Using Algebraic Geometry in Theoretical Physics

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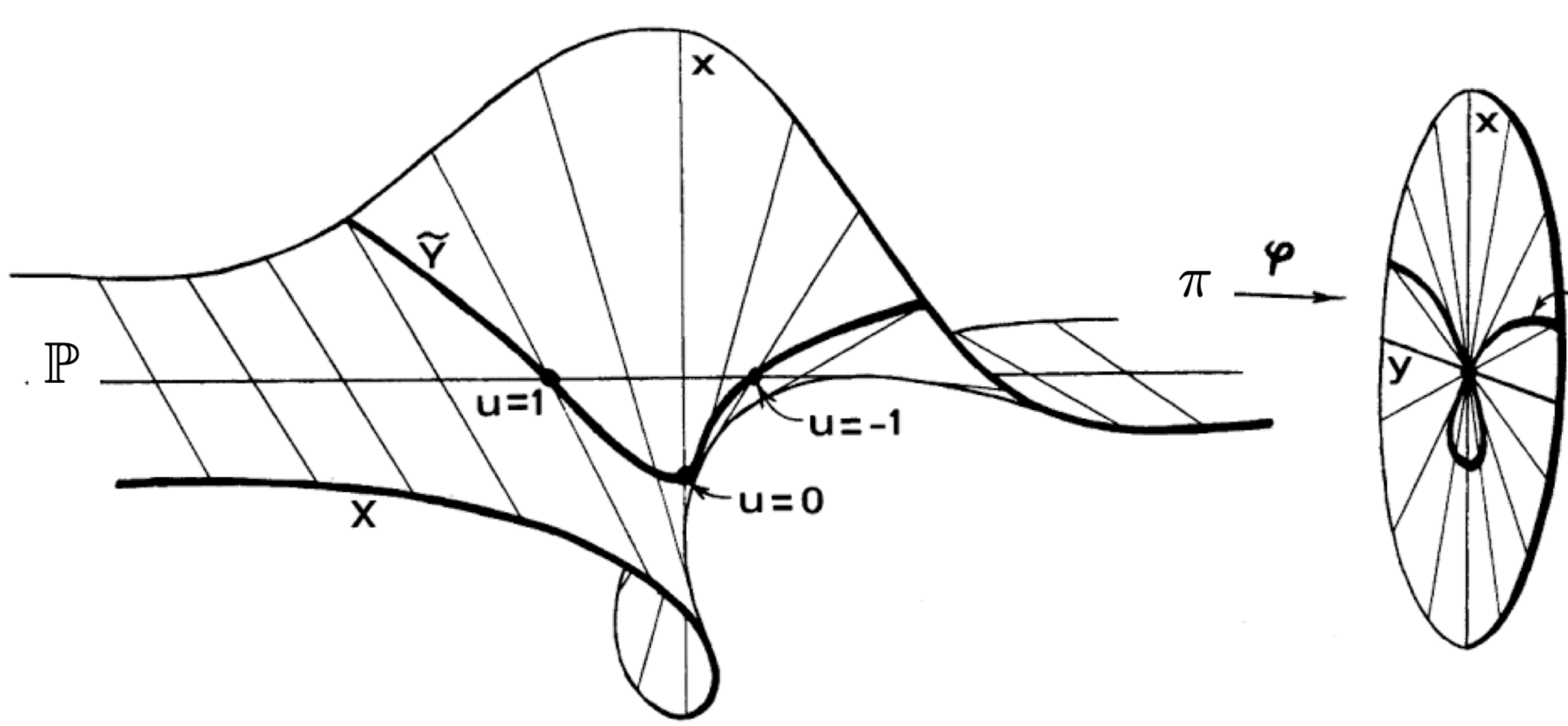
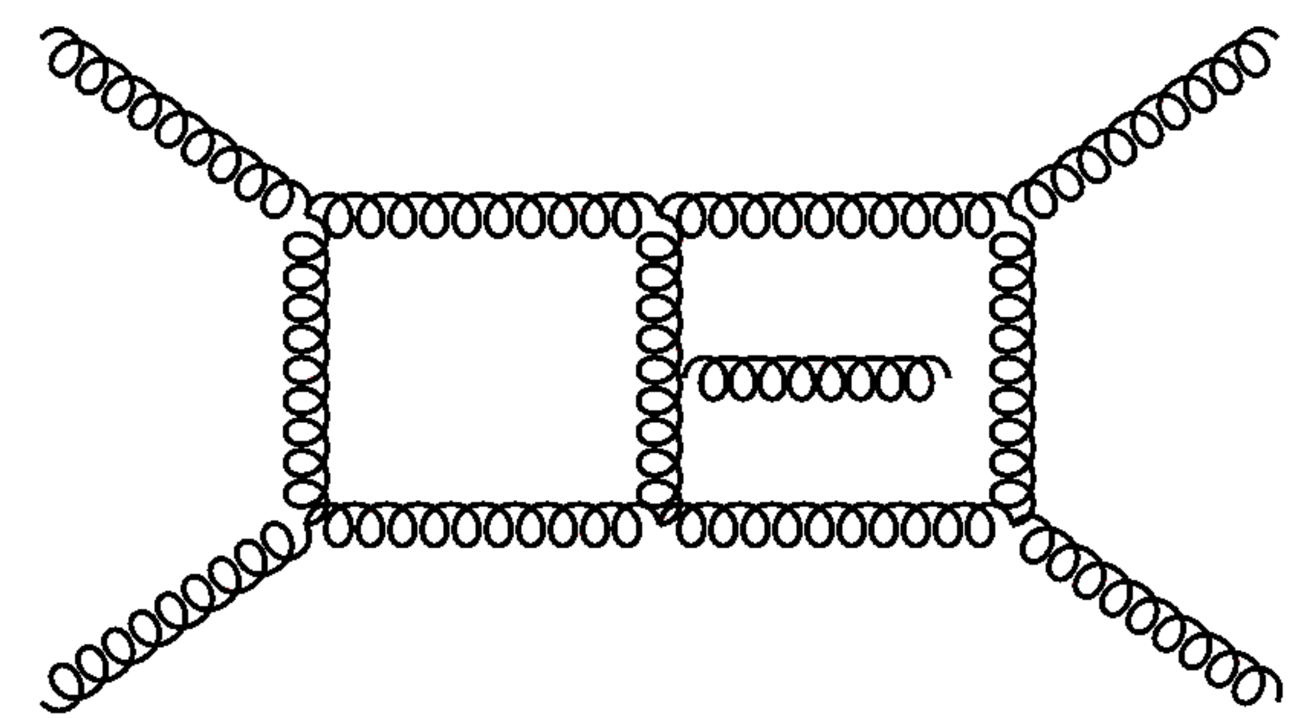
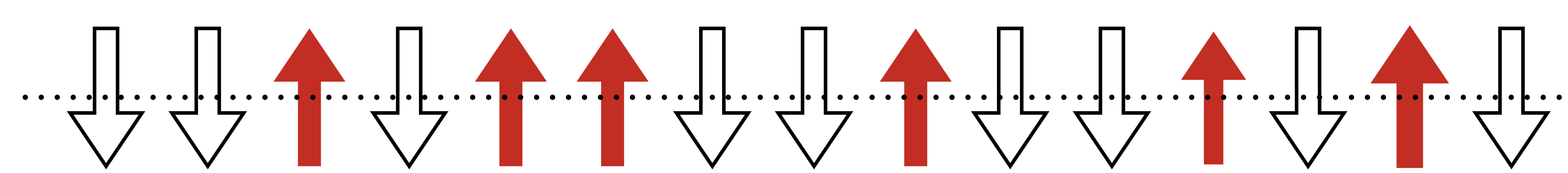
This talk is about using computational algebraic geometry methods for
various areas of theoretical physics

- Feynman integrals
- Integrable Spin Chain







For details, please have a look at my lecture in SAGEX Network 2021

https://www.youtube.com/watch?v=0Q-1Q0lIIKY

Motivation

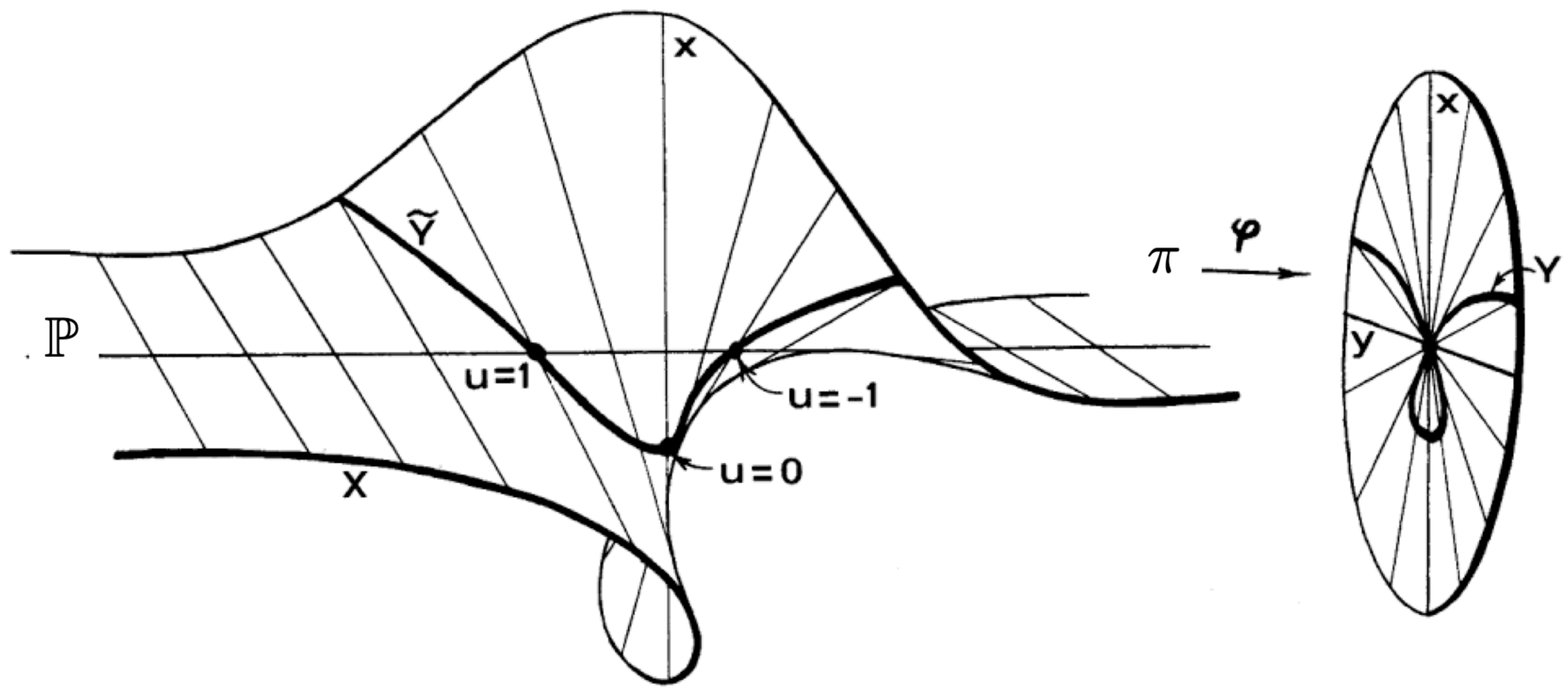
In the research direction of QFT and integrable spin chain
we have a lot of physical quantities which can be difficult to calculate

Multi-loop scattering amplitude
Feynman integrals
Partition function

Somehow, they all have roots from polynomials/rational functions

Therefore, we try computational algebraic geometry
a modern mathematical branch to deal with polynomials

Algebraic Geometry



Algebraic geometry originates from the study of curves/surfaces defined by multivariate polynomial equations.

Modern algebraic geometry generalised these geometric objects to abstract objects (like scheme), and has been applied to number theory, complex analysis , topology and physics.

Computational Algebraic Geometry (CAG)

