Spectrum of quantum KdV hierarchy in the semiclassical limit

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Based on

- **JHEP 09(2022)169 [2208.01062]**
- **JHEP 05(2020)041 [2002.08368]**

with **Anatoly Dymarsky, Ashish Kakkar, Kirill Pavlenko**

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Introduction

The AdS/CFT correspondence

Black hole formation in AdS $\Longleftrightarrow$ Thermalization in CFT

- Solving the dynamics of CFTs is hard.
- 2D CFT is slightly tractable because the dynamics is highly constrained by the conformal symmetry.
- The conf sym in 2D is infinite dimension (the Virasoro sym).
- Any 2D CFT has an infinite number of commuting conserved charges known as quantum KdV charges.

(Korteweg & de Vries)

Virasoro algebra

- 2D CFT is controlled by Virasoro algebra.

  We consider (1+1)D CFT on a circle $\varphi \sim \varphi + 2\pi$.

  Stress energy tensor $T(\varphi) = \sum_n L_n e^{-in\varphi} - \frac{c}{24}$

  L_n satisfy the Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

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$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

- The sector of stress-energy tensor is integrable.

  Any 2D CFT has an infinite number of commuting conserved charges known as **quantum KdV charges**.

  (Korteweg & de Vries)

Quantum KdV charges

Quantum KdV charges $Q_{2k-1} = \int_0^{2\pi} \frac{d\varphi}{2\pi} T_{2k}(\varphi), \quad k = 1, 2, \cdots$

- Local densities

$$T_2 = T\,, \quad T_4 =: T^2 :\,, \quad T_6 =: T^3 : + \frac{c+2}{12} : T'^2 :\,, \quad T_8 = \cdots$$

General expressions are not known but can be constructed order by order.

$$Q_1 = \int_0^{2\pi} \frac{d\varphi}{2\pi} T(\varphi) = L_0 - \frac{c}{24} \longleftarrow$$

usual Hamiltonian

$$Q_3 = L_0^2 + 2\sum_{n=1} L_{-n} L_n - \frac{c+2}{12} L_0 + \frac{c}{24}\left(\frac{c}{24} + \frac{11}{60}\right)$$

They commute with each other. $[Q_{2k-1}, Q_{2\ell-1}] = 0$

Generalized Gibbs ensemble in CFT

■ Equilibration with conserved charges

- We expect that a generic excited state is described by an equilibrium ensemble after a sufficient time.

- If the system has a conserved charge, the equilibrium state is generally not the Gibbs ensemble $e^{-\beta H}$.
We need to introduce the chemical potential $e^{-\beta H - \mu Q}$.

Generalized Gibbs ensemble (GGE)

■ 2D CFT has infinite # of charges Q_{2k-1} $\left(H = L_0 - \frac{c}{24} = Q_1\right)$

$\rho_{\text{GGE}} = e^{-\sum_{k=1}^{\infty} \mu_{2k-1} Q_{2k-1}}$ is typical rather than $e^{-\beta H}$

Spectrum of KdV charges

- To study GGE $\rho_{\mathrm{GGE}} = e^{-\sum_{k=1}^{\infty} \mu_{2k-1} Q_{2k-1}}$ in CFT, we need spectrum of KdV charges.

- **Explicit expressions of the spectrum are unknown.**

- A tractable way is large c expansion.

  ⇨ a large c spectrum of KdV charges including (sub)subleading corrections. [2208.01062]

- Large c is also related to gravity.

  ⇨ New black hole solutions corresponding to non-trivial GGE [2002.08368]

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- Introduction
- Review of the classical KdV hierarchy
- 3D gravity and the KdV hierarchy
  [2002.08368]
- Spectrum of quantum KdV charges in large c
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KdV equation

- a mathematical model of waves on shallow water surfaces

$$u(t,x)$$

Fig. from wikipedia

$$\boxed{\dot{u} = 6uu' - 4u'''}$$

Introduced by Boussinesq (1877)
and rediscovered by Korteweg and de Vries (1895).

- Nonlinear but integrable

- Infinite number of local conserved charges

KdV equation on circle

- Let's consider the same eq. on 1-dim circle $\varphi \sim \varphi + 2\pi$

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$$Q_1 = \int_0^{2\pi} \frac{d\varphi}{2\pi} u, \quad Q_3 = \int_0^{2\pi} \frac{d\varphi}{2\pi} (u^2 - \frac{4}{3}u''), \quad Q_5 = \cdots$$

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General charges are given by the Gelfand-Dikii polynomials

$$\boxed{Q_{2k-1} = \int_0^{2\pi} \frac{d\varphi}{2\pi} R_k}$$

$$R_0 = 1, \quad R_1 = u, \quad R_2 = u^2 - \frac{4}{3}\partial^2 u, \quad R_3 = u^3 - 4u\partial^2 u - 2(\partial u)^2 + \frac{8}{5}\partial^4 u$$

$$\partial_\varphi R_{k+1} = \frac{k+1}{2k+1} \mathcal{D}_u R_k, \quad \mathcal{D}_u = u' + 2u\partial_\varphi - 2\partial_\varphi^3 \qquad (\dot{u} = \mathcal{D}_u u)$$

KdV hierarchy

An infinite tower of equations

$$\dot{u} = u'$$

$$\dot{u} = 6uu' - 4u''' \qquad \text{KdV eq.}$$

$$\dot{u} = 15u^2u' - 40u'u'' - 60uu''' + 24u'''''$$

$$\cdots$$

$$\dot{u} = k\mathcal{D}_u R_{k-1}$$

KdV hierarchy

An infinite tower of equations

$$\text{KdV hierarchy} \quad \begin{cases} \dot{u} = u' \\ \dot{u} = 6uu' - 4u''' \qquad \longleftarrow \quad \text{KdV eq.} \\ \dot{u} = 15u^2u' - 40u'u'' - 60uu''' + 24u''''' \\ \qquad \cdots \\ \dot{u} = k\mathcal{D}_u R_{k-1} \end{cases}$$

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$$\begin{cases} \dot{u} = u' \\ \dot{u} = 6uu' - 4u''' \\ \dot{u} = 15u^2u' - 40u'u'' - 60uu''' + 24u''''' \\ \quad \cdots \\ \dot{u} = k\mathcal{D}_u R_{k-1} \end{cases}$$

KdV eq.

- Each eq. is integrable and the conserved charges are given by

$$\boxed{Q_{2k-1} = \int_0^{2\pi} \frac{d\varphi}{2\pi} R_k}$$

classical KdV charges

Poisson bracket

Mode expansion on a circle: $u = \sum_n u_n e^{in\varphi}$

Introduce a Poisson bracket for modes

$$\boxed{i\{u_n, u_m\} = (n-m)u_{n+m} + 2n^3\delta_{n+m,0}}$$

Poisson bracket

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Introduce a Poisson bracket for modes

$$\boxed{i\{u_n, u_m\} = (n-m)u_{n+m} + 2n^3\delta_{n+m,0}}$$

Each eq. in the KdV hierarchy is generated by KdV charge

$$\dot{u} = k\mathcal{D}_u R_{k-1} = \{Q_{2k-1}, u\}$$

KdV charge is a Hamiltonian in this sense.

