Rational Q-systems and Linear Quiver Gauge Theories

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Overview

- Bethe/gauge duality: SUSY vacua of 3d $\mathcal{N} = 2^*$ theory on S^1 are given by solutions to BAEs.
- Best way to solve BAEs are through the rational Q-systems.
- Lesson for SUSY field theorists: We generalise construction of rational Q-systems to generic A-type quivers.
- Lesson for integrability experts: We give quick criteria for bispectral duality.

Outline

Overview of 3d $\mathcal{N} = 4$ theories

3d $\mathcal{N} = 2^*$ theories and Bethe/gauge correspondence

Rational Q-systems

- From QQ-relations to BAEs
- Solving Q-system

Mirror symmetry and bispectral duality

Conclusion and outlook

Overview of 3d $\mathcal{N} = 4$ theories

Why 3d $N = 4$ theories?

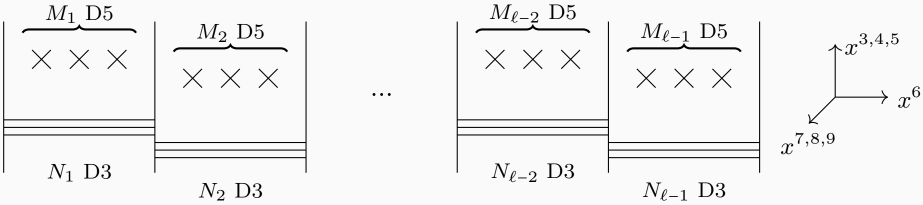
- Relatively easy to understand.
  - Different constructions
  - Moduli spaces
  - Hilbert series
  - Monopole operators, defects
  - Partition functions
  - ...
- Interesting properties: Mirror symmetry, etc.
- Deep connections to quantum integrable systems.

$T[SU(n)]$ theories

- Brane construction in type IIB superstring theory

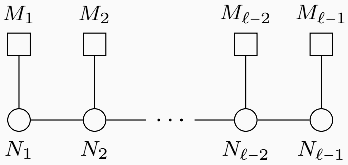
	0	1	2	3	4	5	6	7	8	9
NS5	×	×	×	×	×	×				
D3	×	×	×				×			
D5	×	×	×					×	×	×

$T[SU(n)]$ theories



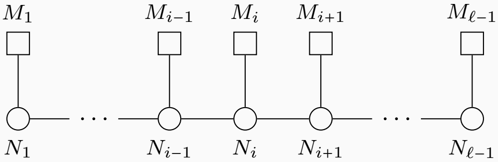
- ℓ NS5 separated along x^6
- N_i D3 suspended between i-th and $(i+1)$-th NS5 branes
- M_i D5 in between i-th and $(i+1)$-th NS5 branes
- 3d $N = 4$ theory lives on $x^{0,1,2,3}$.

Field theory contents



- Round node: dynamic $U(N_i)$ vector multiplets
- Square node: background $U(M_i)$ vector multiplets
- Black lines: hypermultiplets in bifundamental representation

IR theory



- Flow to interacting SCFT in IR if for each gauge node

$$e_i = N_{i-1} + N_{i+1} + M_i - 2N_i \geq 0$$

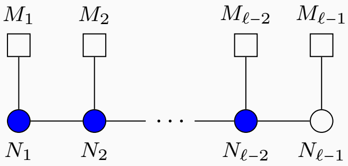
These theories are referred to as “good” in [Gaiotto-Witten].

Global symmetries of SCFT

	0	1	2	3	4	5	6	7	8	9
NS5	×	×	×	×	×	×				
D3	×	×	×				×			
D5	×	×	×					×	×	×

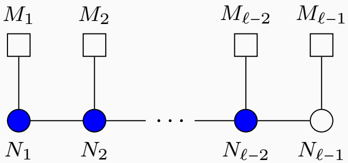
- $\mathcal{N} = 4$ R-symmetry: $SO(4)_R \cong SU(2)_H \times SU(2)_C$, geometrically realised as
  - rotation group $SU(2)_H \cong SO(3)_{7,8,9}$
  - rotation group $SU(2)_C \cong SO(3)_{3,4,5}$

Global symmetries of SCFT



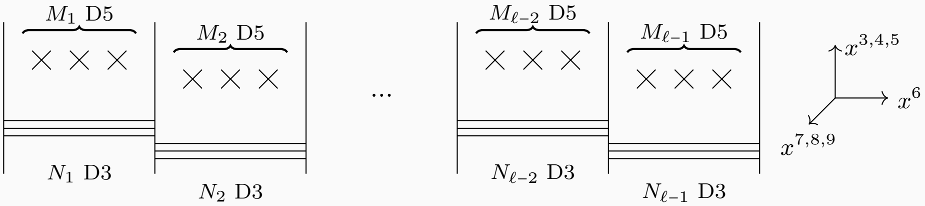
- Flavor symmetry: $G_H = \left(\prod_{j=1}^{\ell-1} U(M_j)\right) / U(1)_{\text{diag}}$

Global symmetries of SCFT



- Flavor symmetry: $G_H = \left(\prod_{j=1}^{\ell-1} U(M_j)\right) / U(1)_{\text{diag}}$
- Coulomb branch symmetry G_C^{IR}
  - $G_C^{\text{UV}} = U(1)_J^{\ell-1}$: topological symmetry $U(1)_J$ from each $U(N_i)$
  - May be enhanced in IR by monopole operators as ladder operators. [Gaiotto-Witten], [Aharony-Hanany-Intriligator-Seiberg-Strassler], [Borokhov-Kapustin-Wu], [Bashkirov]
  - The subset of balanced gauge nodes yield Dynkin diagram of non-Abelian part of G_C^{IR}.
  - Example above: $G_C^{\text{IR}} = SU(\ell - 1) \times U(1)$.