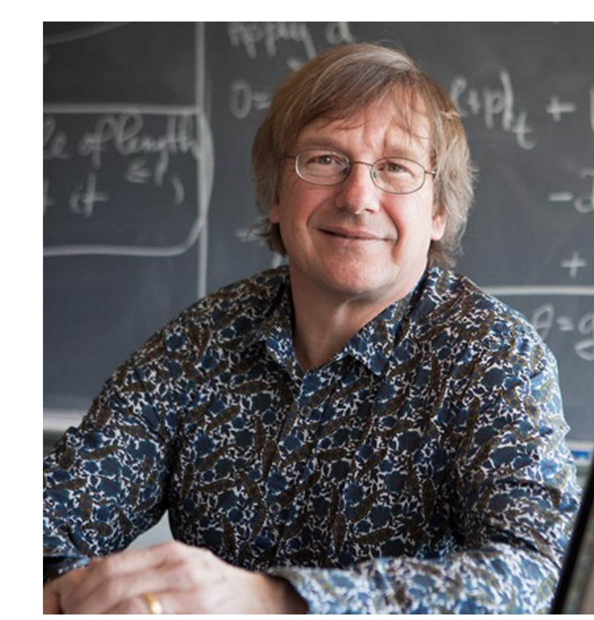
Algebraic Geometry

Computer Algorithms

originated in 1970s
thrive from 2000s

Computational Algebraic Geometry

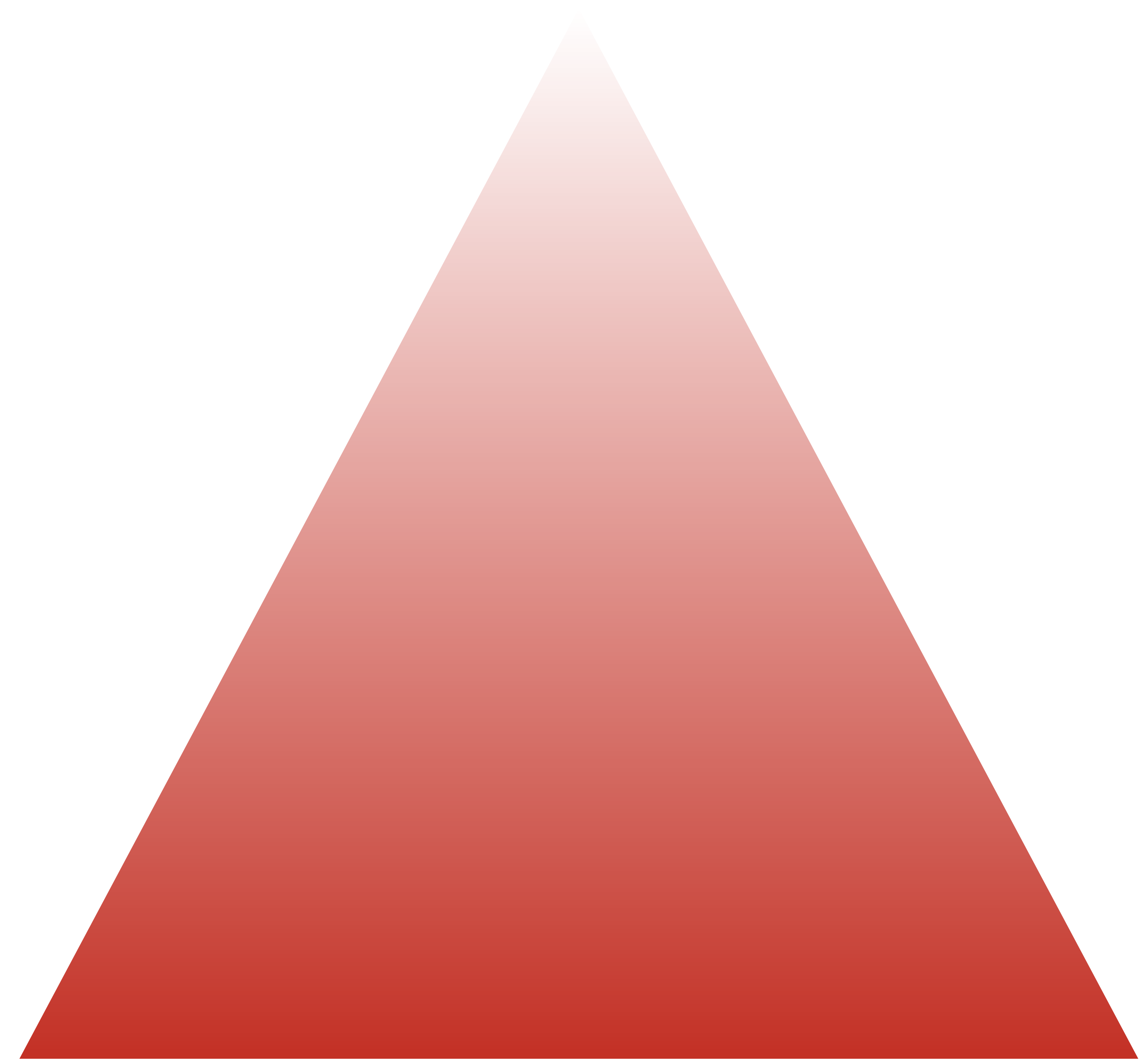
Bruno Buchberger, Frank-Olaf Schreyer, Jean-Charles Faugère
David Eisenbud, Michael Stillman, Daniel Grayson, Wolfram Decker …



Personally I learnt CAG from
Professor Michael Stillman.

CAG in one slide

Groebner basis



Trinity of CAG

Syzygy

Lift

CAG in one slide

“Gaussian elimination”
for several multivariate polynomials

Groebner basis

Buchberger algorithm

Trinity of CAG

Syzygy

Solve homogeneous linear equations
with polynomial solutions

Schreyer algorithm

Lift

Solve inhomogeneous linear equations
with polynomial solutions

Polynomial ring and ideal

Polynomial ring $\quad R = \mathbb{F}[x_1, \ldots, x_n]$

field, $\mathbb{C}, \mathbb{Q}, \mathbb{Z}/p, \mathbb{Q}[c_1, \ldots c_m], \ldots$

An ideal I in the polynomial ring $R = \mathbb{F}[z_1, \ldots z_n]$ is a linear subspace of R such that, For $\forall f \in I$ and $\forall h \in R$, $hf \in I$.

The ideal in the polynomial ring generated by a polynomial set S is the collection of all such polynomials,

$$\sum_i h_i f_i, \quad h_i \in R, \quad f_i \in S.$$

This ideal is denoted as $\langle S \rangle$.

(Noether) Any ideal in a polynomial ring is finitely generated.

Ideal and algebraic set

To solve $f_1 = \ldots = f_k = 0$ is equivalent to solve all polynomials in $\langle f_1, \ldots f_k \rangle$

$$\mathcal{Z}(S) = \mathcal{Z}(\langle S \rangle),$$

common zero set in $\mathbb{F}^n$

$\mathcal{Z}(I)$, with I an ideal is called an *affine algebraic set*.

(*Zariski topology*) Define Zariski topology of $\mathbb{F}^n$ by setting all algebraic set to be topologically closed.

$$\bigcap_i \mathcal{Z}(I_i) = \mathcal{Z}(\bigcup_i I_i), \quad \mathcal{Z}(I_1) \bigcup \mathcal{Z}(I_2) = \mathcal{Z}(I_1 \cap I_2)$$

Zariski topology is *different* from the usual topology defined by Euclidean distance.

For example, the "open" unit disc defined by $D = \{z||z| < 1\}$ is not Zariski open in $\mathbb{C}$. $\mathbb{C} - D = \{z||z| \geq 1\}$ is not Zariski closed, i.e. it cannot be the solution set of one or several complex polynomials in z.

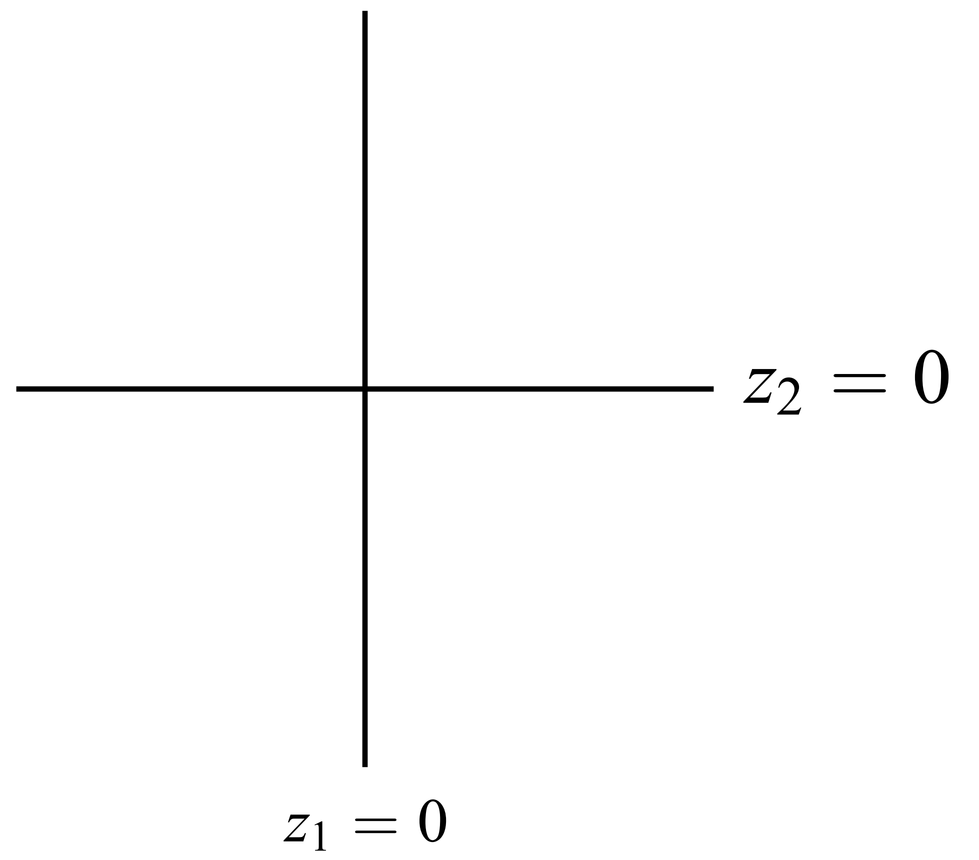
Variety

With Zariski topology, an algebraic set (closed set) may be a union of several closed set

$$V = V_1 \cup \dots V_l$$

If V cannot be decompose as a nontrivial union, then V is called an *affine variety*.

In $\mathbb{C}$, $\{0,1\} = \{0\} \bigcup \{1\}$.
In $\mathbb{C}^2$, $\mathcal{Z}(z_1 z_2) = \mathcal{Z}(z_1) \bigcup \mathcal{Z}(z_2)$



Algebraic geometry is a subject to study the relation between ideals and algebraic sets (varieties).

Hilbert's weak Nullstellensatz

(Hilbert) Let I be an ideal of $\mathbb{F}[x_1, \dots x_n]$ and $\mathbb{F}$ is *algebraically closed*. If $\mathcal{Z}(I) = \emptyset$, then $I = \langle 1 \rangle$.

See Commutative algebra, *Zariski and Samuel*, Chapter 7 for the proof.

- $\mathbb{F} = \mathbb{C}$. $\mathcal{Z}(\langle x^2 - 1, x^3 + x \rangle) = \emptyset$ and the we

$$1 = \underline{(-\frac{x^2}{2} - 1)}(x^2 - 1) + \underline{\frac{x}{2}}(x^3 + x)$$

- $\mathbb{F} = \mathbb{C}$, $\mathcal{Z}(x^2 - y^2, x + y + 1, 2x - y) = \emptyset$ and the we

$$1 = \underline{-3}(x^2 - y^2) + \underline{(1 + 3x - 3y)}(x + y + 1) - \underline{2}(2x - y)$$

- $\underline{\mathbb{F} = \mathbb{Q}}$, $\mathcal{Z}(x^2 - 2) = \emptyset$ and the we **cannot** claim

not algebraically closed

$$\langle x^2 - 2 \rangle = \langle 1 \rangle$$

The theorem itself does not give coefficients,
it is the task of computational algebraic geometry

Hilbert's Nullstellensatz

Given a set U in $\mathbb{F}^n$, we want to go backwards to find all f's in the polynomial ring such that

$$f(p) = 0, \quad \forall p \in U$$

Such f's form an ideal, which is denoted as $\mathcal{I}(U)$. One may naively think that

$$\mathcal{I}(\mathcal{Z}(I)) = I \quad \times$$

For example, $\mathbb{F} = \mathbb{C}$, $I = \langle x^2 \rangle$ and $\mathcal{Z}(I) = \{0\}$. However, $\mathcal{I}(\{0\}) = \langle x \rangle \neq I$.

(Hilbert) Let $\mathbb{F}$ be an algebraically closed field and $R = \mathbb{F}[z_1, \ldots z_n]$. Let I be an ideal of R. If $f \in R$ and,

$$f(p) = 0, \quad \forall p \in \mathcal{Z}(I),$$

then there exists a positive integer k such that $f^k \in I$.

Groebner basis

This is like a "nonlinear" version of Gaussian elimination

For an ideal I in $\mathbb{F}[x_1, \ldots, x_n]$ with a monomial order, a Groebner basis $G(I) = \{g_1, \ldots g_m\}$ is a generating set for I such that for each $f \in I$, there always exists $g_i \in G(I)$ such that,

$$\mathrm{LT}(g_i) | \mathrm{LT}(f) \, .$$

invented by B. Buchburger, in the namesake of his supervisor, W.Groebner

- Polynomial division over a Groebner basis, provide a unique remainder, independent of the polynomial order.
- If $f \in I$, then the remainder of f over the Groebner basis is zero.
  Ideal membership problem is solved.
- The remainder provides a canonical representation of $F[x_1, \ldots, x_n]/I$.
- With a fixed monomial order, the reduced Groebner basis is unique.
  Ideal identification problem is solved.

Buchberger's algorithm

Algorithm 3 Buchberger algorithm

Input: $B = \{f_1 \dots f_n\}$ and a monomial order $\succ$
queue := all subsets of B with exactly two elements
while *queue*! $= \emptyset$ **do**
  $\{f, g\}$:= head of *queue*
  $r := \overline{S(f,g)}^B$
  if $r \neq 0$ **then**
    $B := B \cup r$
    queue $<<$ $\{\{B_1, r\}, \dots \{\text{last of } B, r\}\}$
  end if
  delete head of *queue*
end while
return B (Gröbner basis)

$$S(f_i, f_j) = \frac{\mathrm{LT}(f_j)}{\gcd\big(\mathrm{LT}(f_i), \mathrm{LT}(f_j)\big)} f_i - \frac{\mathrm{LT}(f_i)}{\gcd\big(\mathrm{LT}(f_i), \mathrm{LT}(f_j)\big)} f_j.$$

S-pair

Buchberger algorithm calculates Groebner basis

can be thought as a non-linear generalization of Gaussian elimination

Buchberger algorithm is computationally very heavy, double exponentially in the number of variables.

Groebner basis, A first look in Mathematica

```
PolynomialSet1={x-y ,x+y-1,x-2};
Gr1=GroebnerBasis[PolynomialSet1,{x,y},MonomialOrder->DegreeReverseLexicographic]
PolynomialSet2={x-y ,x y-y+1};
Gr2=GroebnerBasis[PolynomialSet2,{x,y},MonomialOrder->DegreeReverseLexicographic]
```

{1} The equation system has no solution

$\{x - y, 1 - y + y^2\}$

```
PolynomialSet3={x^3-2 x y,x^2 y-2 y^2+x};
Gr3=GroebnerBasis[PolynomialSet3,{x,y},MonomialOrder→DegreeReverseLexicographic]
PolynomialReduce[x^3,PolynomialSet3,{x,y},MonomialOrder→DegreeReverseLexicographic]
PolynomialReduce[x^3,Gr3,{x,y},MonomialOrder→DegreeReverseLexicographic]
```

$\{-x + 2 y^2, x y, x^2\}$

$\{\{1, 0\}, 2 x y\}$ remainder nonzero, x^3 is not in the ideal ?