KdV charges commute with each other in this Poisson bracket.

$$\{Q_{2k-1}, Q_{2\ell-1}\} = 0$$

Classical limit of Virasoro

The Poisson algebra $i\{u_n, u_m\} = (n-m)u_{n+m} + 2n^3\delta_{n+m,0}$

is the classical limit of the Virasoro algebra.

$$\frac{1}{\hbar}[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}$$

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If we set $T(\varphi) = \sum_n L_n e^{-in\varphi} - \frac{c}{24} \equiv \frac{c}{24}\sum_n u_n e^{in\varphi},$

the quantum Virasoro alg can be written as

$$\frac{c}{24\hbar}[u_n, u_m] = (n-m)u_{n+m} + 2n^3\delta_{n+m,0}$$

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$$\frac{c}{24\hbar}[u_n, u_m] = (n-m)u_{n+m} + 2n^3\delta_{n+m,0}$$

Replacement rule $\frac{c}{24i\hbar}[\ ,\ ] \to \{\ ,\ \}$ $\left(\text{usual } \frac{1}{i\hbar}[\ ,\ ] \to \{\ ,\ \}\right)$

The effective Planck const $\hbar_{eff} = \left(\frac{24}{c}\right)\hbar$ is small for large c.

The large c physics is effectively classical even at finite $\hbar$.

Classical and quantum KdV charges

- Classical KdV charges: $Q^{cl}_{2k-1} = \int_0^{2\pi} \frac{d\varphi}{2\pi}(u^k + \cdots)$

- Quantum KdV charges: $Q^{q}_{2k-1} = \int_0^{2\pi} \frac{d\varphi}{2\pi}(:T^k: + \cdots)$

■ In the large c limit, quantum KdV charges are reduced to classical ones.

- Replacement rule $T \sim \frac{c}{24} u$

$$Q^{q}_{2k-1} \sim \left(\frac{c}{24}\right)^k Q^{cl}_{2k-1}$$

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AdS/CFT

- Let's consider GGE in 2D CFT.

$$\rho_{GGE} = \frac{1}{Z_{GGE}} e^{-\sum_k \mu^{\mathrm{CFT}}_{2k-1} Q^q_{2k-1}}$$

- For holographic 2D CFT, the large c physics is described by the classical gravity in 3D AdS.

- **What is the dual geometry of GGE in 2D CFT?**

Generalized Hamiltonian

- partition function $Z_{GGE} = \mathrm{tr}\, e^{-\sum_k \mu^{\mathrm{CFT}}_{2k-1} Q^q_{2k-1}}$
- We can regard this as the usual thermal partition function at $\beta = 1$ but the Hamiltonian is generalized:

$$Z_{GGE} = \mathrm{tr}\, e^{-H_{gen}} \quad \text{with} \quad H_{gen} = \sum_k \mu^{\mathrm{CFT}}_{2k-1} Q^q_{2k-1}$$

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- Partition function can be obtained by the Euclidean path-int with the generalized Hamiltonian

- AdS/CFT says $Z_{CFT} = Z_{\mathrm{grav}}$

■ Dual geometry is saddle of the gravitational path-int with generalized Hamiltonian.

Relation to classical KdV

- Large c CFT is related to classical KdV hierarchy.
- classical 3D gravity should be also related to it.

Indeed, that is the case!

Relation to classical KdV

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Indeed, that is the case!

- We will see that the generalization of Hamiltonian is realized by the generalization of the boundary condition of metric.
- problem is reduced to
  solving Einstein eq. with new boundary conditions.
- We will see this is solving (general) KdV eqs.

3D pure gravity

- Einstein gravity can be formulated as Chern-Simons theory

$$A_{\pm} = \left(\omega^a \pm \frac{1}{\ell} e^a\right) T_a \qquad [T_a, T_b] = \epsilon_{ab}{}^c T_c, \quad \mathrm{Tr}(T_a T_b) = \frac{1}{2}\eta_{ab}$$

$$S = S_{CS}[A_+] - S_{CS}[A_-], \quad S_{CS}[A] = \frac{\ell}{16\pi G}\int \mathrm{Tr}\left(A dA + \frac{2}{3}A^3\right) + (\mathrm{bdry})$$

metric $\quad g_{\mu\nu} = \frac{\ell^2}{2}\mathrm{Tr}[(A_\mu^+ - A_\mu^-)(A_\nu^+ - A_\nu^-)]$

- EoM is equivalent to Einstein eq. with cosmological const.

$F_{\pm} = 0$ bulk dynamics is topological

- General sols are parametrized only by the bdry d.o.f.

Asymptotic AdS condition

- The boundary d.o.f. is called boundary graviton. $u_\pm(t,\varphi)$
  (~ CFT stress tensor)
- The dynamics depends on the choice of the boundary condition.
- Generic asymptotically AdS boundary condition (in CS formulation) is given by [Bunster, Henneaux, Perez, Tempo, Troncoso (2014)].

$$\left[T_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \right]$$

$$\begin{cases} A_r^\pm \sim \mp \dfrac{1}{r} T_1 \\ A_\varphi^\pm \sim \dfrac{r}{\ell} T_\pm - \dfrac{\ell}{4r} \underbrace{u_\pm(t,\varphi)}_{\text{bdry graviton}} T_\mp \\ A_t^\pm \sim \pm \left(f_\pm A_\varphi^\pm \pm f'_\pm T_1 + \dfrac{\ell}{2r} f''_\pm T_\mp \right) \end{cases}$$

EOM for boundary gravitons

- For the gauge fixing, we have

$$F^{\pm}_{\varphi t} = \frac{\ell}{4r}(\dot{u}_{\pm} \mp \mathcal{D}_{u_{\pm}} f_{\pm})$$

- CS EoM $F^{\pm}_{\varphi t} = 0$ $\Rightarrow$ $\dot{u}_{\pm} = \pm \mathcal{D}_{u_{\pm}} f_{\pm}$

- On-shell Hamiltonian is given by $H^{+}[u_{+}] + H^{-}[u_{-}]$

  where $2\pi \frac{\delta H^{\pm}}{\delta u_{\pm}} = f_{\pm}$

- EoM can be written by the Poisson bracket for the classical KdV hierarchy as

$$\boxed{\dot{u}_{\pm} = \pm\{H^{\pm}, u_{\pm}\}}$$

Identification

- boundary gravitons are CFT stress tensors as

$$u^+ = \frac{24}{c}T, \quad u^- = \frac{24}{c}\bar{T}$$

- Generalized CFT Hamiltonian

$$H_{gen} = \sum_k \mu_{2k-1}^{\mathrm{CFT}} Q_{2k-1}^q$$

$$\Rightarrow \quad H^+ = \sum_k \mu_{2k-1}^{cl} Q_{2k-1}^{cl} \quad \text{with} \quad \mu_{2k-1}^{\mathrm{CFT}} = \left(\frac{24}{c}\right)^k \mu_{2k-1}^{cl}$$

$\dot{u}_\pm = \pm\{H^\pm, u_\pm\}$ is the generalized KdV equation.