Deformation parameters



- A triplet of masses $\vec{m} = (m^1, m^2, m^3)$
  - Corresponding to $CSA(G_H)$, $\mathbf{1} \otimes \mathbf{3}$ of $SU(2)_H \times SU(2)_C$
  - Pst of D5 in $x^{3,4,5}$ rotated by $SO(3)_{3,4,5}$
- A triplet of FI parameters $\vec{\omega} = (\omega^1, \omega^2, \omega^3)$
  - Corresponding to $CSA(G_C)$, $\mathbf{3} \otimes \mathbf{1}$ of $SU(2)_H \times SU(2)_C$
  - Relative psts of NS5_i, NS5_{i+1} in $x^{7,8,9}$ rotated by $SO(3)_{7,8,9}$

Two partitions

- The linear quiver can be described by two partitions $\boldsymbol{\rho}, \boldsymbol{\sigma}$; the 3d $\mathcal{N} = 4$ theory denoted by $T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}[SU(n)]$ [Gaiotto-Witten]

$$\boldsymbol{\rho} = (\rho_1, \rho_2, \ldots), \quad \rho_1 \geq \rho_2 \geq \ldots \geq \rho_\ell > 0, \quad |\boldsymbol{\rho}| = \sum_{i=1}^{\ell} \rho_i = n$$

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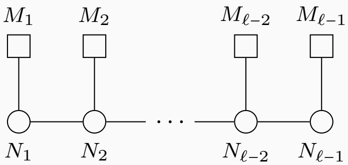


- Let $\boldsymbol{\sigma}^T = (\hat{\sigma}_1, \hat{\sigma}_2, \ldots)$. The two partitions are

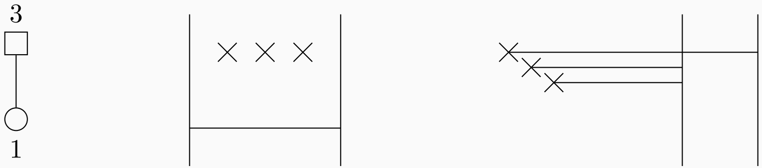
$$\hat{\sigma}_j = \sum_{i=j}^{\ell-1} M_i, \quad \rho_i = \begin{cases} -N_1 + \sum_{j=1}^{\ell-1} M_j, & i = 1 \\ N_{i-1} - N_i + \sum_{j=i}^{\ell-1} M_j, & 1 < i < \ell \\ N_{\ell-1}, & i = \ell \end{cases}$$

- $\boldsymbol{\rho}$ is a partition function if the quiver theory is "good"

$$\rho_i = e_i + e_{i+1} + \ldots e_{\ell-1} + N_{\ell-1}$$

Two partitions

- Physically, we can move all D5 to the left, taking into account brane creation/annihilation [Hanany-Witten]
  - ρ_j are the net numbers D3 ending on NS5
  - σ_j are the net numbers D3 ending on D5
  - $|\boldsymbol{\rho}| = |\boldsymbol{\sigma}| = n$ is the total number of D3 branes
- Example: SQED w/ $\boldsymbol{\rho} = (2, 1)$, $\boldsymbol{\sigma} = (1, 1, 1)$.



3d $\mathcal{N} = 2^*$ theories and Bethe/gauge correspondence

- To make connection with quantum integrable systems, we softly break supersymmetry $\mathcal{N}=4 \to \mathcal{N}=2^*$;
- and compactify on S^1 to get 2d $\mathcal{N}=(2,2)^*$ KK theory

SUSY breaking

- We choose and preserve a $\mathcal{N} = 2$ sub-algebra of $\mathcal{N} = 4$ algebra
- Let j_H^3, j_C^3 be Cartan generators of $SU(2)_H \times SU(2)_C$
- Choose $j_R = j_H^3 + j_C^3$ to be generator of $U(1)_R$ of $\mathcal{N} = 2$

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- $j_\eta = j_H^3 - j_C^3$ generates add'l global non-R-symmetry $U(1)_\eta$ from $\mathcal{N} = 2$ POV
- Breaking $\mathcal{N} = 4 \to \mathcal{N} = 2^*$ by turning on real mass $\tilde{\eta}$, by coupling $\mathcal{N} = 2$ background $U(1)_\eta$ vector multiplet

After SUSY breaking

- $\mathcal{N}=4$ vector $(\varphi_{1,2,3}) \to \mathcal{N}=2$ vector (φ_3) + $\mathcal{N}=2$ chiral $(\Phi = \varphi_1 + \mathrm{i}\varphi_2)$
- Trplets of $\vec{m}, \vec{\omega}$ split to complex and real
  - Real components $m \equiv m^3, \omega \equiv \omega^3$: relevant for CB of effective theory
  - Complex components $m^1 + \mathrm{i}m^2, \omega^1 + \mathrm{i}\omega^2$: irrelevant for CB of effective theory