$\{\{0, 0, x\}, 0\}$

remainder is zero for the Groebner basis
x^3 is is in the ideal !

Application of algebraic geometry for Feynman integrals

Based on

Bendle, Boehm, Heymann, Ma, Rahn, Wittman, Ristau, Wu, **YZ** 2021

"Two-loop five-point integration-by-parts relations in a usable form", 2104.06866

Boehm, Wittmann, Xu, Wu and **YZ**

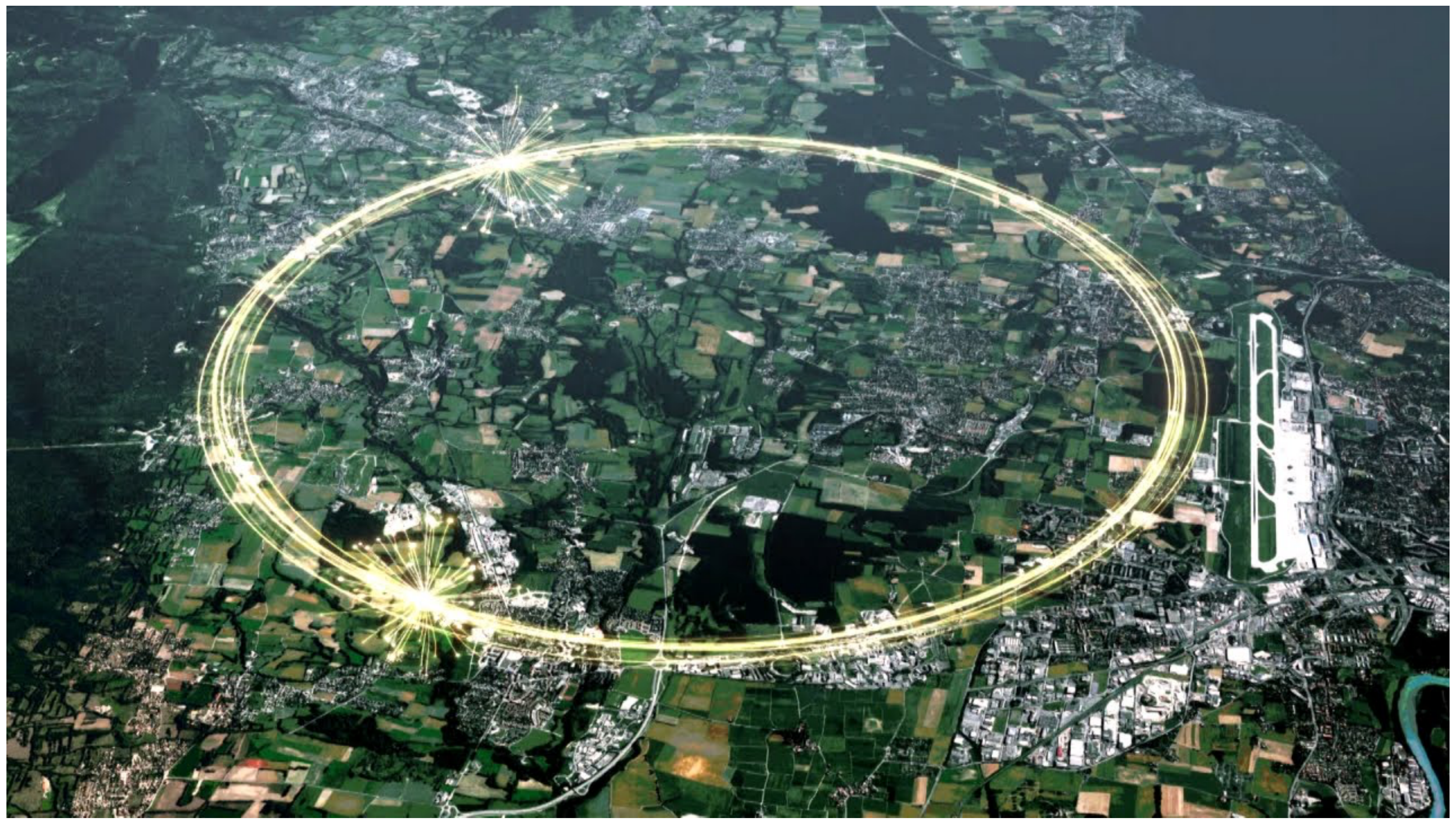
"IBP reduction coefficients made simple" JHEP 12 (2020) 054

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, and **YZ**
"Integration-by-parts reductions of Feynman integrals using Singular and GPI-Space" JHEP 02 (2020) 079

Chicherin, Gehrmann, Henn, Wasser, **YZ**, Zoia

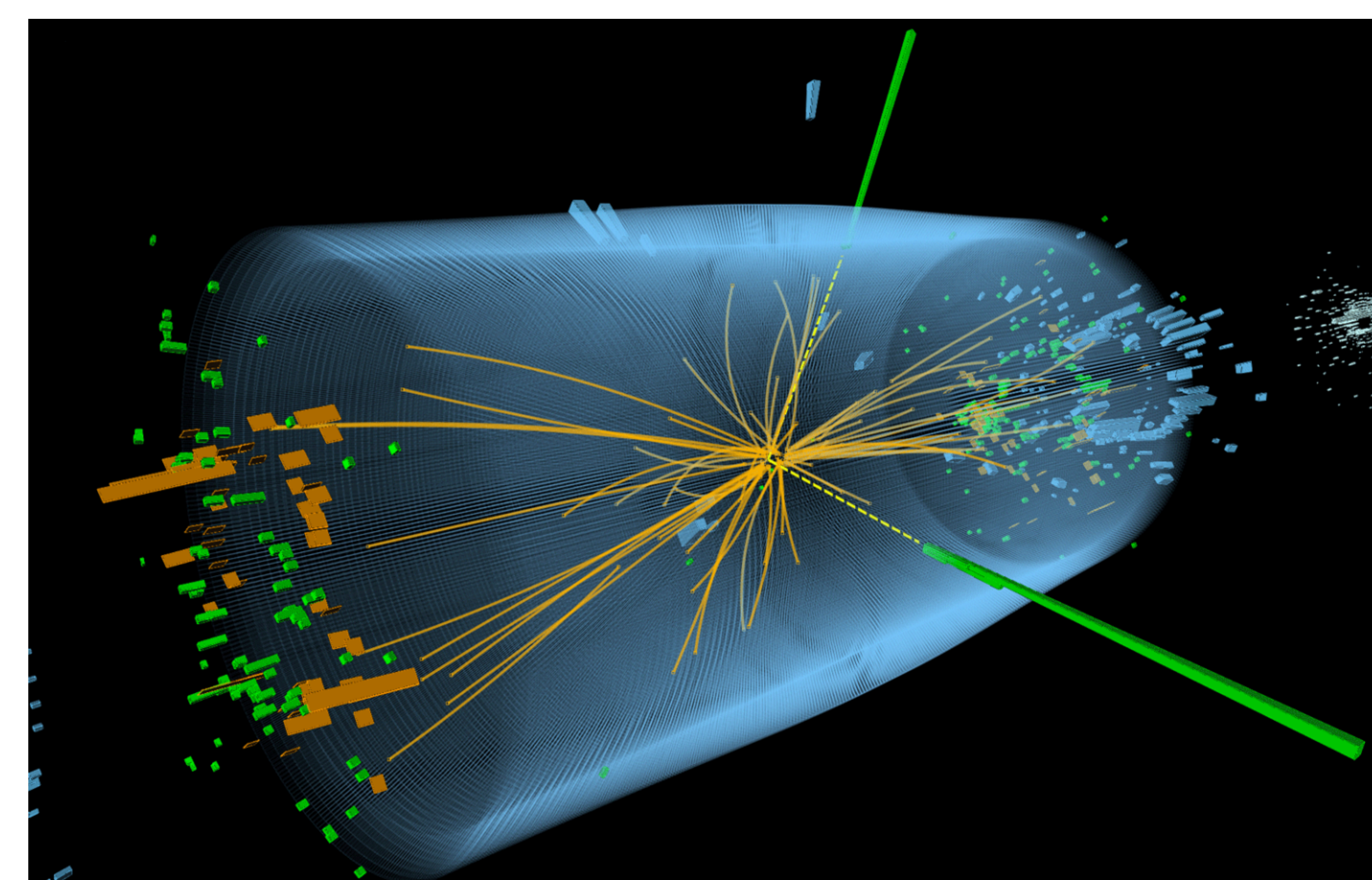
"*All master integrals for three-jet production at NNLO*", PhysRevLett. 123 (2019), no. 4 041603
"*Analytic result for a two-loop five-particle amplitude*", PhysRevLett. 122 (2019), no. 12 121602