$\Rightarrow$ Integrable EOM

Examples

- conventional choice $H^{\pm} = Q_1$

➡ EoM $\dot{u}_{\pm} = \pm u'_{+}$ (left or right mover condition)

$$f_{\pm} = 1 \longrightarrow A_t^{\pm} \sim \pm A_{\varphi}^{\pm}$$

Original Brown-Henneaux boundary cond

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Original Brown-Henneaux boundary cond

- $H^{\pm} = Q_{2k-1}$

$\Rightarrow$ EoM $\dot{u}_{\pm} = \pm\{Q_{2k-1}, u_{\pm}\}$ k-th KdV eq.

- $H^{\pm} = \sum_k \mu_{2k-1}^{cl} Q_{2k-1}^{cl}$

$\Rightarrow$ more complicated but integrable EoM (general KdV eq.)

left-right symmetric cases

- For simplicity, we focus on the case $\begin{cases} u_+ = u_- \ (\equiv u) \\ H^+ = H^- \ (\equiv H) \end{cases}$

- From EoM $\dot{u}_\pm = \pm\{H^\pm, u_\pm\}$

  only static solutions are allowed: $\dot{u} = 0$

- In terms of static $u(\varphi)$, metric is given by

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\varphi\varphi}d\varphi^2$$

[Perez, Tempo, Troncoso (2016)]

$$\begin{cases} g_{tt} = -\left(fr - \dfrac{\ell^2}{4r}(uf - 2f'')\right)^2 \\ g_{rr} = \dfrac{\ell^2}{r^2}, \quad g_{\varphi\varphi} = \left(r + \dfrac{\ell^2}{4r}u\right)^2 \end{cases}$$

$$f = 2\pi \frac{\delta H}{\delta u}$$

Static solutions

$$g_{tt} = -\left(fr - \frac{\ell^2}{4r}\left(uf - 2f''\right)\right)^2, \quad g_{rr} = \frac{\ell^2}{r^2}, \quad g_{\varphi\varphi} = \left(r + \frac{\ell^2}{4r}u\right)^2$$

Static EoM $\dot{u} = D_u f = 0$ Any sol is locally AdS.

e.g. $u = -1$ pure AdS (thermal AdS in Euclidean signature)

$u = u_0 \, (\geq 0)$ BTZ black holes (non-rotating)

Static solutions

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- In general the metric represents non-rotating black holes

horizon: $r_h(\varphi) = \frac{\ell}{2}\sqrt{\frac{fu - 2f''}{f}}$

temperature: $T = \frac{1}{2\pi}\sqrt{f^2 u + f'^2 - 2ff''} \longleftarrow$ const.

Conventional case

- conventional choice $H = Q_1$

Static EoM $0 = \{H, u\} = u'$ ➡ $u = \text{const.} \equiv u_0$

$$g_{tt} = -\left(r - \frac{\ell^2 u_0}{4r}\right)^2, \quad g_{rr} = \frac{\ell^2}{r^2}, \quad g_{\varphi\varphi} = \left(r + \frac{\ell^2}{4r} u_0\right)^2$$

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- positive u_0 ⟶ BTZ black holes $r_h = \frac{\ell}{2}\sqrt{u_0}$
- negative u_0 ⟶ Conical sing. at $r = \frac{\ell}{2}\sqrt{-u_0}$

  except for $u_0 = -1$ ⟶ pure AdS

➡ Allowed solutions are pure AdS or BTZ (No hair theorem)

KdV charges of black holes

- conventional choice $H = Q_1$

$$\Rightarrow \quad u = \text{const.} \equiv u_0$$

$$\longrightarrow \quad Q_1^{cl} = u_0, \quad Q_{2k-1}^{cl} = u_0^k = \left(Q_1^{cl}\right)^k$$

- For generic choices of Hamiltonian $H = \sum_k \mu_{2k-1}^{cl} Q_{2k-1}^{cl}$ angle-dependent $u(\varphi)$ is allowed.

$$\longrightarrow \quad Q_{2k-1}^{cl} \neq \left(Q_1^{cl}\right)^k$$

We will call such geometry *KdV-charged black holes*.

First nontrivial case

As a first nontrivial case, let's introduce Q_3

$$H = \mu_1^{cl} Q_1^{cl} + \mu_3^{cl} Q_3^{cl}$$

EoM $\Rightarrow$

$$\underbrace{\alpha u' + 2uu' - \frac{4}{3}u'''}_{\text{total derivative}} = 0 \qquad \alpha = \frac{\mu_1^{cl}}{3\mu_3^{cl}}$$

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Integrate it $\Rightarrow$ $\underbrace{\alpha u + u^2 - \frac{4}{3}u'' = R}_{\text{Newton eq. } u'' = -\partial_u V(u)}$

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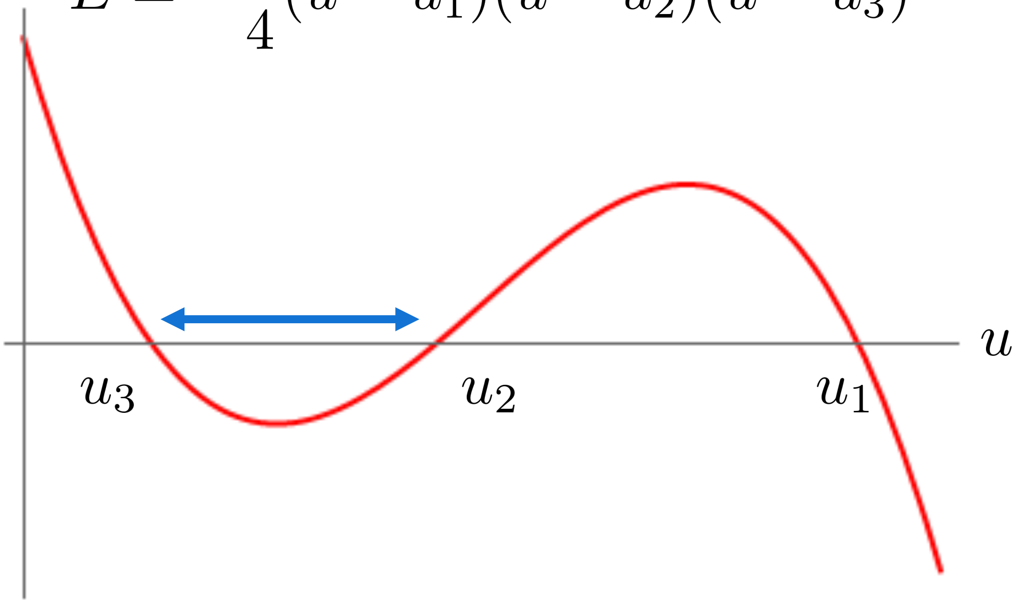
Integrate it again

$$\Rightarrow \quad \frac{1}{2}(u')^2 + V(u) = E, \quad V(u) = -\frac{1}{4}u^3 - \frac{3\alpha}{8}u^2 + \frac{3R}{4}u$$