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- Summary:
  - Global non-R symmetry: $G_H \times G_C \times U(1)_\eta$
  - Ass'ted real parameters: $m, \omega, \tilde{\eta}$

Circle compactification with radius R

- Compactify 3d $\mathcal{N} = 2^*$ on S^1 to get 2d $\mathcal{N} = (2,2)^*$ KK theory.
- Combination of $(m, \omega, \tilde{\eta}/2)$ w/ flavor Wilson lines a_0^F into complex deformation parameters

$$\theta_j = \mathrm{i}R(m_j + \mathrm{i}a^H_{0,j}), \quad t_s = \mathrm{i}R(\omega_s + \mathrm{i}a^C_{0,s}), \quad \eta = \mathrm{i}R(\tilde{\eta} + \mathrm{i}a^\eta_0).$$

These are twisted masses of 2d effective $\mathcal{N} = (2,2)^*$ theory.

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- Define exponentiated variables as Wilson lines a_0^F are periodic

$$y_j = \mathrm{e}^{2\pi\mathrm{i}\theta_j}, \quad \epsilon_s = \mathrm{e}^{2\pi\mathrm{i}t_s}, \quad q = \mathrm{e}^{\pi\mathrm{i}\eta}.$$

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- 3d dynamic $\mathcal{N} = 2$ vector multiplets contain real scalar $\sigma_k \equiv \varphi_{3,k}$ $(k = 1, \ldots, \mathrm{rk}(G))$, which can be combined with Wilson line $a_{0,k}$ along S^1 and exponentiated

$$x_k = \mathrm{e}^{2\pi\mathrm{i}u_k}, \quad u_k = \mathrm{i}R(\sigma_k + \mathrm{i}a_{0,k}).$$

2d effective theory

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- Turning on generic twisted masses, moduli space breaks down to isolated points.
- To find the supersymmetric vacua:
  - Move into Coulomb branch of moduli space
  - The dynamics of 2d Abelian field strength is controled by effective twisted superpotential $\widetilde{W}_{\text{eff}}(u_k)$
  - Turn on twisted masses $\boldsymbol{z} = (\theta_j, t_s, \eta)$, integrate out massive fields, compute contributions to $\widetilde{W}_{\text{eff}}(u_k; \boldsymbol{z})$
  - Find SUSY vacua by minima of $\widetilde{W}_{\text{eff}}(u_k; \boldsymbol{z})$.

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  where $\ell(u)$ is defined by $\frac{\partial}{\partial u}\ell(u) = -\frac{1}{2\pi \mathsf{i}} \log 2 \sinh(-\pi \mathsf{i} u)$.
- Contribution from FI term of $G = U(N)$ w/ a single $U(1)_J$ topological symmetry

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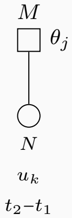
- Supersymmetry vacua [Nekrasov-Shatashvili]

$$\frac{\partial}{\partial u_k}\widetilde{W}_{\text{eff}} = 0 \quad \Rightarrow \quad \exp 2\pi\mathsf{i}\frac{\partial \widetilde{W}_{\text{eff}}}{\partial u_k} = 1,\; k = 1, 2, \ldots, \text{rk}(G).$$

Example: $U(N)$ SQCD w/ M fundamental hypermultiplets



- Effective twisted superpotential

$$\widetilde{W}_{\text{eff}} = \sum_{k=1}^{N}\sum_{j=1}^{M} \ell(u_k - \theta_j + \tfrac{1}{2}\eta) + \ell(-u_k + \theta_j + \tfrac{1}{2}\eta) \quad \leftarrow \text{chirals in hyper}$$

$$+ \sum_{k,l=1}^{N} \ell(u_k - u_l - \eta) \quad \leftarrow \text{chirals in vector}$$

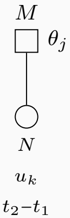
$$+ (t_2 - t_1)\sum_{k=1}^{N} u_k \quad \leftarrow \text{FI terms}$$

- Color coding: gauge charge, flavor charge, $U(1)_\eta$ charge

Example: $U(N)$ SQCD w/ M fundamental hypermultiplets

- Supersymmetric vacua ($k = 1, \ldots, N$)

$$1 = \mathrm{e}^{2\pi \mathrm{i} \partial_{u_k} \widetilde{W}_{\mathrm{eff}}} = (-1)^{N+M-1} \frac{\epsilon_2}{\epsilon_1} \prod_{\substack{l=1 \\ l \neq k}}^{N} \frac{x_k q - x_l q^{-1}}{x_l q - x_k q^{-1}} \prod_{j=1}^{M} \frac{x_k - y_j q}{y_j - x_k q}.$$



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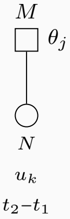

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- Alternatively formulation by Baxter's Q-functions