Particle Physics, precision era



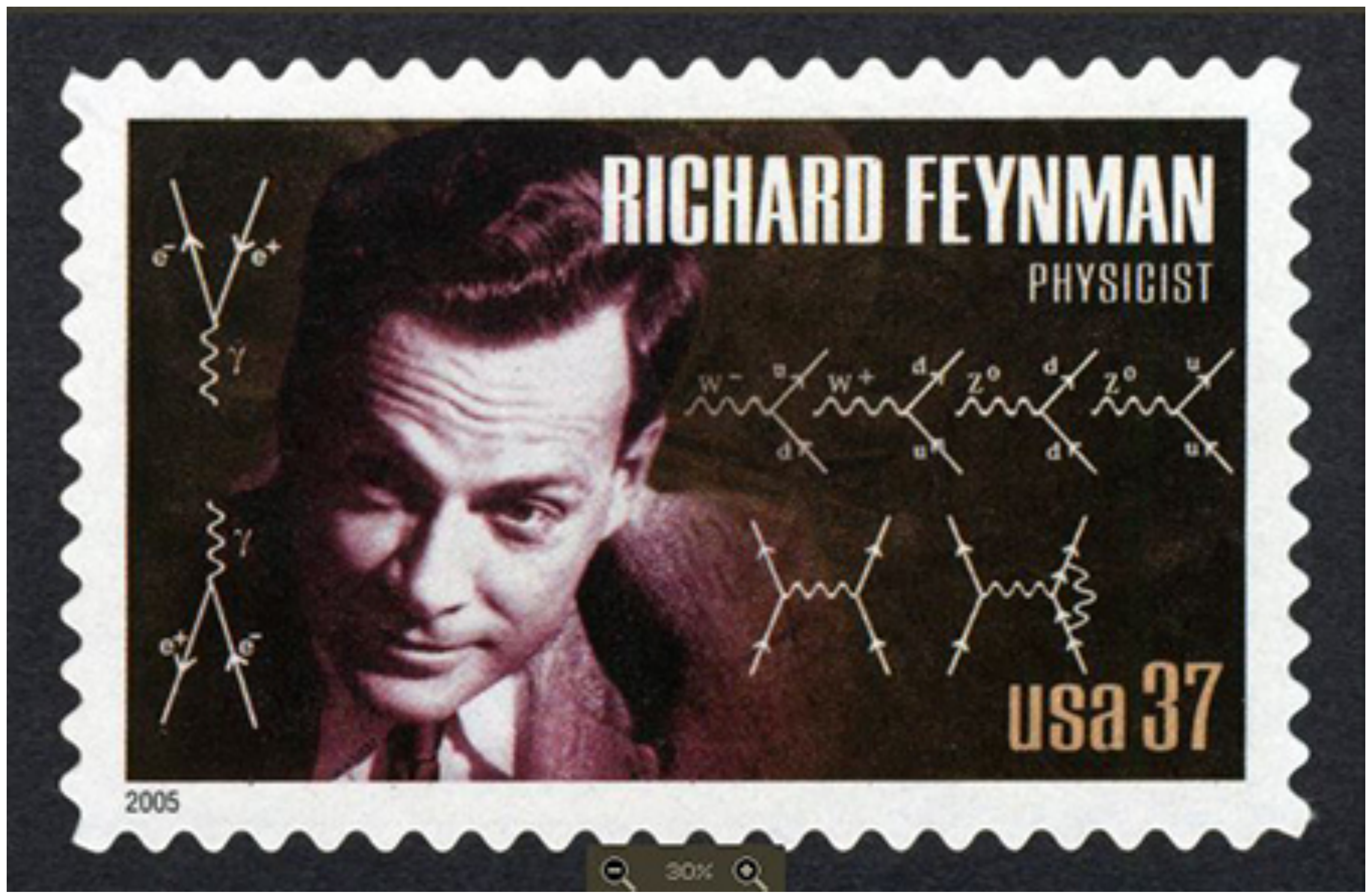
particle scattering

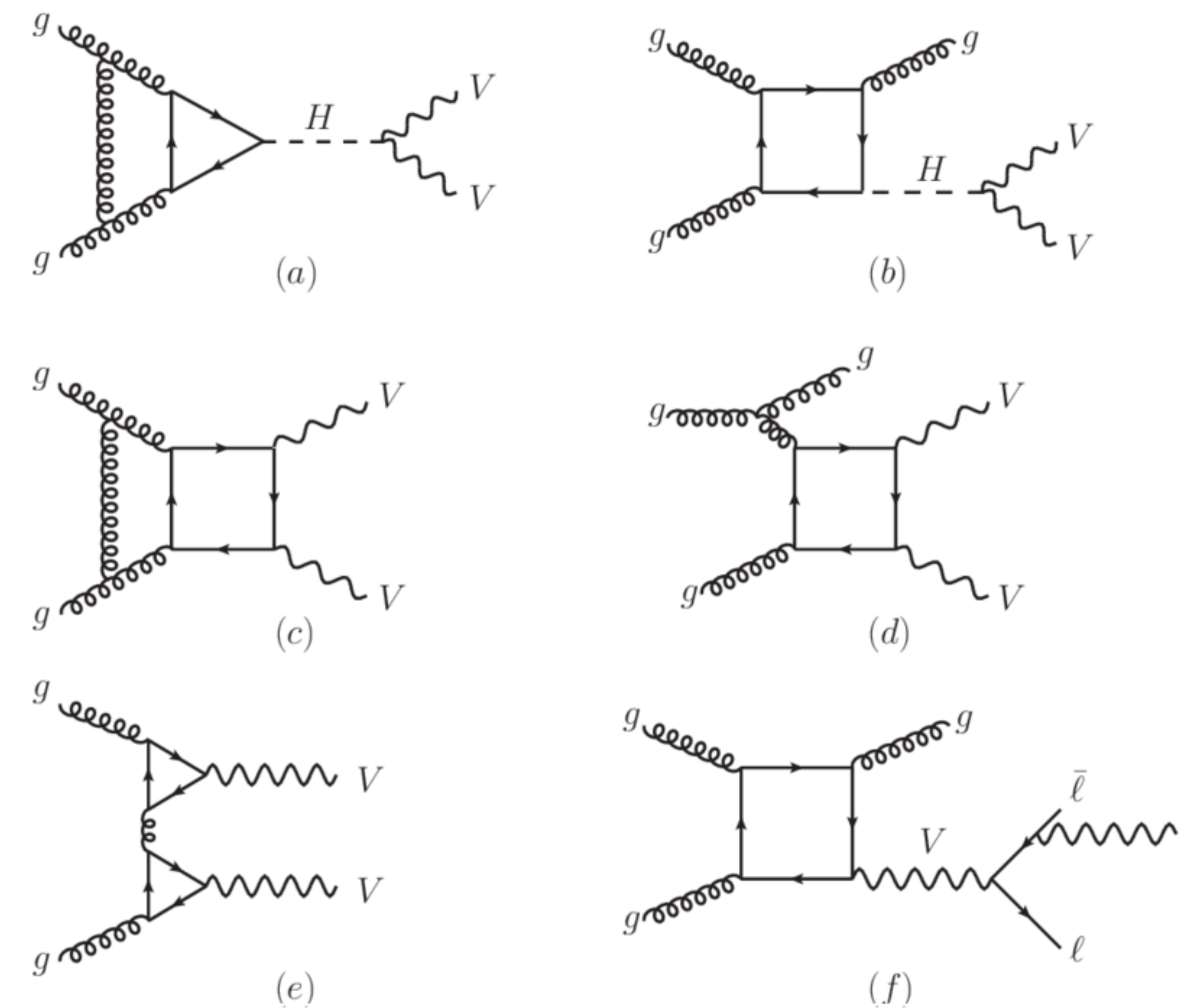
Large Hadron Collider Run III
(Geneva, Switzerland)



HL-LHC in progress
CEPC, FCC, ILC proposed

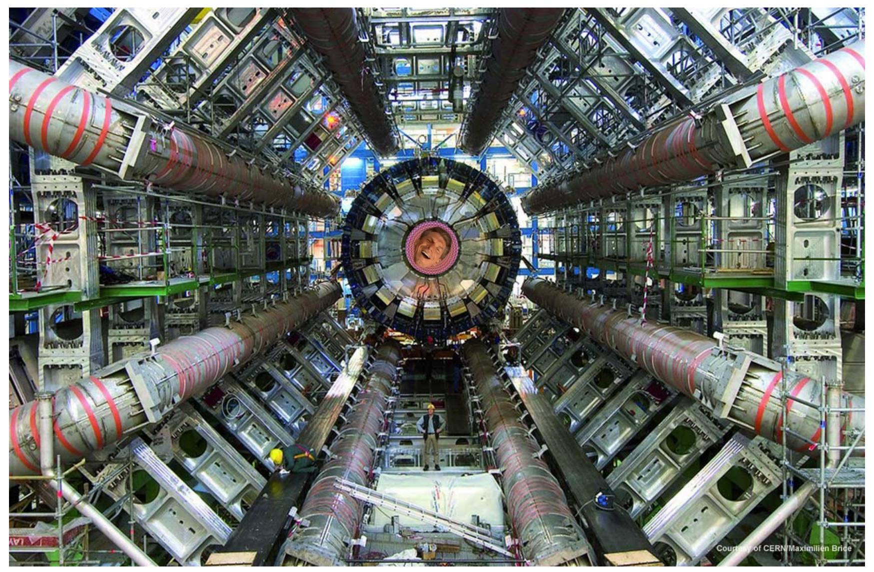
Feynman integrals



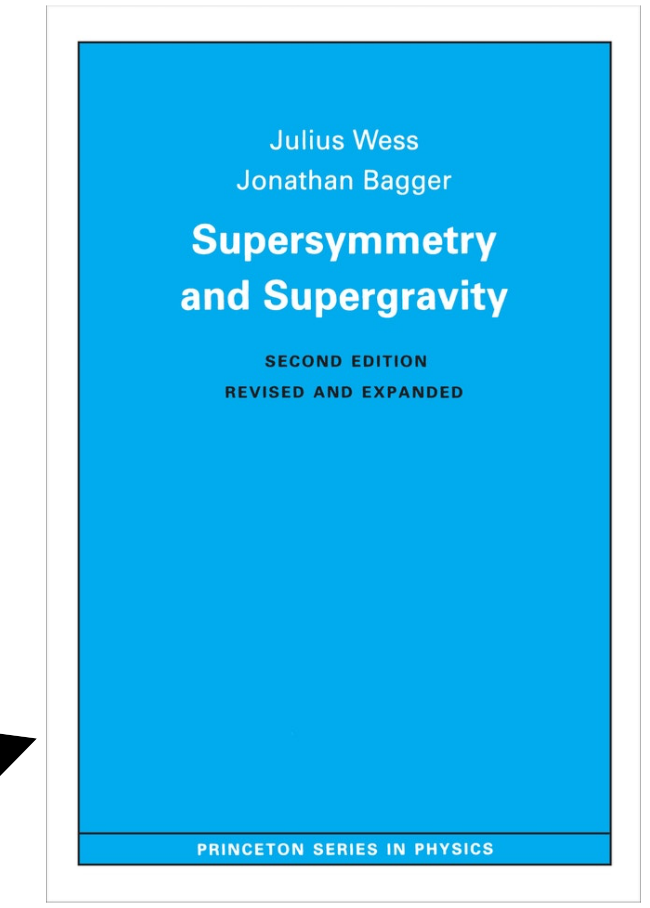


Basic tool for the theoretical prediction in particle physics

Feynman integrals, nowadays



precision
particle
physics



formal
theory

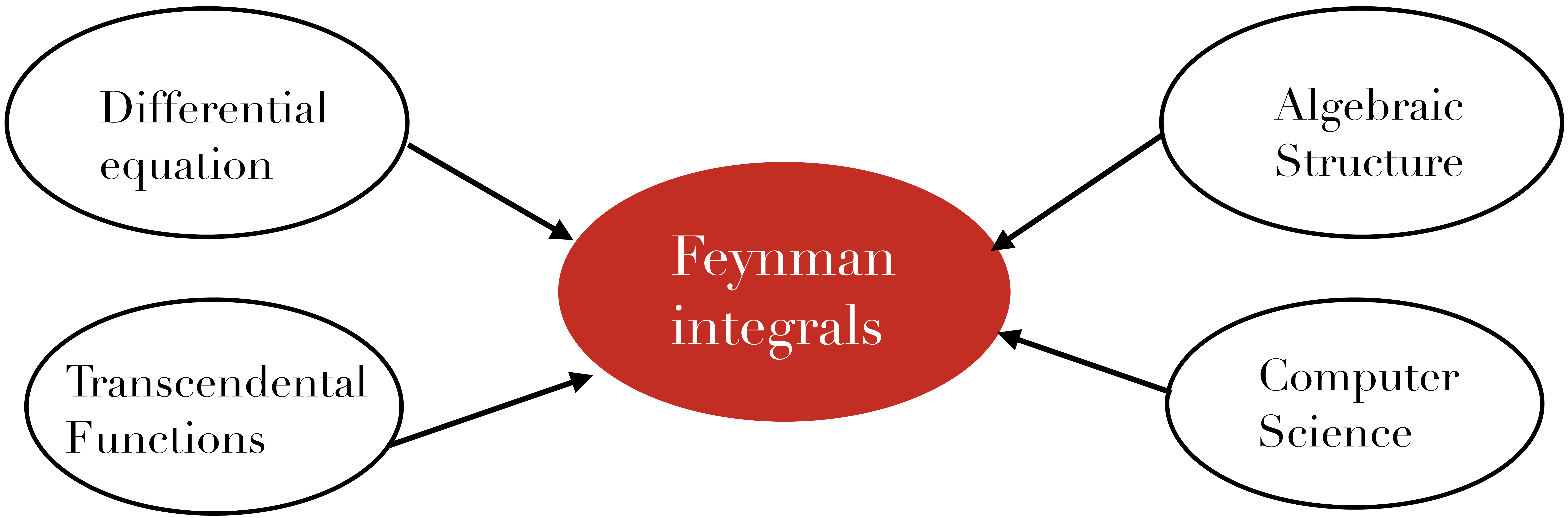
N=8 supergravity
is UV-finite until
five loop

Feynman
integrals

Gravitational wave

Feynman integrals, nowadays

It is still a basic tool in quantum field theory;
Crucially for precision high-energy physics, formal theories,



Significant progress after 2010

Why analytic Feynman integrals?

- Once the analytic expression is obtained, the phase point generation is extremely fast
- Avoid unstable numeric phase points
- Understand the deep structure and hidden symmetry in quantum field theory

and yes, we can.

Main stream Feynman integral computation method

Canonical Differential Equation for Analytic Feynman integrals

Feynman integrals

↓

master integrals

↓

UT Feynman integrals

↓

Canonical Differential Equation

↓

Analytic Feynman Integrals

Syzygy method

Module Lift method

Computational algebraic geometry

It is usually easier to compute
Feynman integrals with differential equations
than by the direct integration.

From Feynman integrals to master integrals

For a scattering process, there are a huge number of integrals

After a tensor reduction

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}}, \quad \alpha_i \in \mathbb{Z}$$

negative index means the numerator

IBP reduction

at the two/three loop orders,

IBP reduction can reduce **millions** of Feynman integrals to **hundreds** of master integrals.

$$\boxed{\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0}$$

Chetyrkin, Tkachov 1981

IBP reduction

$$\boxed{\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{\partial}{\partial l_i^\mu} \frac{v_i^\mu}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} = 0}$$

Usually by choosing different vectors, we also have a huge number of IBP relations
To get the complete reduction: Gaussian Elimination (Laporta algorithm 2000)

Two issues:

1. The Gaussian Elimination is computationally heavy, sometimes the most time consuming step for a scattering amplitude computation

2. The IBP reduction coefficients may be too large to use.

Use computational algebraic geometry for help!

Our method

module intersection

IBP in Baikov representation

$$\int \frac{d^D l_1}{i\pi^{D/2}} \cdots \int \frac{d^D l_L}{i\pi^{D/2}} \frac{1}{D_1^{\alpha_1} \ldots D_k^{\alpha_k}} \quad \propto \int_\Omega dz_1 \ldots dz_k \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}}$$

Baikov 1996

$$\boxed{0 = \int_\Omega dz_1 \ldots dz_k \sum_{i=1}^{k} \frac{\partial}{\partial z_i} \left(a_i(z) \frac{F^{\frac{D-L-E-1}{2}}}{z_1^{\alpha_1} \ldots z_k^{\alpha_k}} \right)}$$

No boundary term
feel free to set some of z's to zero (unitary cut)

IBP in Baikov representation with constraints

polynomials

Require

1. no shifted exponent: $$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$ These $\big(a_1(z), \ldots a_k(z)\big)$ form a module $M_1 \subset R^k$.

2. no propagator degree increase: $a_i(z) \in \langle z_i \rangle, \quad 1 \leq i \leq m$ These $\big(a_1(z), \ldots a_k(z)\big)$ form a module $M_2 \subset R^k$.

Larsen, YZ 2015
YZ 2016

Both M_1 and M_2 are pretty simple ...

$$\boxed{M_1 \cap M_2}$$

Intersection of two modules
a typical

Determine the first module

$$\sum_{j=1}^{k} a_j(z)\frac{\partial F}{\partial z_j} + \beta(z)F = 0$$

More Advanced

- syzygy for the $\{\frac{\partial F}{\partial z_1},\ldots,\frac{\partial F}{\partial z_k},F\}$ Roman Lee's trick
- $\mathrm{Ann}(F^s)$, annihilator of F^s in Weyl algebra. Bitoun, Bogner, Klausen, Panzer Lett.Math.Phys. 109 (2019) no.3, 497-564

If F is a determinant matrix whose elements are free variables, this kind of syzygy module is simple.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

equivalent to
canonical IBP
in momentum space

Laplace expansion

$$\sum_j a_{k,j}\frac{\partial(\det A)}{\partial a_{i,j}} - \delta_{k,i}\cdot \det A = 0$$

Get all first order annihilator, proved by Gulliksen–Negard and Jozefiak exact sequences

Boehm, Georgoudis, Larsen, Schulze, YZ 2017

Module Intersection

computational
algebraic geometry
problem

very similar
to linear space intersection,
but only polynomials
are allowed

$M_1 = \langle v_1, v_2, \dots, v_m \rangle$ each v_i : t-dim row vector

$M_2 = \langle u_1, u_2, \dots, u_n \rangle$ each u_j : t-dim row vector

$(m+n) \times (m+n+t)$

$$\left(\begin{array}{cccccccccc} v_1 & 1 & & & & & & & \\ v_2 & & 1 & & & & & & \\ \cdots & & & \cdots & & & & & \\ v_m & & & & 1 & & & & \\ u_1 & & & & & 1 & & & \\ u_2 & & & & & & 1 & & \\ \cdots & & & & & & & \cdots & \\ u_n & & & & & & & & 1 \end{array}\right)$$