Potential problem

$$\frac{1}{2}(u')^2 + V(u) = E, \quad V(u) = -\frac{1}{4}u^3 - \frac{3\alpha}{8}u^2 + \frac{3R}{4}u$$

$$V(u) - E = -\frac{1}{4}(u-u_1)(u-u_2)(u-u_3)$$



$u(\varphi)$ oscillates in the range $u_3 \leq u \leq u_2$

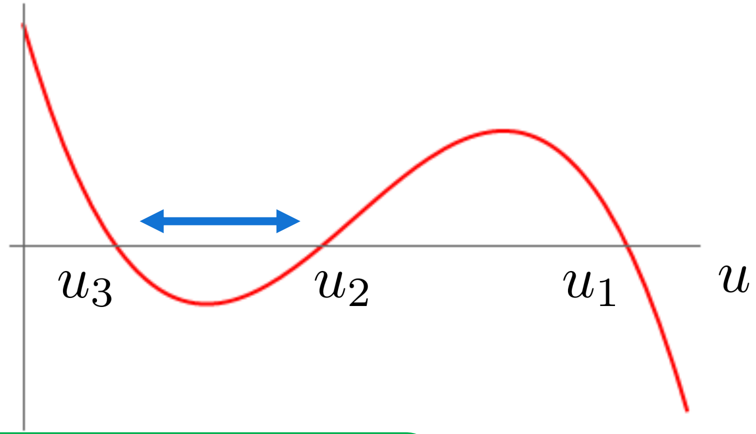
$$\varphi \propto \int \frac{du}{\sqrt{(u-u_1)(u-u_2)(u-u_3)}}$$

Elliptic integral

Periodicity

$u(\varphi)$ should have the period 2π

It should go around with period $2\pi/k$



$$\boxed{\frac{2\pi}{k} = 2\sqrt{2}\int_{u_3}^{u_2} \frac{du}{\sqrt{(u-u_1)(u-u_2)(u-u_3)}}}$$

- This imposes a condition on u_1, u_2, u_3.

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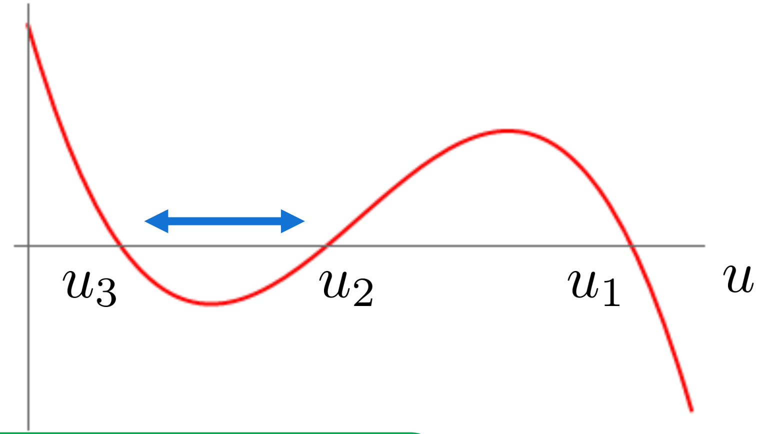

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- This imposes a condition on u_1, u_2, u_3.
- Another condition $\boxed{u_1 + u_2 + u_3 = -\frac{3}{2}\alpha}$

One parameter is still free. $m = \frac{u_2 - u_3}{u_1 - u_3}$

➡ non-trivial KdV charged black holes parametrized by m

Leading saddle?

- Our question is what geometry is the leading saddle of

  $Z_{GGE} = \mathrm{tr}\, e^{-H_{gen}}$

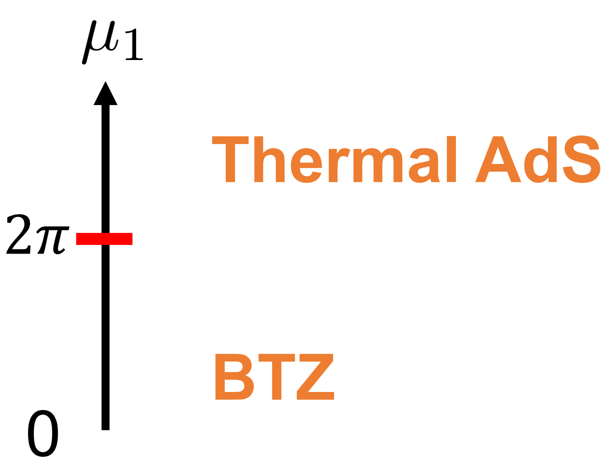
- Non-trivial KdV charged black holes can be saddles but it is not sure whether they are leading saddles because BTZ is also a saddle.

- Leading saddle is a classical solution minimizing the (generalized) free energy: $F = H_{gen} - S$

Hawking-Page phase transition

- conventional case $H = \mu_1^{cl} Q_1^{cl}$

- the leading saddle is
  (Euclidean) BTZ black hole or thermal AdS.



1st order phase transition at $\mu_1 = 2\pi$

Hawking-Page phase transition

New phase ?

Q: Can KdV-charged black holes be the leading saddle when we turn on chemical potentials?

$$H = \sum_k \mu_{2k-1} Q_{2k-1}$$

- **No**, if we have only μ_1 and μ_3.