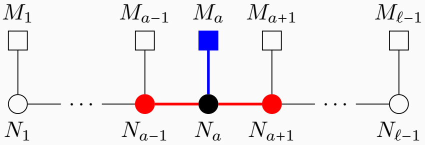
$$-1 = \tau \frac{Q^{++}(x_k)}{Q^{--}(x_k)} \frac{B^{-}(x_k)}{B^{+}(x_k)},$$

with $\tau = \epsilon_2/\epsilon_1$ and

$$Q(x) = \prod_{l=1}^{N} \left(\sqrt{\frac{x}{x_l}} - \sqrt{\frac{x_l}{x}} \right), \; B(x) = \prod_{j=1}^{M} \left(\sqrt{\frac{x}{y_j}} - \sqrt{\frac{y_j}{x}} \right),$$

$$f^{\pm}(x) = f(x q^{\pm 1}).$$

Generic A-type quiver



- For generic A-type quiver ($a = 1, \ldots, \ell - 1$; $k = 1, \ldots, N_a$)

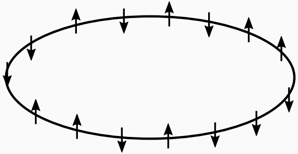
$$\tau^{(a)} \frac{Q_a^{++}(x_k^{(a)})}{Q_a^{--}(x_k^{(a)})} \frac{Q_{a-1}^{-}(x_k^{(a)}) Q_{a+1}^{-}(x_k^{(a)})}{Q_{a-1}^{+}(x_k^{(a)}) Q_{a+1}^{+}(x_k^{(a)})} \frac{B_a^{-}(x_k^{(a)})}{B_a^{+}(x_k^{(a)})} = -1$$

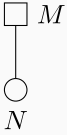
with

$$Q_a(x) = \prod_{l=1}^{N_a} \left(\left(\frac{x}{x_l^{(a)}} \right)^{1/2} - \left(\frac{x_l^{(a)}}{x} \right)^{1/2} \right), \quad B_a(x) = \prod_{j=1}^{M_a} \left(\left(\frac{x}{y_j^{(a)}} \right)^{1/2} - \left(\frac{y_j^{(a)}}{x} \right)^{1/2} \right)$$

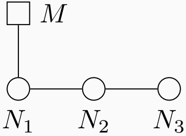
$$f^{\pm}(x) = f(xq^{\pm 1})$$

- These are Bethe Ansatz Equations (BAEs) for XXZ Heisenberg spin chains





$SU(2)$ spin chain with M sites and N magnons



$SU(4)$ spin chain with M sites

Difficulty with BAEs

- BAEs are difficult to solve in practise: numerical instability, unphysical solutions (coinciding Bethe roots, some singular solutions,...)
- The best way to solve BAEs is through rational Q-systems [Marboe-Volin]
  - Originally constructed for $GL(M|N)$ invariant XXX spin chain
  - Various generalisation to XXZ spin chain: triangular quivers
- We construct rational Q-systems for generic A-type quivers

Rational Q-systems

Young diagram

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Young diagram

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- Use $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_\ell)$ to define $\mathbb{Q}$-functions

$$\rho_i = \begin{cases} -N_1 + \sum_{j=1}^{\ell-1} M_j, & i = 1 \\ N_{i-1} - N_i + \sum_{j=i}^{\ell-1} M_j, & 1 < i < \ell \\ N_{\ell-1}, & i = \ell \end{cases}$$

Young diagram

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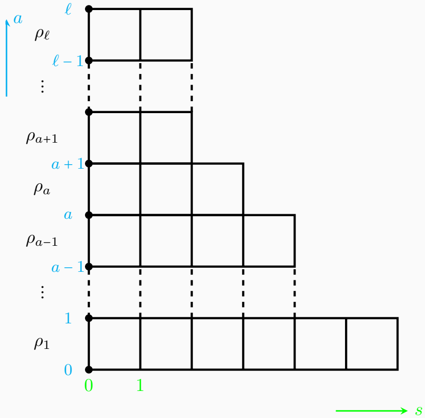
- Construct a Young diagram with ρ_i boxes on each row

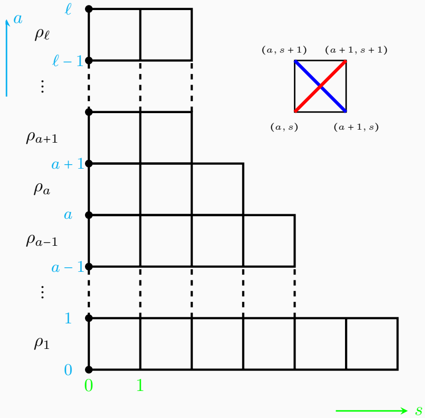


$\mathbb{Q}$ functions

- Place $\mathbb{Q}_{a,s}(x)$ at each point (a, s): a Laurent polynomial in x
- QQ-relations: at each box

$$\mathbb{Q}_{a+1,s}\mathbb{Q}_{a,s+1} = \mathbb{Q}^{+}_{a+1,s+1}\mathbb{Q}^{-}_{a,s} - \epsilon_a \mathbb{Q}^{-}_{a+1,s+1}\mathbb{Q}^{+}_{a,s}$$

with $f^{\pm}(x) = f(xq^{\pm 1})$



Boundary conditions

- Use $\boldsymbol{\sigma}^T = (\hat{\sigma}_1, \ldots, \hat{\sigma}_{\ell-1})$ to specify boundary conditions