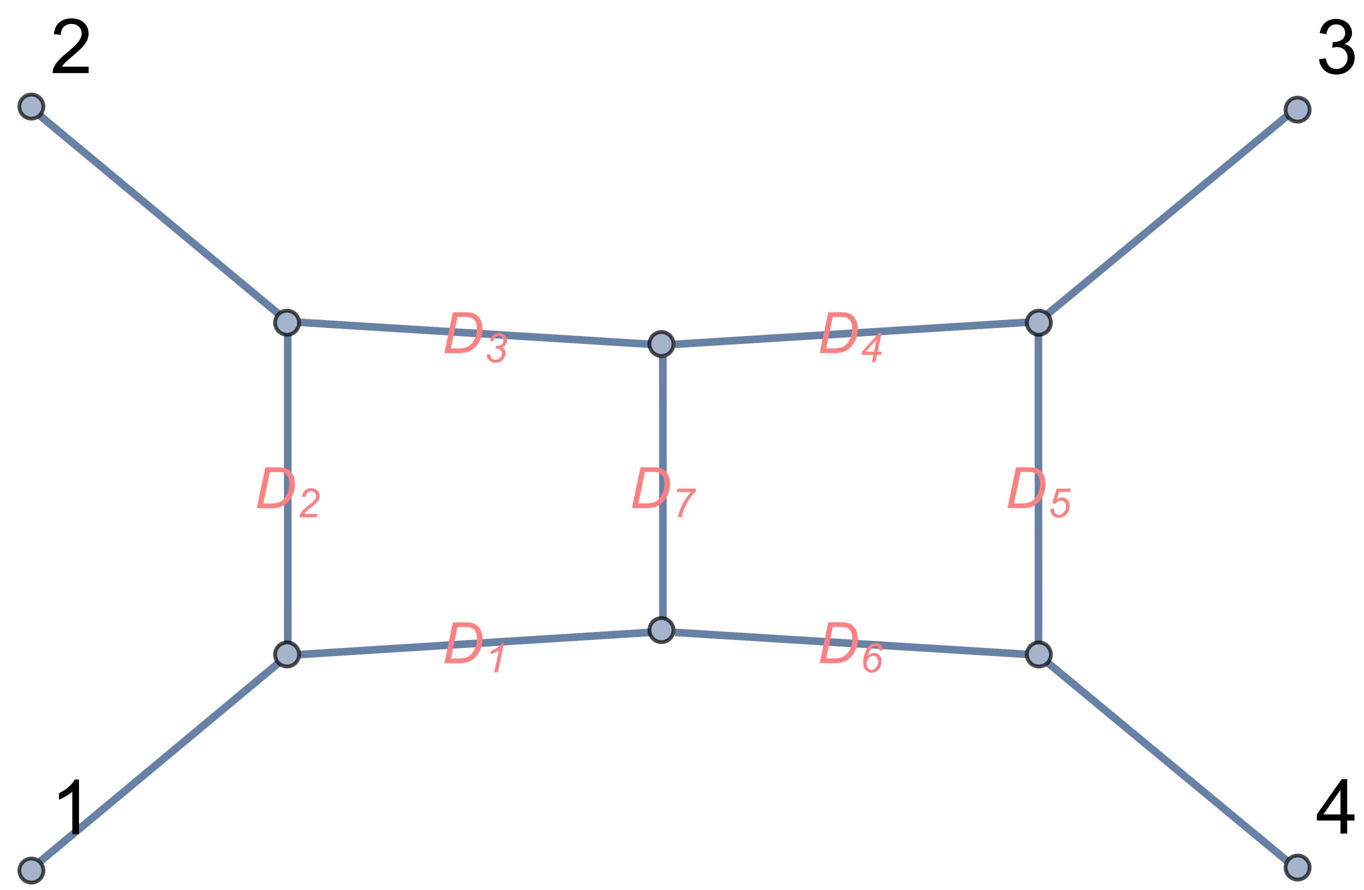
$\uparrow$ t columns $\qquad$ $\uparrow$ $m+n$ columns

Gröbner Basis computation to eliminate
first t components of a row

Example, massless double box

$\mathbb{Q}(s,t)[z_1,\dots z_9]$: 2 parameters, 9 variables

(Each row is a module generator)

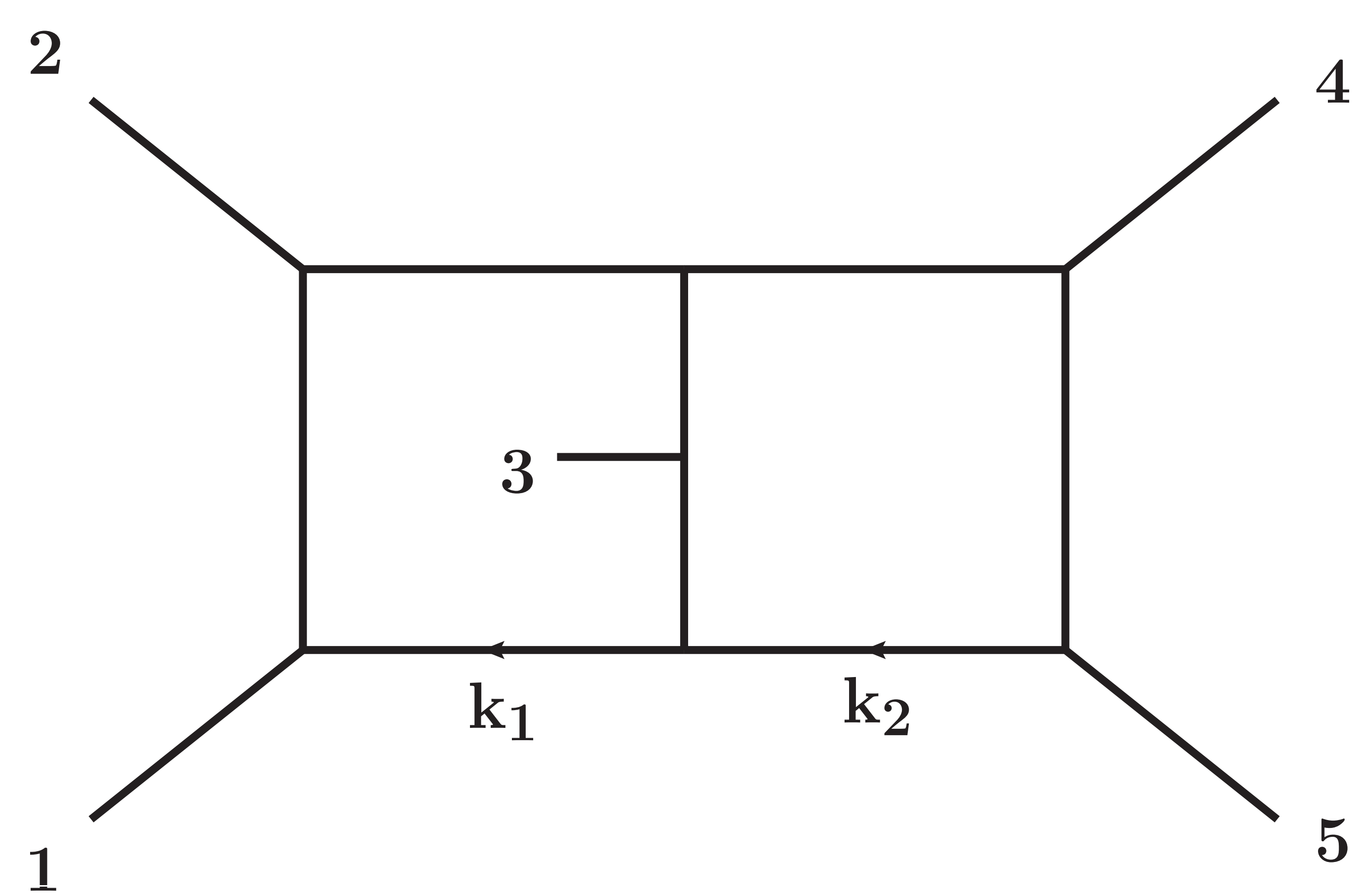


$$M_2 = \begin{pmatrix} z_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & z_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & z_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & z_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & z_7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} z_1-z_2 & z_1-z_2 & -s+z_1-z_2 & 0 & 0 & 0 & z_1-z_2-z_6+z_9 & t+z_1-z_2 & 0 \\ 0 & 0 & 0 & s-z_6+z_9 & -t-z_6+z_9 & -z_6+z_9 & z_1-z_2-z_6+z_9 & 0 & -z_6+z_9 \\ s+z_2-z_3 & z_2-z_3 & z_2-z_3 & 0 & 0 & 0 & z_2-z_3+z_4-z_9 & -t+z_2-z_3 & 0 \\ 0 & 0 & 0 & z_4-z_9 & t+z_4-z_9 & -s+z_4-z_9 & z_2-z_3+z_4-z_9 & 0 & z_4-z_9 \\ -z_1+z_8 & -t-z_1+z_8 & s-z_1+z_8 & 0 & 0 & 0 & -z_1-z_5+z_6+z_8 & -z_1+z_8 & 0 \\ 0 & 0 & 0 & -s-z_5+z_6 & -z_5+z_6 & -z_5+z_6 & -z_1-z_5+z_6+z_8 & 0 & t-z_5+z_6 \\ 2z_1 & z_1+z_2 & -s+z_1+z_3 & 0 & 0 & 0 & z_1-z_6+z_7 & z_1+z_8 & 0 \\ 0 & 0 & 0 & s-z_3-z_6+z_7 & -z_6+z_7-z_8 & -z_1-z_6+z_7 & z_1-z_6+z_7 & 0 & -z_2-z_6+z_7 \\ -z_1-z_6+z_7 & -z_1+z_7-z_9 & s-z_1-z_4+z_7 & 0 & 0 & 0 & -z_1+z_6+z_7 & -z_1-z_5+z_7 & 0 \\ 0 & 0 & 0 & -s+z_4+z_6 & z_5+z_6 & 2z_6 & -z_1+z_6+z_7 & 0 & z_6+z_9 \end{pmatrix}$$

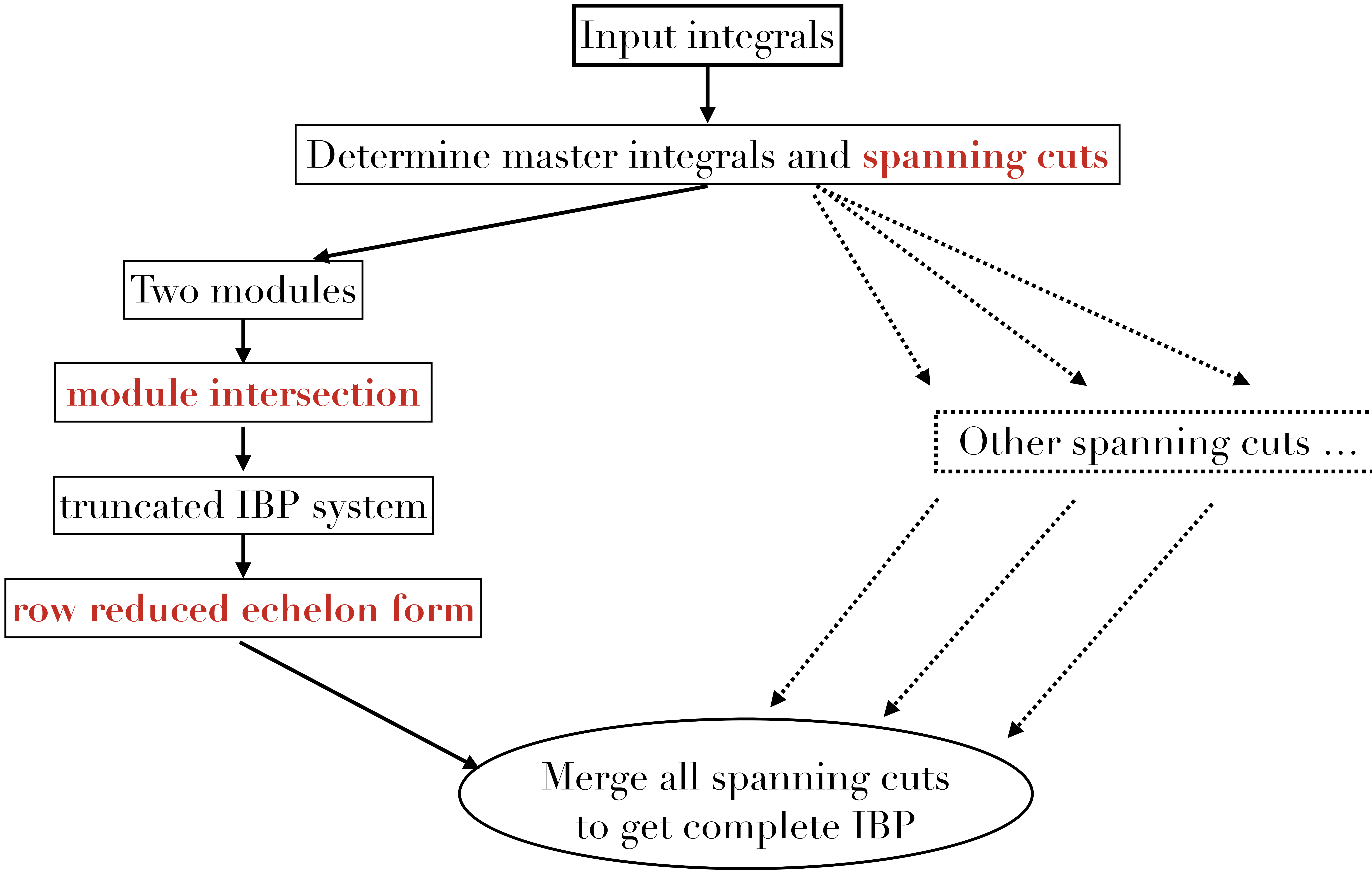
$M_1 \cap M_2$ is computed within seconds, with **Singular** 4.1.2's **intersect**