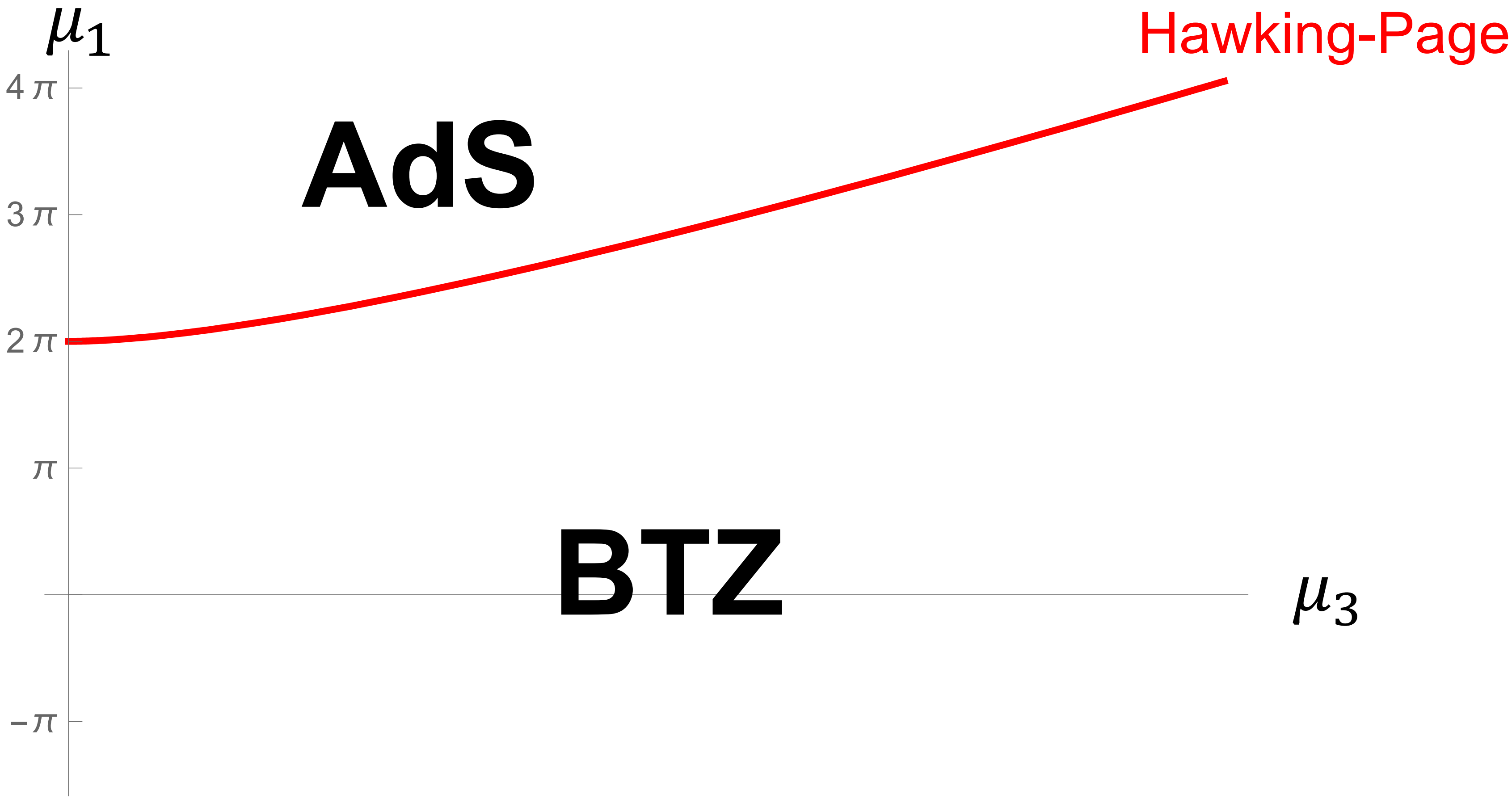
  The leading saddle is BTZ or thermal AdS.

- **Yes**, if we also add μ_5.

  New phase!

Phase-diagram on μ_1- μ_3 plane

- Leading saddle is BTZ or thermal AdS



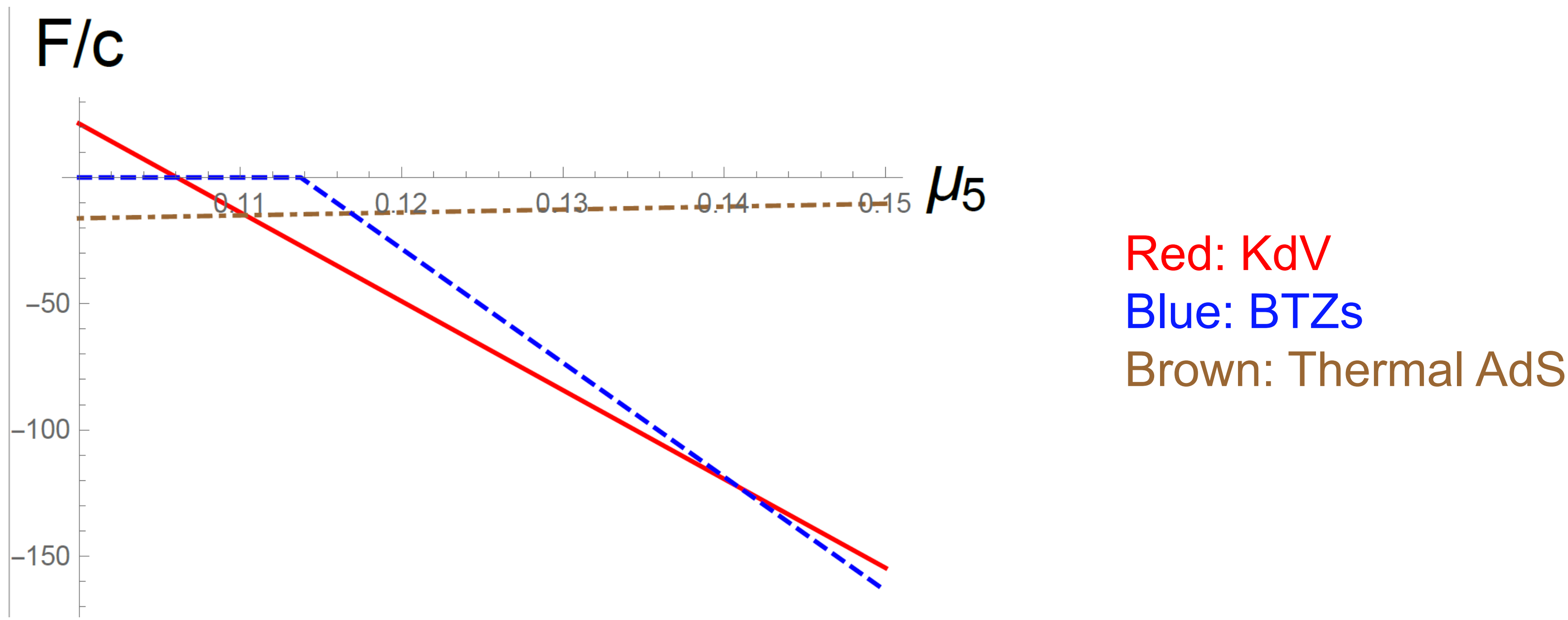
No new phase

Turn μ_5 on

- Difficult to draw phase diagram on $\boldsymbol{\mu_1}$-$\boldsymbol{\mu_3}$-$\boldsymbol{\mu_5}$ space.
- There exists a parameter region where a non-trivial solution has smaller free energy than BTZ and thermal AdS.

Strategy

- Pick up a KdV solution
- μ_1 & μ_3 are fixed in terms of μ_5
- Compare free energies



Non-trivial KdV black holes can be a leading saddle for specific parameters.

New phase (translation broken)

Outline

- Introduction
- Review of the classical KdV hierarchy
- 3D gravity and the KdV hierarchy
- Spectrum of quantum KdV charges in large c
- Summary

Spectrum of KdV charges

- To study GGE $\rho_{\mathrm{GGE}} = e^{-\sum_{k=1}^{\infty} \mu_{2k-1} Q_{2k-1}}$ in CFT, we need spectrum of KdV charges.

- Explicit spectra of general Q_{2k-1} are not known.