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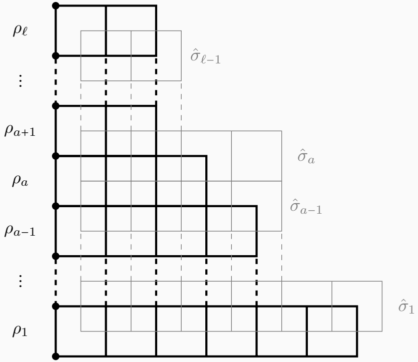
- Top boundary: $\mathbb{Q}_{\ell,s}(x) = 1$
- Left boundary: $\mathbb{Q}_{a,0}(x) = f_a(x) Q_a(x)$

$$Q_a(x) = \prod_{j=1}^{N_a} \left(\left(\frac{x}{x_j^{(a)}} \right)^{1/2} - \left(\frac{x_j^{(a)}}{x} \right)^{1/2} \right)$$



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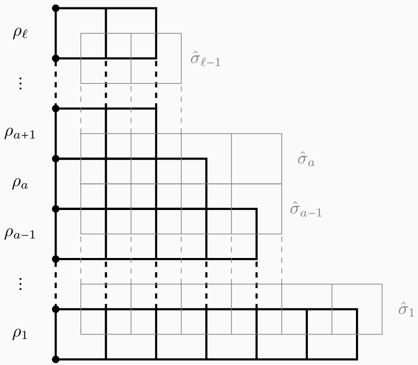
- Left boundary: $\mathbb{Q}_{a,0}(x) = f_a(x) Q_a(x)$

$$f_a(x) = \prod_{k=1}^{\ell-1-a} B_{a+k}(xq^{-\frac{k-1}{2}}) \ldots B_{a+k}(xq^{\frac{k-1}{2}})$$

$$B_a(x) = \prod_{j=1}^{M_a} \left(\left(\frac{x}{y_j^{(a)}} \right)^{1/2} - \left(\frac{y_j^{(a)}}{x} \right)^{1/2} \right)$$



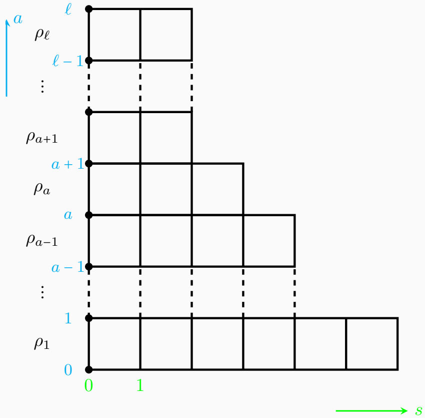
From QQ-relaitons to BAEs

- Consider QQ-relations for nodes $(a-1,0)$

$$\mathbb{Q}_{a,0}\mathbb{Q}_{a-1,1} = \mathbb{Q}^{+}_{a,1}\mathbb{Q}^{-}_{a-1,0} - \epsilon_{a-1}\mathbb{Q}^{-}_{a,1}\mathbb{Q}^{+}_{a-1,0}$$

and take $x = x_k^{(a)}$ so that $\mathbb{Q}_{a,0}(x_k^{(a)}) = 0$

$$\mathbb{Q}^{+}_{a,1}\mathbb{Q}^{-}_{a-1,0} = \epsilon_{a-1}\mathbb{Q}^{-}_{a,1}\mathbb{Q}^{+}_{a-1,0}$$



From QQ-relaitons to BAEs

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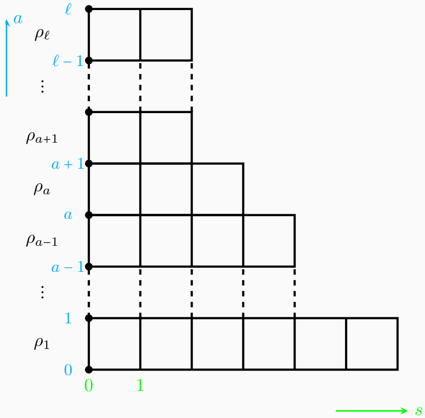
$$\mathbb{Q}_{a,0}\mathbb{Q}_{a-1,1} = \mathbb{Q}^+_{a,1}\mathbb{Q}^-_{a-1,0} - \epsilon_{a-1}\mathbb{Q}^-_{a,1}\mathbb{Q}^+_{a-1,0}$$

and take $x = x_k^{(a)}$ so that $\mathbb{Q}_{a,0}(x_k^{(a)}) = 0$

$$\mathbb{Q}^+_{a,1}\mathbb{Q}^-_{a-1,0} = \epsilon_{a-1}\mathbb{Q}^-_{a,1}\mathbb{Q}^+_{a-1,0}$$

- Assuming none vanishes

$$\frac{\mathbb{Q}^-_{a-1,0}(x_k^{(a)})}{\mathbb{Q}^+_{a-1,0}(x_k^{(a)})}\frac{\mathbb{Q}^+_{a,1}(x_k^{(a)})}{\mathbb{Q}^-_{a,1}(x_k^{(a)})}\frac{1}{\epsilon_{a-1}} = 1.$$



