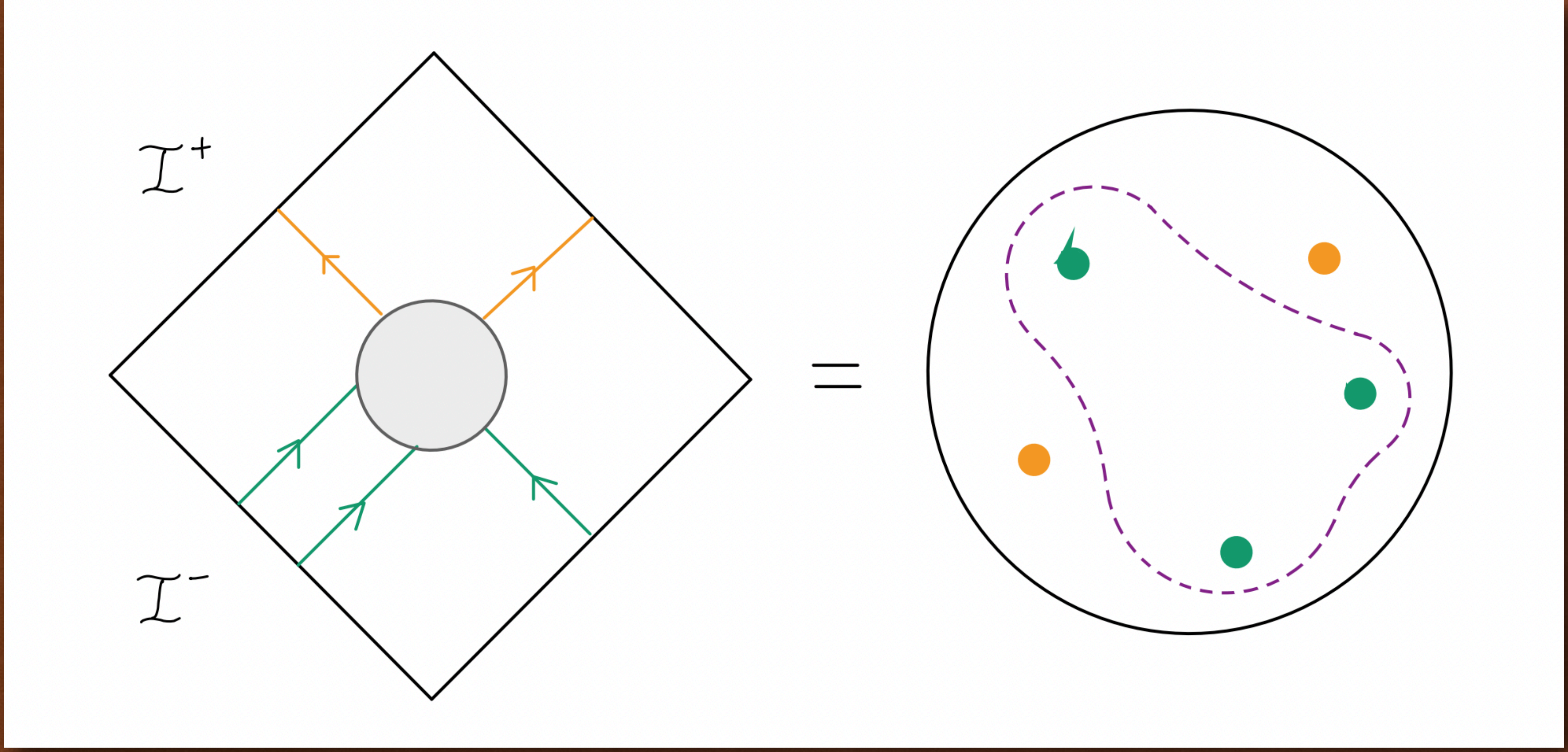
Based on 2208.11635 with Sabrina Pasterski

PERIMETER INSTITUTE

OUTLINE

- Review of Celestial Holography
- Celestial CFT vs BCFW
- From Celestial OPE to Large-z behavior
- Infinitesimal-z story
    - BCFW as Hard Superrotation
    - BCFW as Soft Insertion

REVIEW OF CELESTIAL HOLOGRAPHY



[Pasterski's talk @GR23, '22]

CELESTIAL HOLOGRAPHY

- Holography: quantum gravity in AFS & codim-2 CCFT
- Symmetries, reorganize observables
  - ∞-diml symmetry enhancements
  - Central objects of study: celestial amplitudes
- For concreteness, I will focus on massless scattering in 4D
- Global Symmetries
  - $SO(1,3) \simeq SL(2,\mathbb{C}) \qquad SO(2,2) \simeq SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$
  - Celestial amplitudes as correlation functions of a 2D CFT

CELESTIAL AMPLITUDE

- Massless: $P^\mu = \epsilon\, \omega\, q^\mu$
  - $\epsilon = \pm 1$ incoming/outgoing in (1,3)
  - $q^\mu = (1 + z\bar{z}, z + \bar{z}, -i(z - \bar{z}), 1 - z\bar{z})$
  - Overall scaling ω = energy
  - Map between spinor-helicity variables & celestial variables

$$\lambda_i^\alpha = |\lambda_i\rangle^\alpha = \epsilon_i \sqrt{2\omega_i t_i} \begin{pmatrix} 1 \\ z_i \end{pmatrix} \quad , \quad \lambda_{i\alpha} = \langle\lambda_i|_\alpha = \epsilon_i \sqrt{2\omega_i t_i} \begin{pmatrix} -z_i \\ 1 \end{pmatrix} \quad ,$$

$$\tilde{\lambda}_{i,\dot{\alpha}} = |\tilde{\lambda}_i]_{\dot{\alpha}} = \sqrt{2\omega_i t_i^{-1}} \begin{pmatrix} -\bar{z}_i \\ 1 \end{pmatrix} \quad , \quad \tilde{\lambda}_i^{\dot{\alpha}} = [\tilde{\lambda}_i|^{\dot{\alpha}} = \sqrt{2\omega_i t_i^{-1}} \begin{pmatrix} 1 \\ \bar{z}_i \end{pmatrix} \quad ,$$

t_i is the little group scaling

CELESTIAL AMPLITUDE

- Find the basis of solutions to the EOM which diagonalize L_0 and $\bar{L}_0$ simultaneously → conformal partial wave

$$m = 0 \qquad \Phi^{|J|}_{\Delta,J} \sim \int_0^\infty d\omega\, \omega^{\Delta-1}\, \varepsilon_{\mu_1\cdots\mu_{|J|}}\, e^{\pm i\omega q\cdot X_\pm}$$