- Our strategy is using large c limit and relation to classical KdV charges.

$$Q^{q}_{2k-1} \sim \left(\frac{c}{24}\right)^{k} Q^{cl}_{2k-1}$$

Rep. of Virasoro (Verma module)

- Primary state (highest weight state) $|\Delta\rangle$

$$L_0 |\Delta\rangle = \Delta |\Delta\rangle , \quad L_{n>0} |\Delta\rangle = 0$$

- Descendant states

$$|\Delta, \{n_k\}\rangle = (L_{-1})^{n_1} (L_{-2})^{n_2} \cdots (L_{-k})^{n_k} \cdots |\Delta\rangle$$

$$L_0 \, |\Delta, \{n_k\}\rangle = (\Delta + \underbrace{\sum_k k n_k}_{\text{level}}) \, |\Delta, \{n_k\}\rangle$$

- level 0 $\quad |\Delta\rangle$ $\quad$ dim = 1
- level 1 $\quad L_{-1} |\Delta\rangle$ $\quad$ dim = 1
- level 2 $\quad (L_{-1})^2 |\Delta\rangle , \quad L_{-2} |\Delta\rangle$ $\quad$ dim = 2
- level ℓ $\quad (L_{-1})^{\ell} |\Delta\rangle , \quad (L_{-1})^{\ell-2} L_{-2} |\Delta\rangle , \cdots$ $\quad$ dim = p(ℓ)

KdV charges on Verma module

- Spectrum of Q_1 is easy

$$Q_1 = L_0 - \frac{c}{24} \Rightarrow Q_1 \left|\Delta, \{n_k\}\right\rangle = \left(\Delta - \frac{c}{24} + \sum_k k n_k\right) \left|\Delta, \{n_k\}\right\rangle$$

- KdV charges do not change the level.

$$[L_0, Q_{2n-1}] = 0$$

Different levels are not mixed by KdV charges.

$\Rightarrow$ **Block diagonal** $\quad Q_{2n-1} = \begin{pmatrix} Q^{(0)}_{2n-1} & & & \\ & Q^{(1)}_{2n-1} & & \\ & & Q^{(2)}_{2n-1} & \\ & & & \ddots \end{pmatrix}$

However, it's hard to obtain the matrix elements of each block.
Even $Q^{(0)}_{2n-1}$ is not known for general n.

KdV charges on primary state

$$Q_{2n-1}\left|\Delta\right\rangle = Q^{(0)}_{2n-1}\left|\Delta\right\rangle$$

The exact expressions are known up to $Q^{(0)}_{15}$

[Bazhanov, Lukyanov, Zamolodchikov (1994)]

$$I_1^{vac}(\Delta) = \Delta - \frac{c}{24}\ ,$$

$$I_3^{vac}(\Delta) = \Delta^2 - \frac{c+2}{12}\ \Delta + \frac{c\ (5c+22)}{2880}\ ,$$

$$I_5^{vac}(\Delta) = \Delta^3 - \frac{c+4}{8}\ \Delta^2 + \frac{(c+2)\ (3c+20)}{576}\ \Delta - \frac{c\ (3c+14)\ (7c+68)}{290304}\ ,$$

$$I_7^{vac}(\Delta) = \Delta^4 - \frac{c+6}{6}\ \Delta^3 + \frac{15c^2+194c+568}{1440}\ \Delta^2 - \frac{(c+2)\ (c+10)\ (3c+28)}{10368}\ \Delta + \frac{c\ (3c+46)\ (25c^2+426c+1400)}{24883200}$$

Large c KdV charges on primary state

The known results satisfy

$$Q^{(0)}_{2k-1}(\Delta) = \left(\Delta - \frac{c}{24}\right)^k + \mathcal{O}(c^{k-1})$$

where I suppose that conf dim is also large $\Delta \sim c$

This reminds us of $Q^{cl}_{2k-1} = \left(Q^{cl}_1\right)^k$ for $u = \text{const.} = u_0$

$$\Delta - \frac{c}{24} \sim \frac{c}{24} u_0$$

We extend this to non-const $u = \sum_n u_n e^{in\varphi}$

Classical phase space

$$u(\varphi) = \sum_n u_n e^{in\varphi}$$

Phase space is parametrized by $u_n \quad (n = 0, \pm 1, \pm 2, \cdots)$

This space has the Poisson structure:

$$i\{u_n, u_m\} = (n-m)u_{n+m} + 2n^3 \delta_{n+m,0}$$

There is a conserved quantity: $\{h, u_n\} = 0$ for any u_n

s.t. $h = u_0 + \mathcal{O}(u^2)$

Inversely, u_0 is parametrized by $h, u_n \, (n \neq 0)$

Action-angle variables

We can take more convenient coordinates $I_k, \theta_k \, (k = 1, 2, \cdots)$ rather than $u_n \, (n \neq 0)$ such that

$$\{\theta_k, I_\ell\} = \delta_{k,\ell} \, , \quad u_0 = h + \sum_k k I_k$$

The relation between the variables are complicated.

$$\sqrt{I_k} e^{-i\theta_k} = \frac{1}{\sqrt{2k(k^2+h)}} \left(u_k + \frac{1}{4} \sum_{\substack{p_1+p_2=k \\ p_i \neq 0}} \frac{1}{p_1 p_2} u_{p_1} u_{p_2} + \mathcal{O}(u^3) \right)$$

The KdV charges only depend on the action variables.

$$Q_1 = u_0 = h + \sum_k k I_k, \quad Q_{2n-1} = h^n + \mathcal{O}(I_k)$$

Old quantization

- Classical KdV charges

$$Q_{2n-1} = h^n + \sum_{k=1}^{\infty} \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} h^{n-1-j} k^{2j+1} I_k + \mathcal{O}(I_k^2).$$

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We will guess the spectrum of the quantum KdV charges by using the semiclassical quantization.

- Bohr-Sommerfeld quantization
  (Einstein-Brillouin-Keller method)

$$\{\theta_k, I_\ell\} = \delta_{k,\ell}$$

Classical trajectory is a closed circle.

$\hbar_{eff}$

➡ Action variable is quantized as $I_k = \frac{1}{2\pi} \oint I_k d\theta_k = \frac{24}{c} \left(n_k + \frac{1}{2} \right)$

We will see it works at O(1/c).

Semiclassical quantization

- Naïve guess is $Q^q_{2n-1} = \left(\frac{c}{24}\right)^n Q^{cl}_{2n-1}$

  with replacement $I_k = \frac{24}{c}\left(n_k + \frac{1}{2}\right)$

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with replacement $I_k = \frac{24}{c}\left(n_k + \frac{1}{2}\right)$

For exapmple, $n = 1$,

$$\left(\frac{c}{24}\right) Q^{cl}_1 = \left(\frac{c}{24}\right)(h + \sum_{k=1}^{\infty} k I_k)$$

$I_k = \frac{24}{c}\left(n_k + \frac{1}{2}\right)$

$$= \frac{c}{24} h + \sum_{k=1}^{\infty} k\left(n_k + \frac{1}{2}\right)$$

zeta function regularization

$\zeta(-1) = -\frac{1}{12}$

$$= \frac{c}{24} h - \frac{1}{24} + \sum_{k=1}^{\infty} k n_k$$

It reproduces $Q^{q}_1 = \Delta - \frac{c}{24} + \sum_{k=1}^{\infty} k n_k$ if we set $h = \frac{24}{c}\left(\Delta - \frac{c-1}{24}\right)$