From QQ-relaitons to BAEs

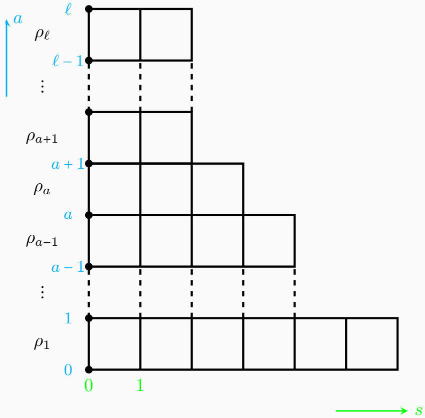
- Consider QQ-relations for nodes $(a,0)$

$$\mathbb{Q}_{a+1,0}\mathbb{Q}_{a,1} = \mathbb{Q}^+_{a+1,1}\mathbb{Q}^-_{a,0} - \epsilon_a \mathbb{Q}^-_{a+1,1}\mathbb{Q}^+_{a,0}$$

shift $x \to xq^{\pm 1}$ to get

$$\mathbb{Q}^+_{a+1,0}\mathbb{Q}^+_{a,1} = \mathbb{Q}^{++}_{a+1,1}\mathbb{Q}_{a,0} - \epsilon_a \mathbb{Q}_{a+1,1}\mathbb{Q}^{++}_{a,0}$$
$$\mathbb{Q}^-_{a+1,0}\mathbb{Q}^-_{a,1} = \mathbb{Q}_{a+1,1}\mathbb{Q}^{--}_{a,0} - \epsilon_a \mathbb{Q}^{--}_{a+1,1}\mathbb{Q}_{a,0}$$

and set $x = x^{(a)}_k$ so that red terms vanish



From QQ-relaitons to BAEs

- Consider QQ-relations for nodes $(a, 0)$

$$\mathbb{Q}_{a+1,0}\mathbb{Q}_{a,1} = \mathbb{Q}^{+}_{a+1,1}\mathbb{Q}^{-}_{a,0} - \epsilon_a \mathbb{Q}^{-}_{a+1,1}\mathbb{Q}^{+}_{a,0}$$

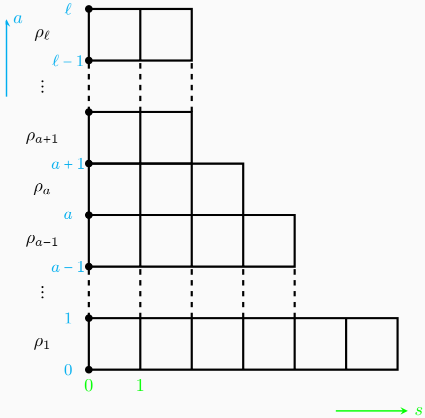
shift $x \to xq^{\pm 1}$ to get

$$\mathbb{Q}^{+}_{a+1,0}\mathbb{Q}^{+}_{a,1} = \mathbb{Q}^{++}_{a+1,1}\mathbb{Q}_{a,0} - \epsilon_a \mathbb{Q}_{a+1,1}\mathbb{Q}^{++}_{a,0}$$
$$\mathbb{Q}^{-}_{a+1,0}\mathbb{Q}^{-}_{a,1} = \mathbb{Q}_{a+1,1}\mathbb{Q}^{--}_{a,0} - \epsilon_a \mathbb{Q}^{--}_{a+1,1}\mathbb{Q}_{a,0}$$

and set $x = x^{(a)}_k$ so that red terms vanish

- Take ratio

$$\frac{\mathbb{Q}^{+}_{a,1}(x^{(a)}_k)}{\mathbb{Q}^{-}_{a,1}(x^{(a)}_k)} = -\epsilon_a \frac{\mathbb{Q}^{++}_{a,0}(x^{(a)}_k)\mathbb{Q}^{-}_{a+1,0}(x^{(a)}_k)}{\mathbb{Q}^{--}_{a,0}(x^{(a)}_k)\mathbb{Q}^{+}_{a+1,0}(x^{(a)}_k)}$$



From QQ-relations to BAEs

- Combining two results

$$\frac{\epsilon_a}{\epsilon_{a-1}}\frac{\mathbb{Q}_{a,0}^{++}(x_k^{(a)})}{\mathbb{Q}_{a,0}^{--}(x_k^{(a)})}\frac{\mathbb{Q}_{a-1,0}^{-}(x_k^{(a)})\mathbb{Q}_{a+1,0}^{-}(x_k^{(a)})}{\mathbb{Q}_{a-1,0}^{+}(x_k^{(a)})\mathbb{Q}_{a+1,0}^{+}(x_k^{(a)})}=-1$$

- Use $\mathbb{Q}_{a,0}(x)=f_a(x)Q_a(x)$

$$\frac{\epsilon_a}{\epsilon_{a-1}}\frac{Q_a^{++}(x_k^{(a)})}{Q_a^{--}(x_k^{(a)})}\frac{Q_{a-1}^{-}(x_k^{(a)})Q_{a+1}^{-}(x_k^{(a)})}{Q_{a-1}^{+}(x_k^{(a)})Q_{a+1}^{+}(x_k^{(a)})}\frac{f_a^{++}(x_k^{(a)})f_{a-1}^{-}(x_k^{(a)})f_{a+1}^{-}(x_k^{(a)})}{f_a^{--}(x_k^{(a)})f_{a-1}^{+}(x_k^{(a)})f_{a+1}^{+}(x_k^{(a)})}=-1$$

From QQ-relations to BAEs

- The construction of $f_a(x)$ guarantees that

$$\frac{f_a^{++}(x_k^{(a)})f_{a-1}^{-}(x_k^{(a)})f_{a+1}^{-}(x_k^{(a)})}{f_a^{--}(x_k^{(a)})f_{a-1}^{+}(x_k^{(a)})f_{a+1}^{+}(x_k^{(a)})} = \frac{B_a^{-}(x_k^{(a)})}{B_a^{+}(x_k^{(a)})}$$