[Pasterski, Shao, Strominger, '16, '17], [Law, Zlotnikov, '20]

- Celestial amplitude

$$\langle \mathcal{O}_{\Delta_1,J_1}(z_1,\bar{z}_1)\cdots\mathcal{O}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle = \left[\prod_i \int_o^\infty d\omega_i\, \omega_i^{\Delta_i-1}\right] A_n(\omega_i, z_i, \bar{z}_i)$$

ASYMPTOTIC SYMMETRIES

- In 1960s, Bondi, Burg, Metzner, and Sachs: $BMS^{\pm}$ group (∞-diml extension of the Poincaré group)
- BMS = Lorentz + supertranslation
- Supertranslation = angle-dep. translation of generators of null infty
- Superrotation = local enhancements of Lorentz

[Bondi, van der Burg, Metzner, '62], [Sachs, '62], [Barnich, Troessaert, '11]

ASYMPTOTIC SYMMETRIES

- Ward identity of supertranslation $\langle \text{out} | [Q_f, S] | \text{in} \rangle = 0 \Leftrightarrow$ Weinberg's soft graviton theorem

- Ward identity of superrotation $\langle \text{out} | [Q_Y, S] | \text{in} \rangle = 0 \Leftrightarrow$ sub-leading soft graviton theorem

[Kapec, Mitra, Raclariu, Strominger, '17]

[He, Kapec, Raclariu, Strominger, '17]

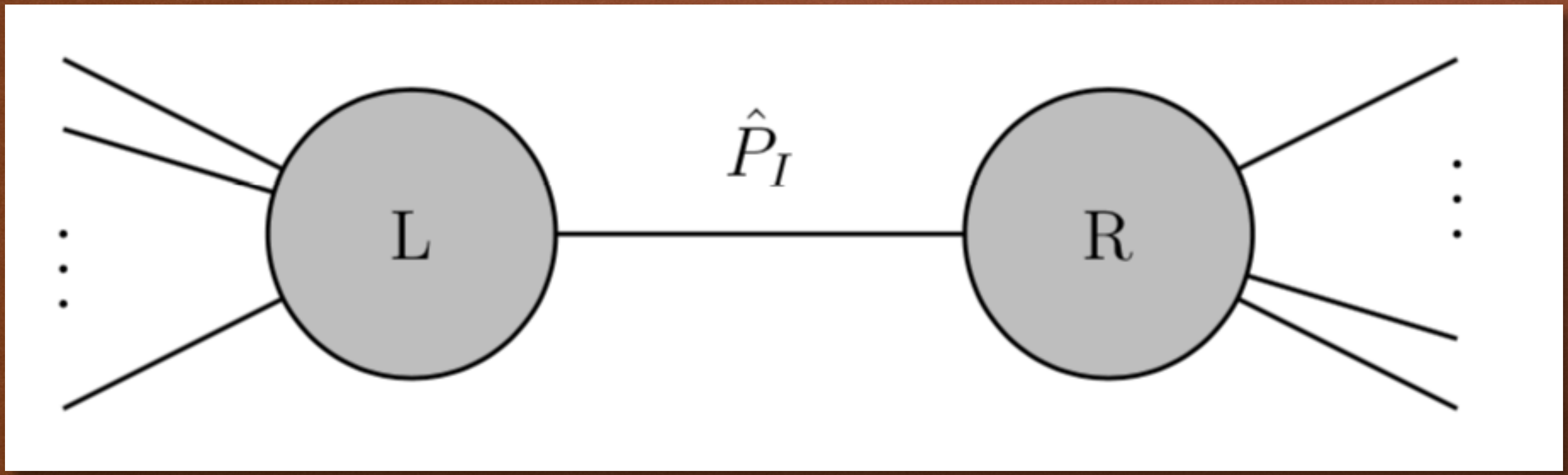
[Donnay, Ruzziconi,'21]

$$\langle \text{out} | [Q_S, S] | \text{in} \rangle = \sum_{k=1}^{n} \left[\frac{\hat{h}_k}{(z - z_k)^2} + \frac{1}{z - z_k} \partial_{z_k} \right] \langle \text{out} | S | \text{in} \rangle$$

- Holographic dual to QG in 4D AFS must admit a 2D conformal symmetry!

- More reviews: [Strominger, 1703.05448], [Pasterski, 2108.04801], [Raclariu, 2107.02075]

CCFT VS BCFW



[Elvang, Huang, '13]

CCFT VS BCFW

- CCFT: symmetries, OPE, spectrum

- BCFW: 3-pt amplitude + unitarity + locality + large-z behavior

$$\begin{cases} \lambda_i \mapsto \lambda_i + z\lambda_j \\ \tilde{\lambda}_j \mapsto \tilde{\lambda}_j - z\tilde{\lambda}_i \end{cases}$$

  - $A(0) = \oint_\gamma \frac{dz}{2\pi i}\frac{A(z)}{z} = -\sum_I \mathrm{Res}_{z\to z_I}\frac{A(z)}{z} + B_\infty$

  - Locality: $A(z)$ only has simple poles at tree-level $\leftrightarrow \frac{1}{P_I^2}$

  - Unitarity: at this pole, $A = A_L \frac{1}{P_I^2} A_R$

  - Recursively generate n-pt amplitude from 3-pt amplitude

LARGE-Z BEHAVIOR

- In order to use BCFW recursion, we assume the boundary term vanishes.

$$\lim_{z\to\infty} A(z) = 0$$

- This condition is far from obvious

- In pure YM, an argument based on background field method

[Arkani-Hamed, Kaplan, '08]

$\langle i, j]$	$\langle -, -]$	$\langle +, -]$	$\langle +, +]$	$\langle -, +]$
$A_n(z) \sim$	$\frac{1}{z}$	$\frac{1}{z}$	$\frac{1}{z}$	z^3

CCFT VS BCFW

- Consider $[2,1\rangle$-shift
  - $p_1' \;=\; p_1 - z\,q \;\;,\;\; p_2' \;=\; p_2 + z\,q \;\;,\;\; q = -\,|2\rangle[1\,|$
- Equivalent to transforming celestial variables as follows $(\varepsilon_1 = \varepsilon_2 = 1)$