Now module intersection is really fast

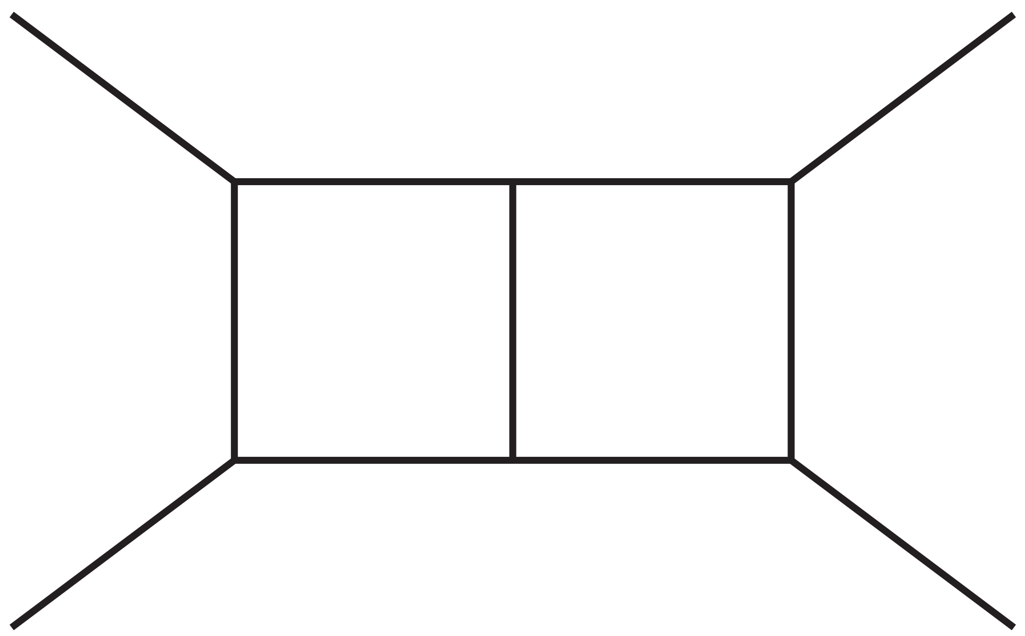


5 Mandelstam variables, with a triple cut
seconds to get the module intersection (and truncated IBPs)

After module intersection



Example, massless double box with spanning cut



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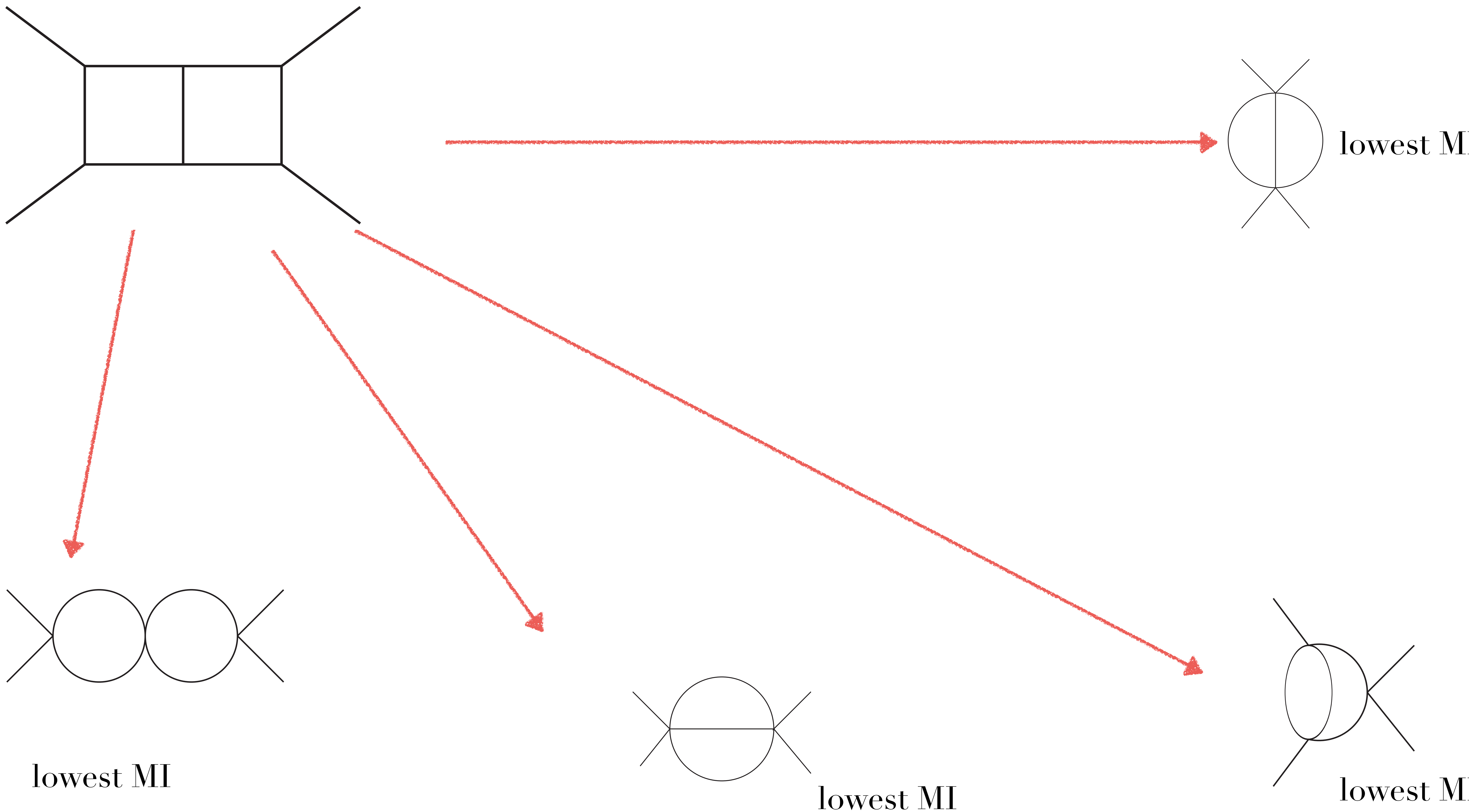

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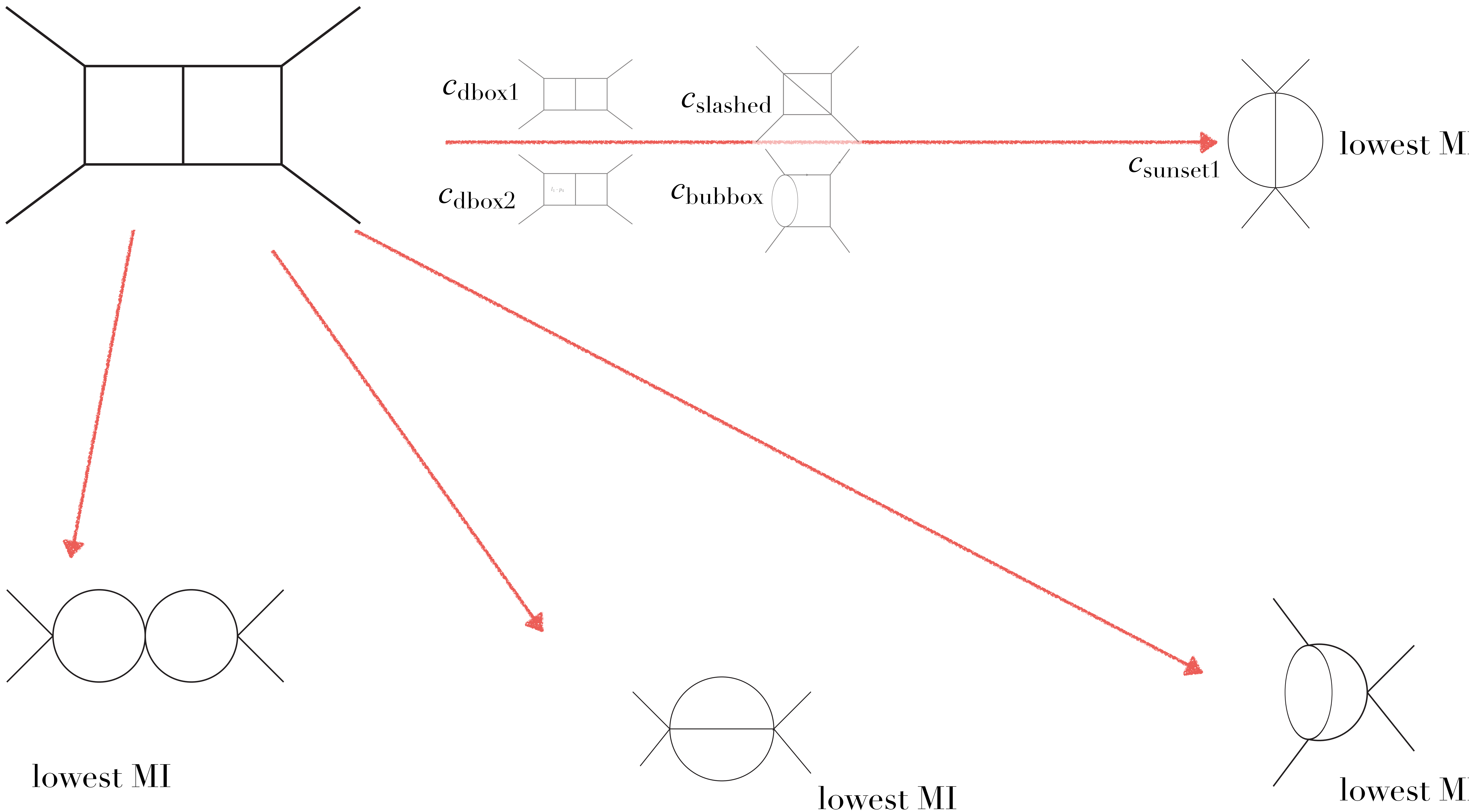

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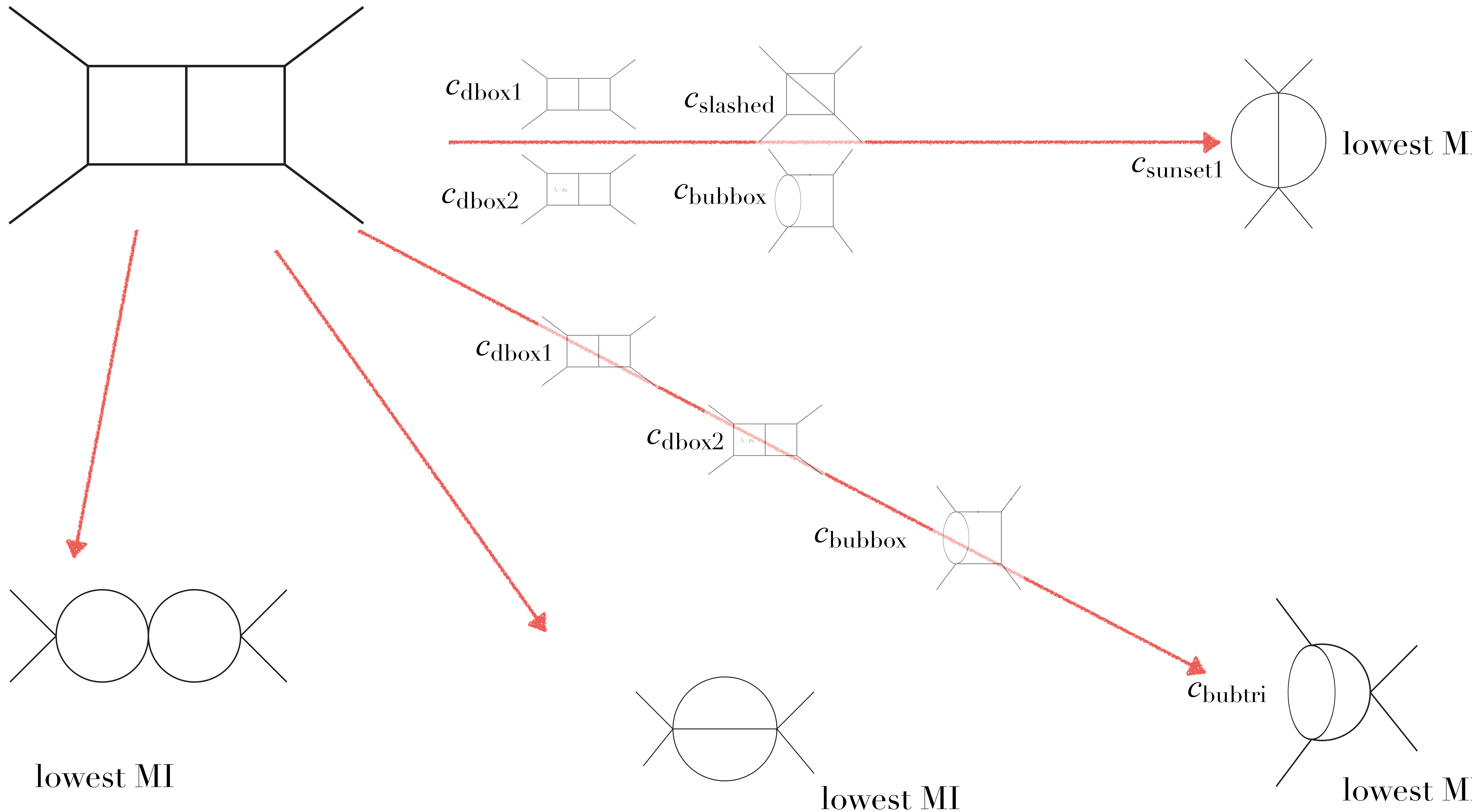