- We then get back to the desired BAEs

$$\frac{\epsilon_a}{\epsilon_{a-1}} \frac{Q_a^{++}(x_k^{(a)})}{Q_a^{--}(x_k^{(a)})} \frac{Q_{a-1}^{-}(x_k^{(a)})Q_{a+1}^{-}(x_k^{(a)})}{Q_{a-1}^{+}(x_k^{(a)})Q_{a+1}^{+}(x_k^{(a)})} \frac{B_a^{-}(x_k^{(a)})}{B_a^{+}(x_k^{(a)})} = -1$$

Solving Q-system

- First solve $Q_a(x)$, then find $x_k^{(a)}$ as roots of $Q_a(x)$.
- We parametrise $Q_a(x)$ by N_a parameters $c_k^{(a)}$, which are symmetric polynomials of $x_k^{(a)}$

$$Q_a(x) = \prod_{j=1}^{N_a} \left(\Big(\frac{x}{x_j^{(a)}}\Big)^{1/2} - \Big(\frac{x_j^{(a)}}{x}\Big)^{1/2} \right) = \Big(\prod_{j=1}^{N_a} x_j^{(a)}\Big)^{-1/2} x^{-N_a/2} \prod_{j=1}^{N_a} (x - x_j^{(a)})$$
$$\equiv (c_0^{(a)})^{-1/2} x^{-N_a/2} (x^{N_a} + c_{N_a-1}^{(a)} x^{N_a-1} + \ldots + c_0^{(a)})$$

Solving Q-system

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$$
\begin{aligned}
Q_a(x) = \prod_{j=1}^{N_a}\left(\Big(\frac{x}{x_j^{(a)}}\Big)^{1/2} - \Big(\frac{x_j^{(a)}}{x}\Big)^{1/2}\right) &= \Big(\prod_{j=1}^{N_a} x_j^{(a)}\Big)^{-1/2} x^{-N_a/2}\prod_{j=1}^{N_a}(x - x_j^{(a)}) \\
&\equiv (c_0^{(a)})^{-1/2} x^{-N_a/2}(x^{N_a} + c_{N_a-1}^{(a)} x^{N_a-1} + \ldots + c_0^{(a)})
\end{aligned}
$$

- $\mathbb{Q}_{a,0}(x) = f_a(x) Q_a(x)$ are thus fixed in terms of $c_k^{(a)}$; the remaining $\mathbb{Q}$ functions can be recursively solved by QQ-relations.
- They are not necessarily Laurent polynomials of x. Imposing this condition leads to algebraic equations for $\{c_k^{(a)}\}$, called "Zero Remainder Conditions" (ZRC).

Solving Q-system

- Top row: $a = \ell, \mathbb{Q}_{\ell,s} = 1$
- Next row: $a = \ell - 1$, QQ-relation

$$\mathbb{Q}_{\ell-1,s+1} = \mathbb{Q}^-_{\ell-1,s} - \epsilon_{\ell-1}\mathbb{Q}^+_{\ell-1,s}$$

$\mathbb{Q}_{\ell-1,s}$ can be solved recursively from $\mathbb{Q}_{\ell-1,0}$.

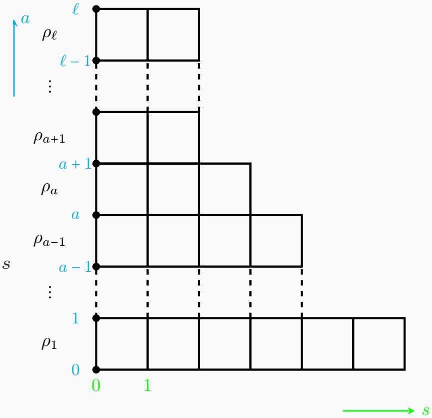
- N-Next row: $a = \ell - 2$, QQ-relation

$$\mathbb{Q}_{\ell-1,s}\mathbb{Q}_{\ell-2,s+1} = \mathbb{Q}^+_{\ell-1,s+1}\mathbb{Q}^-_{\ell-2,s} - \epsilon_{\ell-2}\mathbb{Q}^-_{\ell-1,s+1}\mathbb{Q}^+_{\ell-2,s}$$

leads to

$$\mathbb{Q}_{\ell-2,s+1} = \frac{\mathbb{Q}^+_{\ell-1,s+1}\mathbb{Q}^-_{\ell-2,s} - \epsilon_{\ell-2}\mathbb{Q}^-_{\ell-1,s+1}\mathbb{Q}^+_{\ell-2,s}}{\mathbb{Q}_{\ell-1,s}}$$

Requiring polynomial $\mathbb{Q}_{\ell-2,s+1}(x)$ leads to ZRCs.



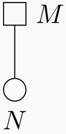
Q-system vs BAEs

- Solving $x_k^{(a)}$ from Q-system is much more superior than from BAEs
  - Unphysical solutions are automatically eliminated.
  - Numerically stable.
  - Much faster.
- Symmetry of $x_k^{(a)}$ is exploited in Q-system.

$$Q_a(x) = \prod_{j=1}^{N_a} \left(\left(\frac{x}{x_j^{(a)}} \right)^{1/2} - \left(\frac{x_j^{(a)}}{x} \right)^{1/2} \right) = c_0^{-1/2} x^{-N_a/2} (x^{N_a} + c_{N_a-1} x^{N_a-1} + \ldots + c_0)$$

Permutations of $x_k^{(a)}$ for fixed a do not change the solution to the Q-system, but lead to different solutions to BAEs.