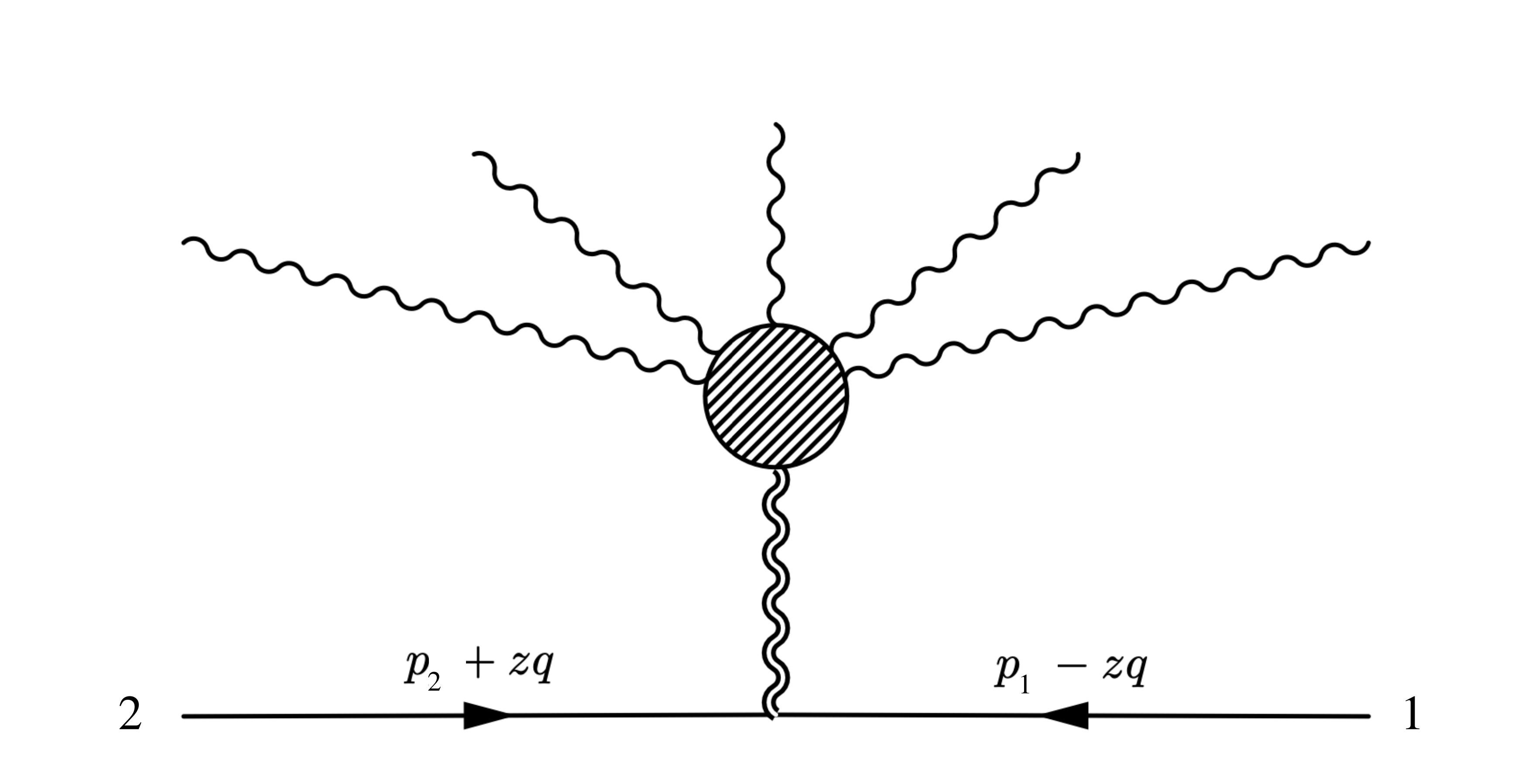
$$\begin{aligned} \varepsilon_1\,\omega_1 &\mapsto \varepsilon_1'\,\omega_1' = \omega_1 - z\sqrt{\omega_1\omega_2} \\ z_1 &\mapsto z_1' = z_1 + \frac{z_{12}\,z}{\frac{\varepsilon_1\sqrt{\omega_1}}{\varepsilon_2\sqrt{\omega_2}} - z} \\ \bar{z}_1 &\mapsto \bar{z}_1' = \bar{z}_1 \end{aligned} \qquad \begin{aligned} \varepsilon_2\,\omega_2 &\mapsto \varepsilon_2'\,\omega_2' = \omega_2 + z\sqrt{\omega_1\omega_2} \\ z_2 &\mapsto z_2' = z_2 \\ \bar{z}_2 &\mapsto \bar{z}_2' = \bar{z}_2 + \frac{\bar{z}_{12}\,z}{\frac{\sqrt{\omega_2}}{\sqrt{\omega_1}} + z} \end{aligned}$$

- Take $z \to \infty$ limit, $\lim_{z\to\infty} z_1' = z_2 = z_2'$ and $\lim_{z\to\infty} \bar{z}_2' = \bar{z}_1 = \bar{z}_1' \Leftrightarrow$ coincident limit on CS

CCFT VS BCFW

- Idea/Plan:
  - $\langle \mathcal{O}_{\Delta_1,J_1}(z_1,\bar{z}_1)\cdots\mathcal{O}_{\Delta_n,J_n}(z_n,\bar{z}_n)\rangle$
  - Implement $[2,1\rangle$-shift
  - Take $z \to \infty$ limit
  - Look at how OPE transforms
  - Extract the large-z scaling

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR



[Arkani-Hamed, Kaplan, '08]

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

STEP 1: HOW THE BCFW SHIFT OPERATIONS ACT ON $\mathcal{O}_{\Delta,J}$

- **Consider the Lorentz transformation acts on the spinors:**

- $$\begin{cases} |\lambda_i\rangle \mapsto \Lambda_i |\lambda_i\rangle \\ |\tilde{\lambda}_i] \mapsto \tilde{\Lambda}_i |\tilde{\lambda}_i] \end{cases} \quad \Lambda_i = \begin{pmatrix} d_i & c_i \\ b_i & a_i \end{pmatrix} \quad \tilde{\Lambda}_i = \begin{pmatrix} \bar{a}_i & -\bar{b}_i \\ -\bar{c}_i & \bar{d}_i \end{pmatrix}$$

- **Work in (2,2) signature**

- $[2,1\rangle$**-shift** $\Leftrightarrow$ $$\begin{cases} \Lambda_i = \mathbb{I}_2 \ (i \neq 1) \\ \tilde{\Lambda}_j = \mathbb{I}_2 \ (j \neq 2) \end{cases}$$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