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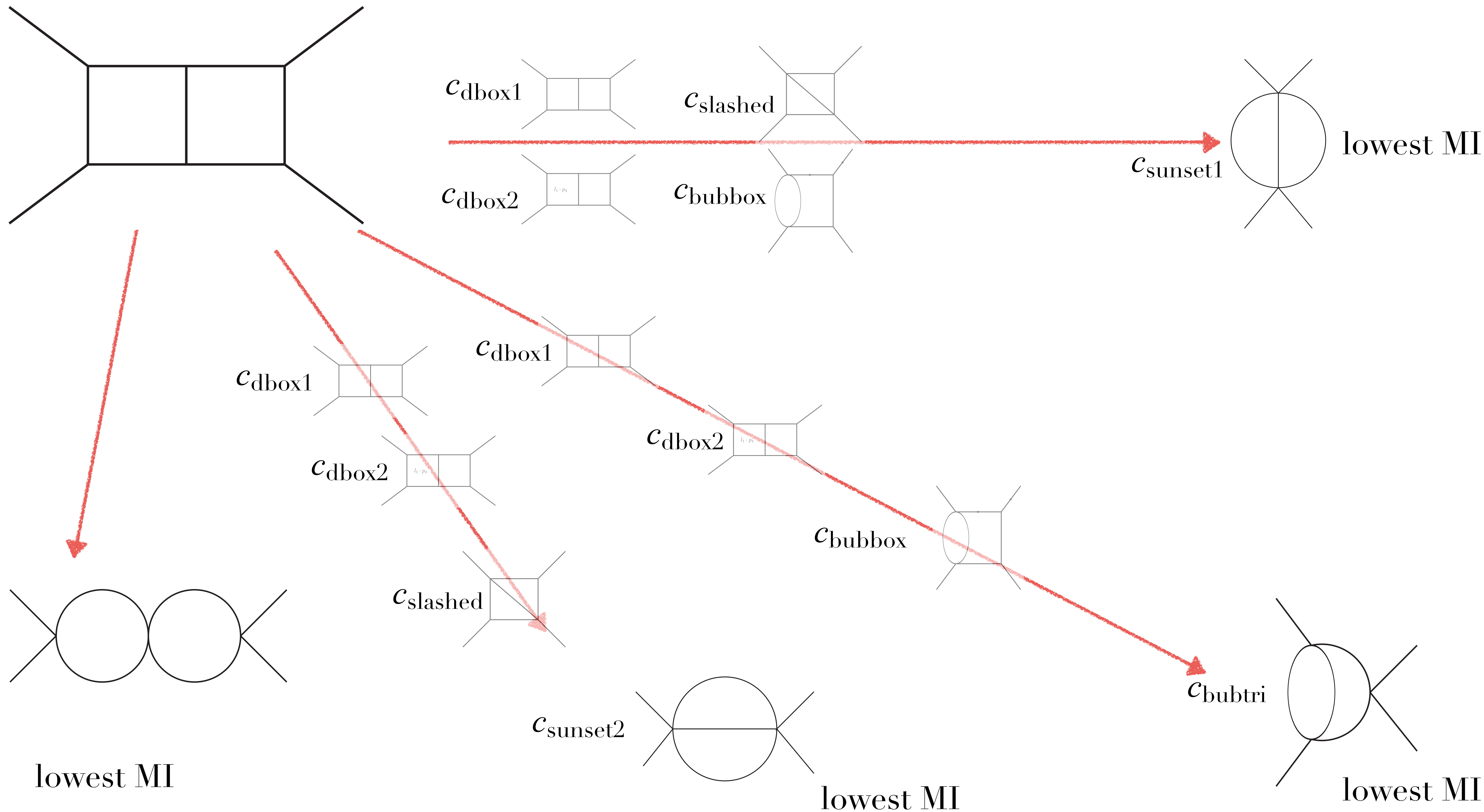
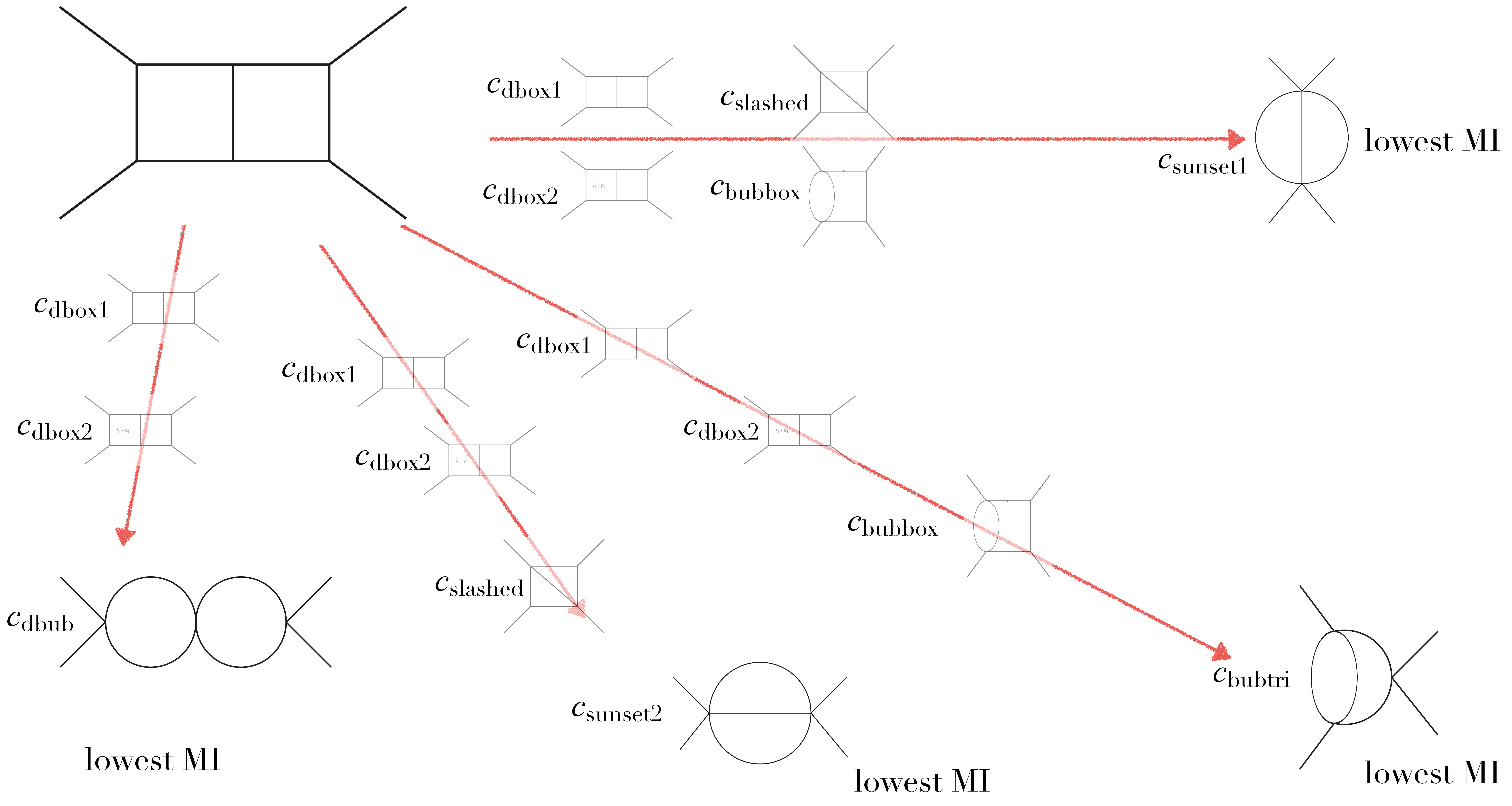
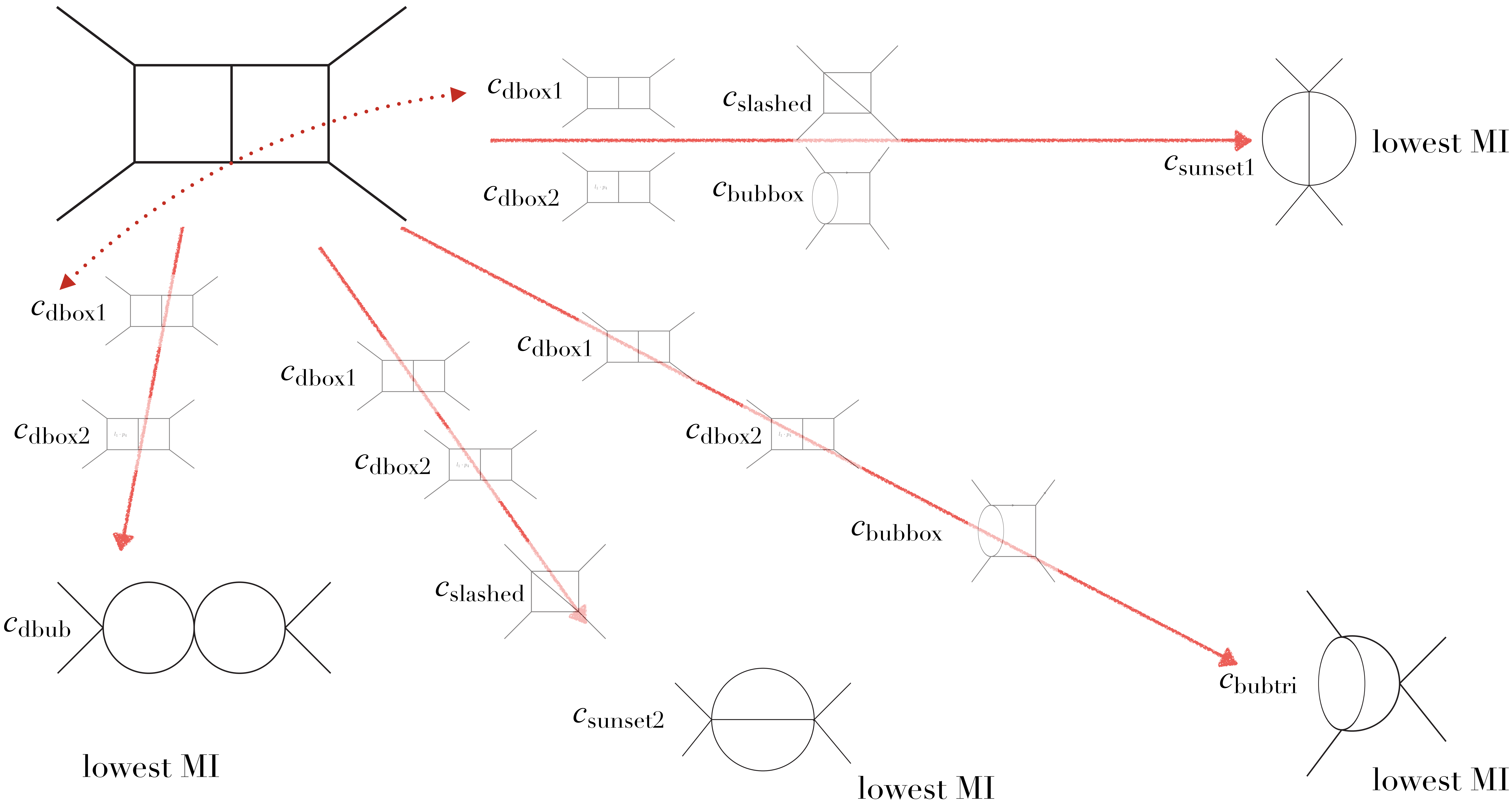


Example, massless double box with spanning cut



Example, massless double box with spanning cut

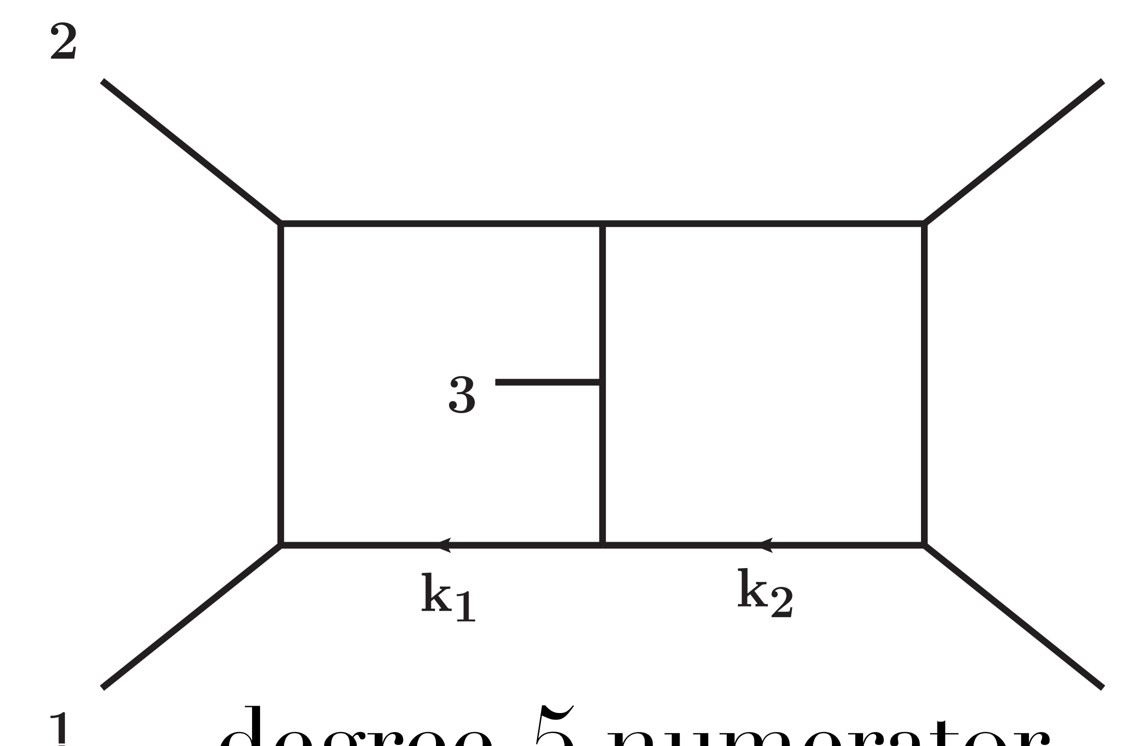


Research example

Bendle, Boehm, Decker, Georgoudis, Pfreundt, Rahn, Wasser, YZ
JHEP 02 (2020) 079

module intersection

Cut	# relations	# integrals	size
{1,5,7}	2723	2749	1.4 MB
{1,5,8}	2753	2777	1.6 MB
{1,6,8}	2817	2822	2.1 MB
{2,4,8}	2918	2921	2.1 MB
{2,5,7}	2796	2805	1.5 MB
{2,6,7}	2769	2814	1.2 MB
{2,6,8}	2801	2821	1.6 MB
{3,4,7}	2742	2771	1.4 MB
{3,4,8}	2824	2849	1.9 MB
{3,6,7}	2662	2674	1.5 MB
{1,3,4,5}	1600	1650	0.72MB



degree-5 numerator reduction

Only 17.2 MB in total

100 times smaller system than the system from the standard program FIRE6

Application of algebraic geometry for integrable spin chain

Based on

Boehm, Jacobsen, Jiang and **YZ**
"Geometric algebra and algebraic geometry of loop and Potts models", JHEP 05 (2022) 068

Jiang, Wen and **YZ**
"Exact Quench Dynamics from Algebraic Geometry", 2109.10568

Bajnok, Jacobsen, Jiang, Nepomechie and **YZ**
"*Cylinder partition function of the 6-vertex model from algebraic geometry*", JHEP 06 (2020) 169

Jacobson, Jiang and **YZ**
"*Torus partition function of the six-vertex model from algebraic geometry*", JHEP 1903 (2019) 152

Jiang and **YZ**
"Algebraic geometry and Bethe ansatz. Part I. The quotient ring for BAE", JHEP 03 (2018) 087

Bethe Ansatz Equation

Heisenberg spin chains are solved by BAE

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L = \prod_{k \neq j}^{M} \frac{u_j - u_k - i}{u_j - u_k + i}, \quad j = 1, \ldots, M$$

Bethe roots

Hans Bethe

Physical quantities are usually symmetric functions of Bethe roots for example:

Symmetric function

$$E \propto \sum_{k=1}^{M} \frac{1}{u_k^2 + 1/4}$$

Solving BAE usually provides numeric roots with errors

Can we get analytic physical quantities from BAE?