KdV spectra at O(1/c)

$$Q^{cl}_{2n-1} = h^n + \sum_{k=1}^{\infty}\sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} h^{n-1-j} k^{2j+1} I_k + \mathcal{O}(I_k^2).$$

$$Q^{q}_{2n-1} = \left(\frac{c}{24}\right)^n Q^{cl}_{2n-1}, \quad h = \frac{24}{c}\tilde{\Delta}, \quad I_k = \frac{24}{c}\left(n_k + \frac{1}{2}\right)$$

$$\tilde{\Delta} = \Delta - \frac{c-1}{24}$$

$$\begin{aligned} Q^q_{2n-1} &= \tilde{\Delta}^n \\ &+ \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} \underbrace{\left(\frac{c}{24}\right)^j \tilde{\Delta}^{n-1-j}}_{\mathcal{O}(c^{n-1})} \left[\sum_k k^{2j+1} n_k + \frac{1}{2}\zeta(-2j-1)\right] \\ &+ \mathcal{O}(c^{n-2}) \end{aligned}$$

It is consistent with explicit diagonalization for small level with small n and also the known results for

$$Q^{(0)}_{2n-1}(\Delta) \quad (n=1,\cdots,8), \qquad \mathrm{Tr}(Q_{2n-1}e^{-\beta Q_1}) \quad (n=1,\cdots,7)$$

Extension to higher orders

- At the 2nd order of the action variables, the KdV charges are

$$Q^{cl}_{2n-1} = h^n + \sum_{k=1} a_k^{(n,1)} I_k + \underbrace{\sum_{k=1} a_k^{(n,2)} I_k^2 + \frac{1}{2}\sum_{k,\ell=1} b_{k,\ell}^{(n)} I_k I_\ell}_{\mathcal{O}\left(I^2\right)} + \mathcal{O}\left(I^3\right)$$

$$a_k^{(n,2)} = -\sum_{m=0}^{n-1} \frac{(2n-1)(2mn+2n-3m-2)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{16\Gamma\left(m+\frac{3}{2}\right)\Gamma(n-m)} h^{n-m-1} k^{2m}$$

$$b_{k,\ell}^{(n)} = \sum_{m=1}^{n-1} \frac{(2n-1)^2\Gamma(n+1)\Gamma\left(\frac{1}{2}\right) h^{n-m-1}}{2^2\Gamma(n-m)\Gamma\left(m+\frac{3}{2}\right)} \sum_{s=0}^{m-1} k^{2(m-s)-1}\ell^{2s+1}$$

Naïve semiclassical results

- No reason that semiclassical quantization works at the subsubleading order. But let’s try it.

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Replacement $Q^{q}_{2n-1} = \left(\frac{c}{24}\right)^n Q^{cl}_{2n-1}, \quad h = \frac{24}{c}\tilde{\Delta}, \quad I_k = \frac{24}{c}\left(n_k + \frac{1}{2}\right)$

gives

$$Q^{\text{naive}}_{2n-1} = \tilde{\Delta}^n + \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(j+\frac{3}{2}\right)\Gamma(n-j)} \left(\frac{c}{24}\right)^j \tilde{\Delta}^{n-1-j} k^{2j+1} \tilde{n}_k$$

$$\mathcal{O}(c^{n-2}) \left[\begin{array}{l} - \displaystyle\sum_k \sum_{m=0}^{n-1} \frac{(2n-1)(2mn+2n-3m-2)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{16\Gamma\left(m+\frac{3}{2}\right)\Gamma(n-m)} \left(\frac{c}{24}\right)^{m-1} \tilde{\Delta}^{n-m-1} k^{2m} \tilde{n}_k^2 \\ + \displaystyle\frac{1}{2} \sum_{k,\ell} \sum_{m=1}^{n-1} \frac{(2n-1)^2\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2^2\Gamma(n-m)\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{c}{24}\right)^{m-1} \tilde{\Delta}^{n-m-1} \sum_{s=0}^{m-1} k^{2(m-s)-1} \ell^{2s+1} \tilde{n}_k \tilde{n}_\ell \end{array} \right.$$

$$+ \mathcal{O}(c^{n-3})$$

$$\tilde{n}_k = n_k + \frac{1}{2}$$

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$$\mathcal{O}(c^{n-2}) \left[\begin{aligned} & -\sum_k \sum_{m=0}^{n-1} \frac{(2n-1)(2mn+2n-3m-2)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{16\Gamma\left(m+\frac{3}{2}\right)\Gamma(n-m)} \left(\frac{c}{24}\right)^{m-1} \tilde{\Delta}^{n-m-1} k^{2m}\tilde{n}_k^2 \\ & + \frac{1}{2}\sum_{k,\ell} \sum_{m=1}^{n-1} \frac{(2n-1)^2\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2^2\Gamma(n-m)\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{c}{24}\right)^{m-1} \tilde{\Delta}^{n-m-1} \sum_{s=0}^{m-1} k^{2(m-s)-1}\ell^{2s+1}\tilde{n}_k\tilde{n}_\ell \end{aligned} \right.$$

$$+\,\mathcal{O}(c^{n-3})$$

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It fails because it is **not consistent** with the known results.

Thermal 1pt function

The differences of thermal 1-pt functions

$$\langle Q_3\rangle_q - \langle Q_3\rangle_{sc} = -\frac{3\left(2E_2+1\right)\tilde{\Delta}}{2c} + \frac{1}{288}E_4$$

$$\langle Q_5\rangle_q - \langle Q_5\rangle_{sc} = \frac{\left(-15E_2 - \frac{15}{2}\right)\tilde{\Delta}^2}{c} + \frac{7}{96}E_4\tilde{\Delta} - \frac{E_6 c}{2592}$$

$$\langle Q_7\rangle_q - \langle Q_7\rangle_{sc} = \frac{\left(-42E_2 - 21\right)\tilde{\Delta}^3}{c} + \frac{259}{720}E_4\tilde{\Delta}^2 - \frac{E_6 c\tilde{\Delta}}{200} + \frac{127E_8c^2}{2592000}$$

$$\cdots$$

$$\langle Q_{13}\rangle_q - \langle Q_{13}\rangle_{sc} =$$

Guess

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$$\cdots$$

$$\langle Q_{13}\rangle_q - \langle Q_{13}\rangle_{sc} =$$

Our guess about general terms:

$$-\frac{n(n-1)(2n-1)}{4}(2E_2+1)\frac{\tilde{\Delta}^{n-1}}{c}$$

Guess

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$$\frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})\zeta(-2j-1)}{16\Gamma(j+\frac{3}{2})\Gamma(n-j)}\left((2n-1)\sum_{\ell=0}^{j}\frac{1}{2\ell+1} + \frac{13j+6}{6}\right) \times E_{2j+2}\left(\frac{c}{24}\right)^{j-1}\tilde{\Delta}^{n-1-j}$$

KdV spectra at O(1/c²)