STEP 1: HOW THE BCFW SHIFT OPERATIONS ACT ON $\mathcal{O}_{\Delta,J}$

$$|\lambda_1'\rangle \;=\; \Lambda_1\,\epsilon_1\sqrt{2\omega_1}\begin{pmatrix}1\\ z_1\end{pmatrix} \;=\; \sqrt{\left|c_1z_1+d_1\right|}\;\epsilon_1'\sqrt{2\omega_1'}\begin{pmatrix}1\\ z_1'\end{pmatrix}$$

$$|\tilde\lambda_2'] \;=\; \tilde\Lambda_2\sqrt{2\omega_1}\begin{pmatrix}-\bar z_1\\ 1\end{pmatrix} \;=\; \sqrt{\left|\bar c_2\bar z_2+\bar d_2\right|}\;\sqrt{2\omega_2'}\begin{pmatrix}-\bar z_2'\\ 1\end{pmatrix}$$

- ***where***

$$z_1' \;=\; \frac{a_1z_1+b_1}{c_1z_1+d_1} \qquad \bar z_2' \;=\; \frac{\bar a_2\bar z_2+\bar b_2}{\bar c_2\bar z_2+\bar d_2}$$

$$\omega_1' \;=\; \omega_1\,|c_1z_1+d_1| \qquad \omega_2' \;=\; \omega_2\,|\bar c_2\bar z_2+\bar d_2|$$

- ***Little group scaling + Jacobian coming from ω-scaling***

$$\Lambda_1\,\tilde\Lambda_2\left\langle \mathcal{O}_{\Delta_1,J_1}(\epsilon_1,z_1,\bar z_1)\mathcal{O}_{\Delta_2,J_2}(\epsilon_2,z_2,\bar z_2)\cdots\right\rangle$$

$$=\;|c_1z_1+d_1|^{-2h_1}\,|\bar c_2\bar z_2+\bar d_2|^{-2\bar h_2}\left\langle \mathcal{O}_{\Delta_1,J_1}(\epsilon_1',z_1',\bar z_1')\mathcal{O}_{\Delta_1,J_1}(\epsilon_2',z_2',\bar z_2')\cdots\right\rangle$$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

STEP 2: TAKE $z \to \infty$ LIMIT

- $\mathcal{O}_1$ and $\mathcal{O}_2$ go to coincident limit

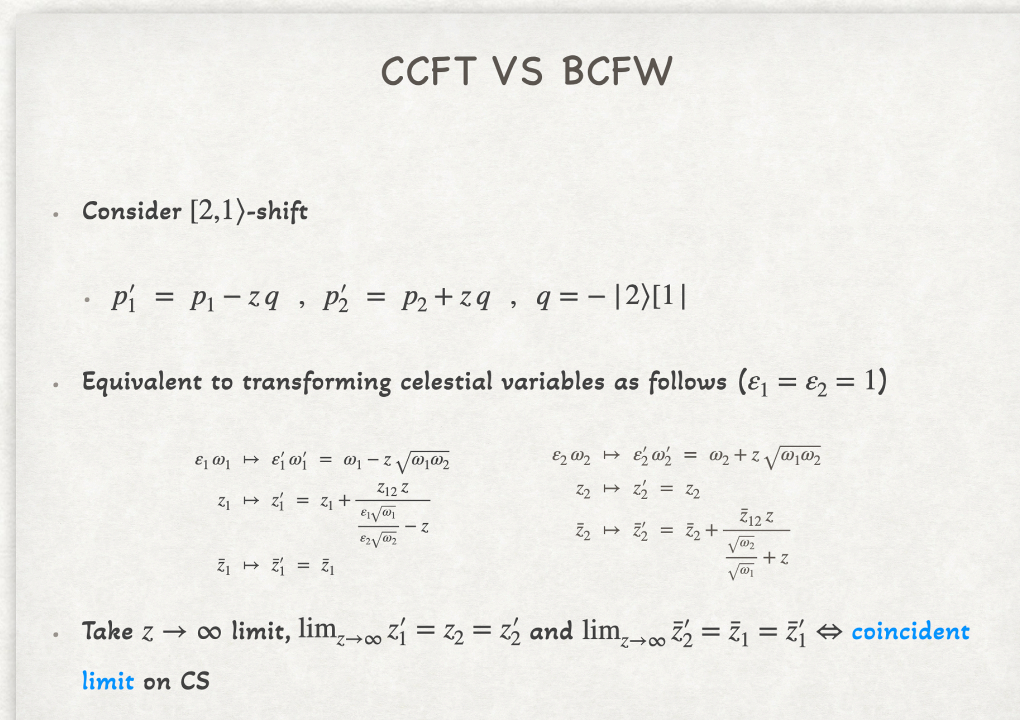
$$\Lambda_1 \tilde{\Lambda}_2 \left\langle \prod_j \mathcal{O}_{\Delta_j, J_j}(\epsilon_j, z_j, \bar{z}_j) \right\rangle = |c_1 z_1 + d_1|^{-2h_1} |\bar{c}_2 \bar{z}_2 + \bar{d}_2|^{-2\bar{h}_2} \left\langle \mathcal{O}_{\Delta_1, J_1}(z_1', \bar{z}_1') \mathcal{O}_{\Delta_1, J_1}(z_2', \bar{z}_2') \cdots \right\rangle$$

$$= |c_1 z_1 + d_1|^{-2h_1} |\bar{c}_2 \bar{z}_2 + \bar{d}_2|^{-2\bar{h}_2} \sum_P C_{12P}(z) \left\langle \mathcal{O}_P(z_2', \bar{z}_2') \cdots \right\rangle$$

We know

- $c_1 z_1 + d_1 \sim z \quad \bar{c}_2 \bar{z}_2 + \bar{d}_2 \sim z$

- $z_2' \sim z_2 \quad \bar{z}_2' \sim \bar{z}_1$



CCFT VS BCFW

- Consider $[2,1\rangle$-shift
  - $p_1' = p_1 - zq \ , \ p_2' = p_2 + zq \ , \ q = -|2\rangle[1|$
- Equivalent to transforming celestial variables as follows ($\varepsilon_1 = \varepsilon_2 = 1$)

$$\varepsilon_1 \omega_1 \mapsto \varepsilon_1' \omega_1' = \omega_1 - z\sqrt{\omega_1 \omega_2} \qquad \varepsilon_2 \omega_2 \mapsto \varepsilon_2' \omega_2' = \omega_2 + z\sqrt{\omega_1 \omega_2}$$

$$z_1 \mapsto z_1' = z_1 + \frac{z_{12}\, z}{\frac{\varepsilon_1\sqrt{\omega_1}}{\varepsilon_2\sqrt{\omega_2}} - z} \qquad z_2 \mapsto z_2' = z_2$$

$$\bar{z}_1 \mapsto \bar{z}_1' = \bar{z}_1 \qquad \bar{z}_2 \mapsto \bar{z}_2' = \bar{z}_2 + \frac{\bar{z}_{12}\, z}{\frac{\sqrt{\omega_2}}{\sqrt{\omega_1}} + z}$$