Symmetric function of roots

For one univariate equation

$$0 = x^n - e_1 x^{n-1} + e_2 x^{n-2} + \ldots + (-1)^n e_n = \sum_{i=1}^{n} (x - x_i)$$

Any symmetric function of roots are clearly polynomials of e_i's, the coefficients of the equation.

no need to solve the equation

multivariate (multiple) equations

Any symmetric polynomial (rational function) of roots
would also be rational function of the equation coefficients

with the help of computational algebraic geometry

Companion Matrix method

polynomial ring

equations

algebraic
field
extension

$$\dim_{\mathbb{F}}(\underline{R/I}) = \text{number of solutions of } I \text{ in } \overline{\mathbb{F}}$$

quotient ring

The quotient ring is a finite dimensional linear space with multiplication

$$R/I = \mathrm{span}\{b_1, \ldots b_k\}$$

We consider the linear representation of R/I,

$$[f \cdot b_i] = a_{ij}[b_j], \quad [f] \in R/I$$

Companion Matrix, with the size of the solution

Companion Matrix method

A root

$$[f \cdot b_i] = a_{ij}[b_j] \quad \Rightarrow f(p)b_i(p) = a_{ij}b_j(p)$$

Each eigenvalue of the companion matrix is the value of f on a solution

$$\boxed{\mathrm{tr} M_{[f]} = \sum_{p \in \mathcal{Z}(I)} f(p)}$$

Symmetric polynomial (rational) function of roots is the trace of the companion matrix

This key property plays the central role
of evaluating physical quantities from BAE

Jiang, **YZ** 2018

Companion Matrix example

$$x^2 - y - 1 = y^2 + z^2 - x = xz - y^2 + 2 = 0$$

```
MultiplicationMatrix[y^2,Gr,kbasis,{x,y,z},MonomialOrder→DegreeReverseLexicographic]//MatrixForm
Tr[Normal[%]]
```

MatrixForm=

$$\begin{pmatrix} 4 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\ -2 & 2 & -2 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 1 & 1 & 0 & -1 & 0 \\ -3 & 0 & -1 & 3 & 0 & 0 & 0 & -1 \\ 0 & -2 & 3 & -2 & 2 & 0 & -2 & 1 \\ 5 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 1 & 0 & 1 & 0 \\ -3 & 1 & -1 & 4 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{14}$$

$$\sum_{p \in \mathcal{Z}(I)} y^2|_p = 14$$

consistent with the numeric result

```
NSolve[Ideal,{x,y,z}]
{{x → -1.48768 - 0.314318 i, y → 1.11439 + 0.935209 i, z → 0.76724 - 1.5632 i},
 {x → -1.48768 + 0.314318 i, y → 1.11439 - 0.935209 i, z → 0.76724 + 1.5632 i},
 {x → -0.0219091 - 0.450523 i, y → -1.20249 + 0.019741 i, z → 0.164836 - 1.22257 i},
 {x → -0.0219091 + 0.450523 i, y → -1.20249 - 0.019741 i, z → 0.164836 + 1.22257 i},
 {x → 0.0618829 + 1.00477 i, y → -2.00574 + 0.124357 i, z → -0.372019 - 2.0209 i},
 {x → 0.0618829 - 1.00477 i, y → -2.00574 - 0.124357 i, z → -0.372019 + 2.0209 i},
 {x → 1.4477 - 0.0448355 i, y → 1.09384 - 0.129817 i, z → -0.560057 - 0.213516 i},
 {x → 1.4477 + 0.0448355 i, y → 1.09384 + 0.129817 i, z → -0.560057 + 0.213516 i}}
```

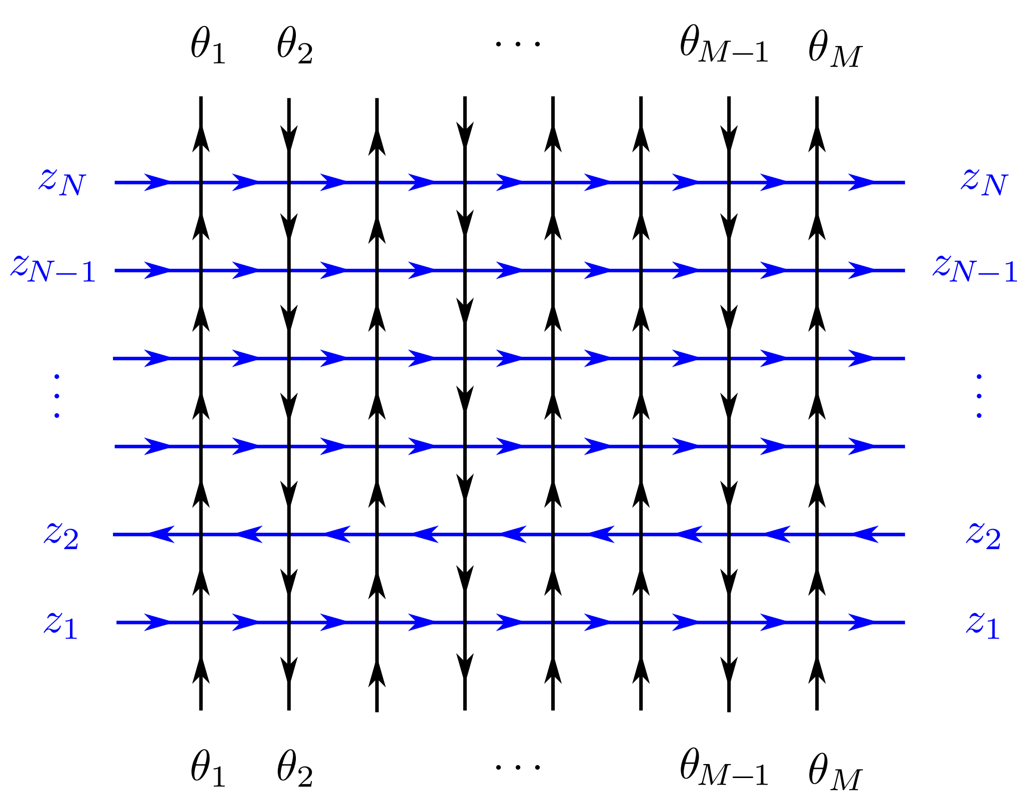
Implement:

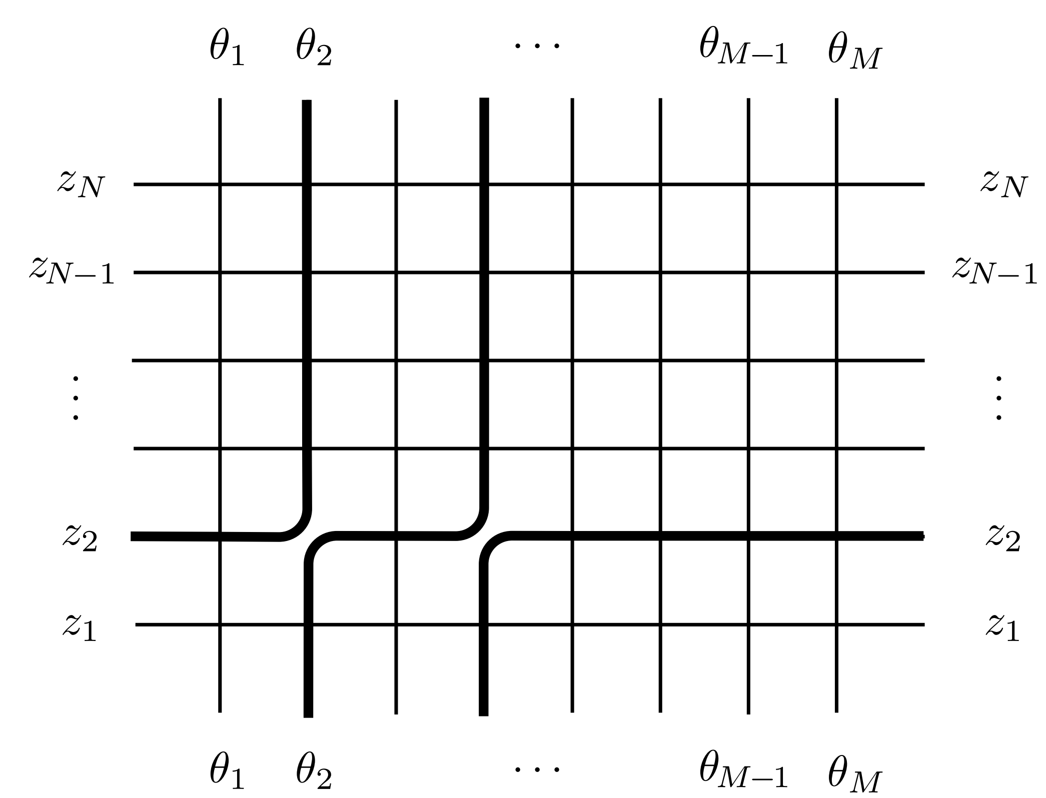
calling Singular from Mathematica

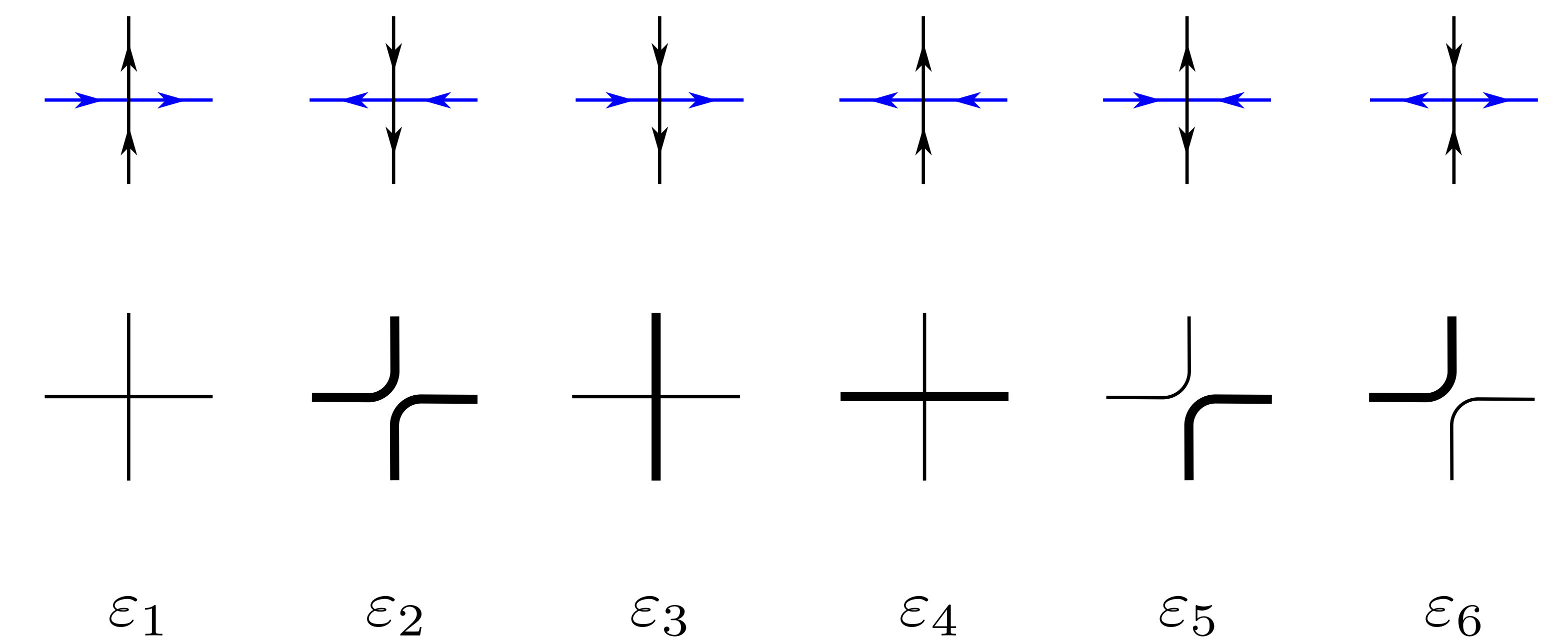
https://www.singular.uni-kl.de

Research level example: 6-vertex model

Jacobson, Jiang, **YZ, 2019**
Bajnok, Jacobson, Jiang, Nepomechie, **YZ,** 2020







a lattice model to describe ice or potassium dihydrogen phosphate
It is mapped to Heisenberg XXX spin chain

✔ Use our algebraic geometry method,
we get the **exact partition function** for the lattice size 100×14

(Brute Force) to compute 100th power of a 16384×16384 matrix, impossible

Summary

- Novel methods for analytic computations in theoretical physics
- Computational algebraic geometry applications:

  Analytic Feynman integral reduction
  Analytic computations with Bethe Ansatz Equation

Vielen Dank!