Our conjectured result:

$$
\begin{aligned}
Q^q_{2n-1} = \tilde{\Delta}^n &+ \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{2\Gamma(j+\frac{3}{2})\Gamma(n-j)} \left(\frac{c}{24}\right)^j \tilde{\Delta}^{n-1-j} k^{2j+1} \tilde{n}_k \\
&\Bigg[- \frac{n(n-1)(2n-1)\tilde{\Delta}^{n-1}}{4c} \\
&+ \sum_k \sum_{j=0}^{n-1} \frac{(2n-1)\Gamma(n+1)\Gamma(\frac{1}{2})}{48\Gamma(j+\frac{3}{2})\Gamma(n-j)} \left(-13j + 6(2n-1)\sum_{\ell=0}^{j} \frac{1}{2\ell+1} - 6\right) \left(\frac{c}{24}\right)^{j-1} \tilde{\Delta}^{n-1-j} k^{2j+1} \tilde{n}_k \\
&- \sum_k \sum_{m=0}^{n-1} \frac{(2n-1)(2mn+2n-3m-2)\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{16\,\Gamma\left(m+\frac{3}{2}\right)\Gamma(n-m)} \left(\frac{c}{24}\right)^{m-1} \tilde{\Delta}^{n-m-1} k^{2m} \tilde{n}_k^2 \\
&+ \frac{1}{2} \sum_{k,\ell} \sum_{m=1}^{n-1} \frac{(2n-1)^2\Gamma(n+1)\Gamma\left(\frac{1}{2}\right)}{2^2\Gamma(n-m)\Gamma\left(m+\frac{3}{2}\right)} \left(\frac{c}{24}\right)^{m-1} \tilde{\Delta}^{n-m-1} \sum_{s=0}^{m-1} k^{2(m-s)-1} \ell^{2s+1} \tilde{n}_k \tilde{n}_\ell \\
&+ \mathcal{O}(c^{n-3})
\end{aligned}
$$

It is consistent with the known results for thermal 1pt functions:

$$\mathrm{Tr}(Q_{2n-1} e^{-\beta Q_1}) \quad (n = 1, \cdots, 7)$$

Check

- Our result at O($1/c^2$) also passes other consistency checks.

  Q^q_{2n-1}

- Explicit diagonalization for small level with small n.

- For primary states, we can compute the large c spectrum of KdV charges recursively from small n using the OED/IM correspondence.
  The result reproduces our conjecture for primary states.

  $n_k = 0$

Outline

- Introduction
- Review of the classical KdV hierarchy
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Summary 1

- Any 2D CFT has an infinite number of commuting local charges (KdV charges).
- The generalized Gibbs ensemble including KdV charges is related to understanding the thermalization in 2D CFT and formation of 3D black holes.

- To analyze the GGE, the spectra of KdV charges are useful.
- We are investigating the spectra at the large c limit.
- Guessed the spectra at $O(c^{-2})$ from the semiclassical quantization.
- Proof?

Summary 2

- We have also investigated the gravitational theory corresponding to the generalized Hamiltonian.
- There are non-trivial black holes corresponding to the solutions of KdV hierarchy.
- What microstates correspond to these solutions?

- There are a lot of things that we have to understand both in CFT and gravity sides.

BLZ's work

- Bazhanov, Lukyanov, Zamolodchikov introduced T-operator which is quantum analog of classical transfer matrix.

[Bazhanov, Lukyanov, Zamolodchikov (1996)]

The T-operator is a generating function of KdV charges.

$$\log \mathbf{T}(\lambda) \simeq \kappa \lambda^{1+\xi} \mathbf{I} - \sum_{n=1}^{\infty} C_n \lambda^{(1-2n)(1+\xi)} Q^q_{2n-1} \qquad (\lambda \to \infty)$$

$$\kappa = \frac{2\sqrt{\pi}\Gamma\left(\frac{1}{2}-\frac{\xi}{2}\right)}{\Gamma\left(1-\frac{\xi}{2}\right)} \left(\Gamma\left(1-\beta^2\right)\right)^{1+\xi}, \quad C_n = \frac{\sqrt{\pi}(1+\xi)\beta^{2n}\Gamma\left((n-\frac{1}{2})(1+\xi)\right)}{\Gamma(n+1)\Gamma\left(1+(n-\frac{1}{2})\xi\right)} \left(\Gamma\left(1-\beta^2\right)\right)^{-(2n-1)(1+\xi)}$$

$$\beta \equiv \sqrt{\frac{1-c}{24}} - \sqrt{\frac{25-c}{24}}, \qquad \xi \equiv \frac{\beta^2}{1-\beta^2}$$

- classical case $T^{cl}(\lambda) = \mathrm{tr}\, M(\lambda)$

M is the monodromy matrix for Schrodinger op. $L = \partial_\varphi^2 + u(\varphi) - \lambda^2$

$$L\psi_i = 0$$

$$(\psi_1(\varphi+2\pi), \psi_2(\varphi+2\pi)) = (\psi_1(\varphi), \psi_2(\varphi))M(\lambda)$$

BLZ's Q operator

- BLZ also introduced Q-operator $[\mathbf{T}(\lambda), \mathbf{Q}(\lambda)] = 0$

For primary states, we define

$$T(\Delta, \lambda) = \langle\Delta|\, \mathbf{T}(\lambda)\, |\Delta\rangle\,, \quad A(\Delta, \lambda) = \lambda^{-2\pi i p/\beta^2}\, \langle\Delta|\, \mathbf{Q}(\lambda)\, |\Delta\rangle$$

$$\Delta = \frac{p^2}{\beta^2} + \frac{c-1}{24}$$

They satisfy $T(\lambda)A(\lambda) = e^{2\pi i p} A(q\lambda) + e^{-2\pi i p} A(q^{-1}\lambda), \qquad q = e^{i\pi\beta^2}$

In principle we can obtain the KdV charges for primary states if we know the function $A(\Delta, \lambda)$.

ODE/IM correspondence 1

[Dorey, Tateo (1998), Bazhanov, Lukyanov, Zamolodchikov (1998)]

- There is a miracle relation between $A(\Delta, \lambda)$ and the spectral determinant for QM on half line.

$$\partial_x^2 \Psi(x) + \left(E - x^{2\alpha} - \frac{l(l+1)}{x^2} \right) \Psi(x) = 0 \qquad (x > 0)$$

$$\alpha = \frac{1}{\beta^2} - 1, \quad l = \frac{2p}{\beta^2} - \frac{1}{2}$$

- Let E_n be the eigenvalues for the following boundary cond:

$$\Psi(x) \to \sqrt{\frac{2\pi}{1+\alpha}} (2+2\alpha)^{-\frac{2l+1}{2+2\alpha}} \frac{x^{l+1}}{\Gamma\left(1 + \frac{2l+1}{2+2\alpha}\right)} + \mathcal{O}(x^{l+3}) \qquad (x \to 0),$$

$$\Psi(x) \to 0 \qquad (x \to \infty)$$

- Spectral determinant: $D(E, l) = \prod_{n=1}^{\infty} \left(1 - \frac{E}{E_n} \right)$

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[Dorey, Tateo (1998), Bazhanov, Lukyanov, Zamolodchikov (1998)]

- Spectral determinant: $D(E,l) = \prod_{n=1}^{\infty}\left(1 - \frac{E}{E_n}\right)$
- Set $E = \rho\lambda$ with $\rho = \left(\frac{2}{\beta^2}\right)^{2-2\beta^2} \Gamma^2(1-\beta^2)$

We then have $A(\Delta, \lambda) = D(\rho\lambda, l)$

■ Determining $A(\Delta, \lambda)$ is equivalent to finding the spectral determinant.