- Take $z \to \infty$ limit, $\lim_{z\to\infty} z_1' = z_2 = z_2'$ and $\lim_{z\to\infty} \bar{z}_2' = \bar{z}_1 = \bar{z}_1' \Leftrightarrow$ coincident limit on CS

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

STEP 3: DETERMINE $C_{12P}(z)$

- **Given the OPE**

$$\mathcal{O}_{\Delta_1,J_1}(z_1',\bar{z}_1')\,\mathcal{O}_{\Delta_2,J_2}(z_2',\bar{z}_2') \sim \sum_{J_P} g_{12P}\frac{(\bar{z}_{12}')^{J_1+J_2-J_P-1}}{z_{12}'}B(\Delta_1-1+J_2-J_P,\Delta_2-1+J_1-J_P)\,\mathcal{O}_{\Delta_P,J_P}$$
$$+\sum_{J_P} g_{12P}\frac{(z_{12}')^{J_P-J_1-J_2-1}}{\bar{z}_{12}'}B(\Delta_1-1-J_2+J_P,\Delta_2-1-J_1+J_P)\,\mathcal{O}_{\Delta_P,J_P}$$

[Pate, Raclariu, Strominger, Yuan, '19], [Himwich, Pate, Singh, '21]

- g_{12P} **is the 3pt coupling constant and** $B(a,b)$ **is the Euler Beta function**

- **We know** $z_{12}' \sim \dfrac{1}{z}$ **and** $\bar{z}_{12}' \sim \dfrac{1}{z}$

- **Inverse Mellin: Beta function tells us** $\omega^{\#} \sim z^{\#}$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

STEP 4: POWER COUNTING

- **Recall that**

$$\Lambda_1 \tilde{\Lambda}_2 \left\langle \prod_j \mathcal{O}_{\Delta_j, J_j}(\epsilon_j, z_j, \bar{z}_j) \right\rangle = |c_1 z_1 + d_1|^{-2h_1} |\bar{c}_2 \bar{z}_2 + \bar{d}_2|^{-2\bar{h}_2}$$

$$\left[\sum_{J_P} \frac{C^{(1)}_{12P}}{z'_{12}} + \sum_{J_P} \frac{C^{(2)}_{12P}}{\bar{z}'_{12}} \right] \left\langle \mathcal{O}_P(z_2, \bar{z}_1) \cdots \right\rangle$$

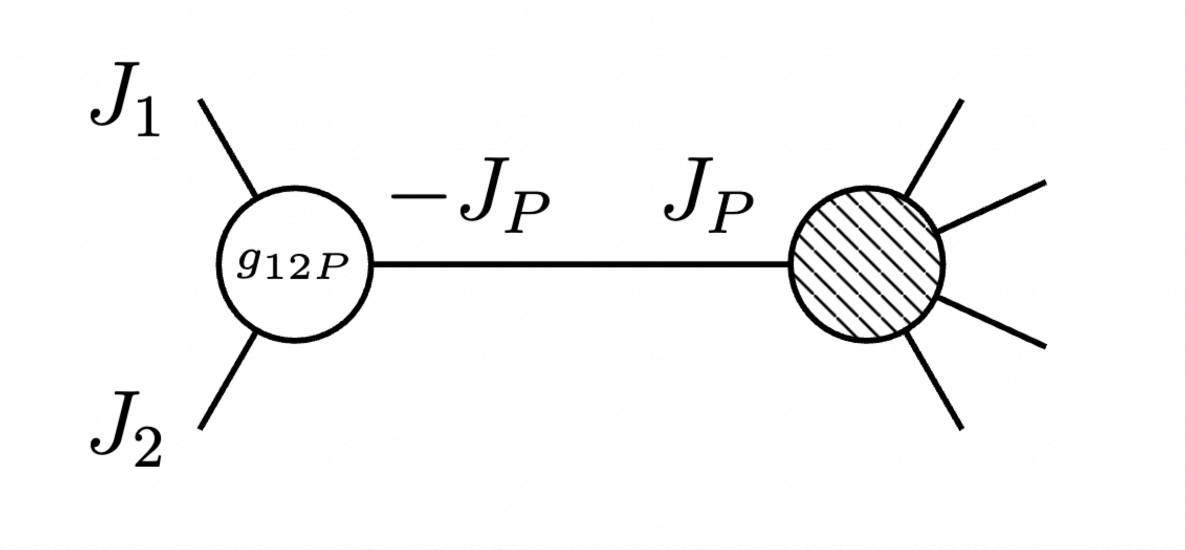
- $\dfrac{1}{z_{12}}$ **term ~** $z^{J_2 - J_1 - J_P}$

- $\dfrac{1}{\bar{z}_{12}}$ **term ~** $z^{J_2 - J_1 + J_P}$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

EXAMPLE: PURE YANG-MILLS

- *3-point interaction selects* J_P



- $\dfrac{1}{z_{12}}$ *term ~* $\overline{\text{MHV}}$ *vertex* $J_P = J_1 + J_2 - 1$
- $\dfrac{1}{\bar{z}_{12}}$ *term ~* MHV *vertex* $J_P = J_1 + J_2 + 1$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

EXAMPLE: PURE YANG-MILLS

	$\frac{1}{z_{12}}$ term		$\frac{1}{\bar{z}_{12}}$ term	
$\langle J_1, J_2]$	$J_P = J_1 + J_2 - 1$	$z^{J_2 - J_1 - J_P}$	$J_P = J_1 + J_2 + 1$	$z^{J_2 - J_1 + J_P}$
$\langle +,+]$	1	$\frac{1}{z}$	3	
$\langle +,-]$	-1	$\frac{1}{z}$	1	$\frac{1}{z}$
$\langle -,+]$	-1	z^3	1	z^3
$\langle -,-]$	-3		-1	$\frac{1}{z}$

- *Match the ones expected [Arkani-Hamed, Kaplan, '08]*

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

- **Comments:**
  - **For Yang-Mills, the color factors were suppressed in the OPE** $\mathcal{O}_1^a \mathcal{O}_2^b \sim i f^{abc} \mathcal{O}_P^c$
  - **Color-ordered: leading singularity comes from 1 and 2 adjacent**
  - **Look at the final result** $z^{J_2 - J_1 \pm J_P}$**: double copy relation is manifest via** $z_{GR} = z_{YM}^2$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

COMMENTS

- Celestial OPE ≃ Splitting function

$$\mathrm{Split}^{s_3=s_1+s_2-p-1}_{s_1,s_2} = \frac{\bar{z}_{12}^{p}}{z_{12}}(\varepsilon_1\omega_1)^{p-s_1}(\varepsilon_2\omega_2)^{p-s_2}(\varepsilon_1\omega_1+\varepsilon_2\omega_2)^{s_3}$$

$$\mathrm{Split}^{s_3=s_1+s_2+p+1}_{s_1,s_2} = \frac{z_{12}^{p}}{\bar{z}_{12}}(\varepsilon_1\omega_1)^{p+s_1}(\varepsilon_2\omega_2)^{p+s_2}(\varepsilon_1\omega_1+\varepsilon_2\omega_2)^{-s_3}$$

[Fan, Fotopoulos, Taylor, '19]

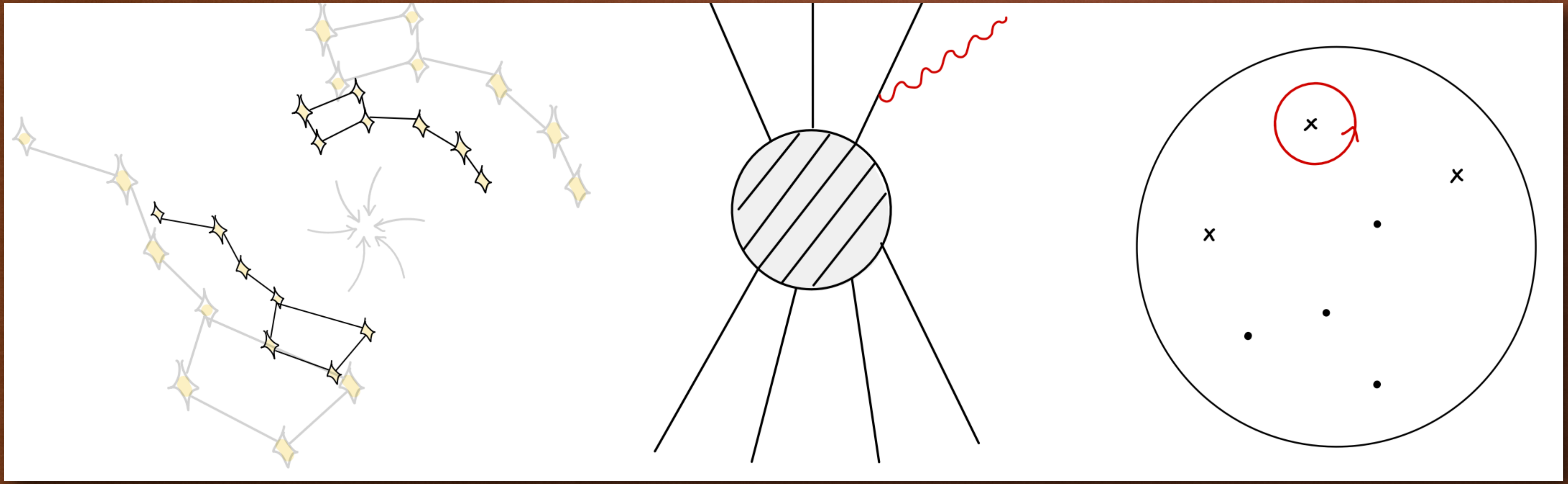
- $\mathrm{Split}^{s_3}_{s_1,s_2} \sim z^{\pm s_3}$

    - Proper little group factors to remove the z-dep. from (n-1)-pt amplitude $\Rightarrow z^{s_2-s_1\pm s_3}$

FROM CELESTIAL OPE TO LARGE-Z BEHAVIOR

- Outlook/Open questions
  - Unitarity and locality encoded in CCFT data?
  - Can we use CCFT data to constrain the bulk interaction?
  - Can we relate celestial BCFW and conformal block decomposition?
  - ...

INFINITESIMAL-Z STORY



[Pasterski's talk @ Strings 2021]

BG EQUATION VS BCFW

- Story begins with [Hu, Ren, Yelleshpur Srikant, Volovich, '21]
  - Color-ordered Banerjee-Ghosh equation is equivalent to infinitesimal BCFW shift acting on Parke-Taylor formula
  - BG equation derived in [Banerjee, Ghosh, '20] can be used to constrain MHV gluon correlators
  - Null states, lean to celestial bootstrap program,…

BG EQUATION VS BCFW

- In this work, we continue this exploration
    - BCFW as angle-dependent Lorentz transformation
    - BCFW interpreted as energy-dependent generalization of the hard superrotation transformation
    - Impletment BCFW shifts to all operators → recast as soft insertion → BG equation
    - Extend this story for super-BCFW

REVIEW OF BG EQUATION

- **Soft and collinear limit commutes**

- **Leading soft gluon current $j^{+,a} = \lim_{\Delta \to 1} (\Delta - 1)\, \mathcal{O}_{\Delta,J}$ is a Kac-Moody current**

  - **Ward Identity:**

$$\left\langle j^{+,a}(z) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{\Delta_i,J_i}(z_i, \bar{z}_i) \right\rangle = - \sum_{k=1}^{n} \frac{\mathcal{T}^a_k}{z - z_k} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{\Delta_i,J_i}(z_i, \bar{z}_i) \right\rangle$$

  - **$\mathcal{T}^a_k$ is the SU(N) generator: $\mathcal{T}^a_k \mathcal{O}^b_j = \delta_{kj}\, i f^{abc} \mathcal{O}^c_j$**

  - **Restatment of the leading soft gluon theorem**

REVIEW OF BG EQUATION

- **Subleading soft gluon current** $S^{+,a} = \lim_{\Delta \to 0} \Delta\, \mathcal{O}_{\Delta,J}$
  - **Ward Identity:**

$$\left\langle S^{+,a}(z,\bar{z}) \prod_{i=1}^{n} \mathcal{O}^{a_i}_{\Delta_i,J_i}(z_i,\bar{z}_i) \right\rangle$$

  - $$= \sum_{k=1}^{n} \epsilon_k \frac{2\bar{h}_k - 1 + (\bar{z}_k - \bar{z})\bar{\partial}_k}{z - z_k} \mathcal{T}^a_k T_k^{-1} \left\langle \prod_{i=1}^{n} \mathcal{O}^{a_i}_{\Delta_i,J_i}(z_i,\bar{z}_i) \right\rangle$$
  - T_k **is the conformal weight shifting operator:** $T_k\, \mathcal{O}_{\Delta_j,J_j} = \delta_{kj}\, \mathcal{O}_{\Delta_j+1,J_j}$
  - **Restatement of the subleading soft gluon theorem**

REVIEW OF BG EQUATION

CONSTRAINT ON OPE

- *From the Ward Identity, we can extract the OPE* $S^{+,a}(z,\bar{z})\,\mathcal{O}^{+,b}(z_1,\bar{z}_1)$

$$S^{+,a}(z,\bar{z})\,\mathcal{O}^{+,b}(z_1,\bar{z}_1) \;\sim\; \left[\frac{1}{z-z_1}(\cdots) + \sum_{p=0}^{\infty}(z-z_1)^p(\cdots)\right]\mathcal{O}^{+,b}(z_1,\bar{z}_1)$$

$$+\;(\bar{z}-\bar{z}_1)\left[\frac{1}{z-z_1}(\cdots) + \sum_{p=0}^{\infty}(z-z_1)^p(\cdots)\right]\mathcal{O}^{+,b}(z_1,\bar{z}_1)$$

- *On the other hand, the OPE of two hard gluons is determined by asymptotic symmetries / collinear limits*

- $$S^{+,a}(z,\bar{z})\,\mathcal{O}^{+,b}(z_1,\bar{z}_1) \;=\; \lim_{\Delta\to 0}\,\Delta\,\mathcal{O}^{+,a}_{\Delta}(z,\bar{z})\,\mathcal{O}^{+,b}_{\Delta_1}(z_1,\bar{z}_1)$$

REVIEW OF BG EQUATION

- Equating these two ($\bar{z} = \bar{z}_1$) gives us
- $\Psi^a \;\equiv\; \mathscr{D}\, \mathcal{O}^{+,a}_{\Delta_1}(z_1, \bar{z}_1) \;=\; 0$
- Ψ^a is a null state: $L_1 \Psi^a = \bar{L}_1 \Psi^a = j^{+,a}_m \Psi^b = 0$ (BMS primary)
- Inserting this into a correlation function yields the differential equation
- $\left\langle \Psi^a\, \mathcal{O}^{a_2}_{\Delta_2, J_2} \cdots \mathcal{O}^{a_n}_{\Delta_n, J_n} \right\rangle \;=\; 0$

REVIEW OF BG EQUATION

BG EQUATION

- Benerjee-Ghosh equation for MHV gluon:

$$\left[\frac{C_A}{2}\frac{\partial}{\partial z_i} - h_i \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{T_i^a T_j^a}{z_i - z_j} + \frac{1}{2}\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\epsilon_j \left(2\bar{h}_j - 1 - (\bar{z}_i - \bar{z}_j)\frac{\partial}{\partial \bar{z}_j}\right)}{z_i - z_j} T_j^a P_j^{-1} T_i^a P_{-1,-1}(i)\right] \left\langle \prod_{k=1}^{n} \mathcal{O}^{a_k}_{h_k,\bar{h}_k}(z_k, \bar{z}_k) \right\rangle_{\mathrm{MHV}} = 0$$

[Banerjee, Ghosh, '21]

- One equation for each positive helicity gluon

- No obvious momentum space origin

- Easier to deal with color-ordered amplitudes

REVIEW OF BG EQUATION

MOMENTUM SPACE ORIGIN

- **Look at the color-ordered version of the BG:**

$$\left(\partial_i - \frac{\Delta_i}{z_{i-1,i}} - \frac{1}{z_{i+1,i}} + \epsilon_i\epsilon_{i-1}\,\frac{\Delta_{i-1} - J_{i-1} - 1 + \bar{z}_{i-1,i}\bar{\partial}_{i-1}}{z_{i-1,i}} T_i T_{i-1}^{-1}\right)\tilde{A}_n(1,\cdots,n) \;=\; 0$$

- **Map back to the $\{\lambda, \tilde{\lambda}\}$ basis:**

$$D_{i,i-1}A_n := \left(\lambda_{i-1}\frac{\partial}{\partial\lambda_i} - \tilde{\lambda}_i\frac{\partial}{\partial\tilde{\lambda}_{i-1}}\right)A_n = \frac{\langle i-1\,, i+1\rangle}{\langle i+1\,, i\rangle}A_n$$

- **LHS: BCFW shift $\lambda_i \mapsto \lambda_i + z\lambda_{i-1}$, $\tilde{\lambda}_{i-1} \mapsto \tilde{\lambda}_{i-1} - z\tilde{\lambda}_i$ can be implemented via**

$$A(z) = \exp z\, D_{i,i-1}A(0)$$

- **RHS: Parke-Taylor formula**

[Hu, Ren, Yelleshpur Srikant, Volovich, '21]

BCFW AS HARD SUPERROTATION

- **Hard superrotation charge** $Q_H[\xi_Y] = i\mathscr{L}_\xi$
  - $\mathscr{L}_\xi|\omega_k, z_k, \bar{z}_k\rangle \ = \ \Big[Y^z_k \partial_{z_k} - \frac{1}{2} D_z Y^z(-\omega_k \partial_{\omega_k} + s_k) + h.c. \Big] |\omega_k, z_k, \bar{z}_k\rangle$
- **For BCFW shift,** $w \to 0$
  - $z_i \ \mapsto \ z_i + \beta_i w - \alpha_i \beta_i w^2 + \cdots$
  - ω_i **scaling** $\Rightarrow D_z Y^z(z_i) = -2\alpha_i w$
  - $Y^z(z \sim z_i) \ = \ w \Big[\beta_i - 2\alpha_i (z - z_i) + \mathcal{O}((z - z_i)^2) \Big] + \mathcal{O}(w^2)$

BCFW AS HARD SUPERROTATION

- Given $Y^z(z \sim z_i) \;=\; w\left[\beta_i - 2\,\alpha_i\,(z-z_i) + \mathcal{O}((z-z_i)^2)\right] + \mathcal{O}(w^2)$

- We can match this behavior with meromorphic $Y^z(z)$ following [Pasterski, '15]

- Schematically $Y|_{z\sim z_i} \propto w\sqrt{\frac{\omega_j}{\omega_i}}\left[z_{ij} + 2(z-z_i) + \ldots\right]$ for $\lambda_i \mapsto \lambda_i + w\,\lambda_j$

- Energy-dependent generalization of hard superrotation

BCFW AS SOFT INSERTION

- Yang-Mills, gluon correlator
- Consider $\langle i, k]$-shift for all $k \neq i$ with fixed i
- Shift parameter as $z\,\frac{(-2)\omega_i}{\langle ik\rangle}\,\mathcal{T}^a_k$
- This shift can be impletmented by the exponential

$$\exp z\left\{\sum_{k\neq i}\frac{(-2)\omega_i}{\langle ik\rangle}\,\mathcal{T}^a_k\left[\lambda_k\frac{\partial}{\partial\lambda_i}-\tilde{\lambda}_i\frac{\partial}{\partial\tilde{\lambda}_k}\right]\right\}A_n$$

BCFW AS SOFT INSERTION

- Impletment to the celestial correlator as

$$\exp\left\{ z\left[-j_0^a(i)\,L_{-1}(i) + 2\,h_i\,j_{-1}^a(i) + P_{-\frac{1}{2},-\frac{1}{2}}(i)\,\mathcal{J}^a_{-\frac{1}{2},\frac{1}{2}}(i)\right]\right\}\left\langle \mathcal{O}^{a_i}_{\Delta_i,J_i}(z_i,\bar{z}_i)\prod_j \mathcal{O}^{a_j}_{\Delta_j,J_j}(z_j,\bar{z}_j)\right\rangle$$

- $j_0^a(i) \;=\; -\,\mathcal{T}_i^a$ gauge transformation
- $P_{-\frac{1}{2},-\frac{1}{2}}(i)\mathcal{O}^{a_i}_{\Delta_i,J_i} \;=\; \mathcal{O}^{a_i}_{\Delta_i+1,J_i}$ global translation
- $j_{-1}^a(i) \;=\; \sum_{k\neq i}\frac{\mathcal{T}_k^a}{z_{ki}}$ inserting a leading soft $(\Delta\to 1)$ helicity $+1$ gluon collinear with $\mathcal{O}_i$
- $\mathcal{J}^a_{-\frac{1}{2},\frac{1}{2}}(i) \;=\; \sum_{k\neq i}\varepsilon_i\varepsilon_k\,\frac{2\bar{h}_k - 1 + \bar{z}_{ki}\bar{\partial}_k}{z_{ik}}\,T_k^{-1}\,\mathcal{T}_k^a$ inserting a sub-leading soft $(\Delta\to 0)$ helicity $+1$ gluon collinear with $\mathcal{O}_i$

BCFW AS SOFT INSERTION

- *Act on the MHV correlator*

- $\exp\{\, z[\cdots]\,\}\langle\quad\rangle_{\text{MHV}} \;=\; \langle\quad\rangle_{\text{MHV}} + z[\cdots]\langle\quad\rangle_{\text{MHV}} + \mathcal{O}(z^2)$

- *Compare both sides of the equation ⇒ 1st order PDE*

- *Take* $J_i = +1$ *⇒ Banerjee-Ghosh equation*

$$\Big[-j_0^a(i)\, L_{-1}(i) + (2h_i - 1)\, j_{-1}^a(i) \;+\; P_{-\frac{1}{2},-\frac{1}{2}}(i)\, \mathcal{J}^a_{-\frac{1}{2},\frac{1}{2}}(i) \Big] \Big\langle \mathcal{O}^{a_i}_{\Delta_i,+}(z_i,\bar{z}_i) \prod_j \mathcal{O}^{a_j}_{\Delta_j,J_j}(z_j,\bar{z}_j) \Big\rangle_{\text{MHV}} \;=\; 0$$

BCFW AS SOFT INSERTION

- Extend to super-BCFW is straightforward
- $\mathcal{N}=4$ SYM: $\eta_k^A \;\mapsto\; \eta_k^A - z\,\dfrac{(-2)\omega_i}{\langle ik\rangle}\,\mathcal{T}^a_k\,\eta_i^A$ for all $k \neq i$
- Implement to the celestial superamplitude

$$\exp z\left\{\text{same as above} - \frac{1}{\sqrt{2}}\tilde{Q}^A_{-\frac{1}{4},-\frac{1}{4}}(i)\,S^a_{-\frac{3}{4},\frac{1}{4}}(i)\right\}\left\langle\prod_j \Omega_j^{a_j}\right\rangle$$

  - $\tilde{Q}^A_{-\frac{1}{4},-\frac{1}{4}}(i) \;=\; \varepsilon_i\sqrt{2}\,T_i^{\frac{1}{2}}\,\eta_i^A$ i-th supercharge
  - $S^a_{-\frac{3}{4},\frac{1}{4}}(i) \;=\; \displaystyle\sum_{k\neq i}\varepsilon_k T_k^{-\frac{1}{2}}\frac{(-1)^{\sigma_k}\mathcal{T}^a_k}{z_{ik}}\,\frac{\partial}{\partial\eta_k^A}$ inserting a leading soft ($\Delta \to \frac{1}{2}$) helicity $+\frac{1}{2}$ gluino that is collinear with $\mathcal{O}_i$

BCFW AS SOFT INSERTION

- Act on the MHV correlator
- 1st order PDE takes the following form

$$\left[\text{same as above (BG operator)} - \frac{1}{\sqrt{2}} \tilde{Q}^A_{-\frac{1}{4},-\frac{1}{4}}(i)\, S^a_{-\frac{3}{4},\frac{1}{4}}(i) \right] \left\langle \prod_j \Omega_j^{a_j} \right\rangle_{\text{MHV}} = 0$$

- Look at the component level:
  - $J_i = +1 \Rightarrow$ Banerjee-Ghosh equation $\left[\text{BG operator} \right] \left\langle \mathcal{O}^{a_i}_{i,+} \prod_j \mathcal{O}^{a_j}_j \right\rangle = 0$
  - $J_i = -1 \Rightarrow \left[\text{BG operator} \right] \left\langle \mathcal{O}^{a_i}_{i,-} \mathcal{O}^{a_s}_{s,-} \prod_j \mathcal{O}^{a_j}_{j,+} \right\rangle = \sum_{k \neq i,s} (\cdots) \left\langle \tilde{\lambda}^{a_i}_{i,A} \lambda^{a_s,A}_k \prod_j \mathcal{O}^{a_j}_j \right\rangle$
  - $\tilde{Q}^A(i) \Rightarrow$ BG operator as raising the helicity of $\mathcal{O}_i$

INFINITESIMAL-Z STORY

- Outlook:
  - BCFW vs BG equation for MHV Graviton?
  - Extend to NMHV or higher order?
  - Extend to finite-z?
  - …

THANK YOU!