Efficiency test



(M,N)	BAEs	Q-systems
$(4,2)$	0.238	0.105
$(5,2)$	0.512	0.155
$(6,3)$	199.7	0.385
$(7,3)$	1803	2.092
$(8,4)$	–	6.322
$(9,4)$	–	29.85
$(10,5)$	–	1145

Mirror symmetry and bispectral duality

Mirror symmetry

- S-duality in type IIB superstring theory: D5 $\leftrightarrow$ NS5, D3 $\leftrightarrow$ D3.
- Mirror symmetry of 3d $\mathcal{N} = 4$ theories

	0	1	2	3	4	5	6	7	8	9
NS5	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$			$\vec{\omega}$	
D3	$\times$	$\times$	$\times$				$\times$			
D5	$\times$	$\times$	$\times$		$\vec{m}$			$\times$	$\times$	$\times$

- Mapping of parameters

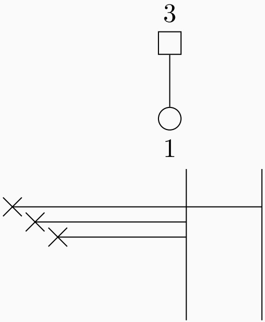
$$\vec{m} \leftrightarrow \vec{\omega}$$

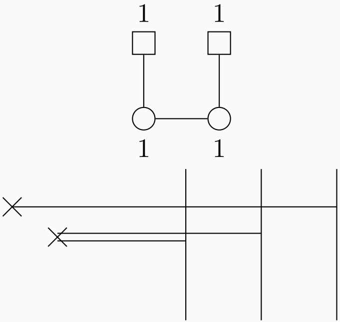
$$SU(2)_C \leftrightarrow SU(2)_H$$

$$G_C \leftrightarrow G_H$$

Mirror symmetry

- Mirror symmetry: D5 $\leftrightarrow$ NS5, $x^6 \leftrightarrow -x^6$ [Hanany-Witten]





- Mirror symmetry by exchange of partitions: $\boldsymbol{\rho} \leftrightarrow \boldsymbol{\sigma}$, $T_{\boldsymbol{\rho}}^{\boldsymbol{\sigma}}[SU(n)] \leftrightarrow T_{\boldsymbol{\sigma}}^{\boldsymbol{\rho}}[SU(n)]$.

Mirror symmetry for $\mathcal{N} = 2^*$ theory on S^1

- Exponentiated complex parameters

$$y = \mathrm{e}^{2\pi \mathrm{i}\theta_j}, \quad \epsilon_s = \mathrm{e}^{2\pi \mathrm{i} t_s}, \quad q = \mathrm{e}^{\pi \mathrm{i}\eta}.$$

  with

$$\theta_j = \mathrm{i}R(m_j + \mathrm{i}a^H_{0,j}), \quad t_s = \mathrm{i}R(\omega_s + \mathrm{i}a^C_{0,s}), \quad \eta = \mathrm{i}R(\tilde{\eta} + \mathrm{i}a^\eta_0).$$

- Mapping of parameters under mirror symmetry between theory $\mathcal{T} = T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}[SU(n)]$ and theory $\mathcal{T}^\vee = T^{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}[SU(n)]$: (Recall $j_\eta = j^3_H - j^3_C$)

$$y_i \leftarrow \epsilon^\vee_i, \quad \epsilon_a \leftarrow y^\vee_a, \quad q \leftrightarrow \frac{1}{q^\vee}$$

A-, B-twisted indices

- A,B-twisted index w/ genus g and twisted masses $z_i = \{y_j, \epsilon_s\}$

  [Closset-Kim],[Closset-Kim-Willett], [Benini-Zaffaroni]

$$I_{g,A/B}(q, z_i) = \mathrm{Tr}_{\Sigma_g^{A/B}} \left((-1)^F q^{Q_\eta} \prod_i z_i^{Q_i} \right)$$

- Partition function of 3d $\mathcal{N} = 2^*$ theory on $\Sigma_g \times S^1$ w/ topological A-,B-twists turned on to preserve half supersymmetry
  - A-twist: Lorentz group $(SU(2)_L \times SU(2)_H)_{\mathrm{diag}}$
  - B-twist: Lorentz group $(SU(2)_L \times SU(2)_C)_{\mathrm{diag}}$

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  - A-twist: Lorentz group $(SU(2)_L \times SU(2)_H)_{\mathrm{diag}}$
  - B-twist: Lorentz group $(SU(2)_L \times SU(2)_C)_{\mathrm{diag}}$
- Reduces via localisation to summation over Bethe roots [Closset-Kim], [Closset-Kim-Willett]

$$I_{g,A/B} = \frac{1}{|W_G|} \sum_{\hat{\boldsymbol{x}} \in \mathcal{S}_{BE}} \mathcal{H}_{A/B}(\hat{\boldsymbol{x}})^{g-1}$$

with

$$\mathcal{H}_{A/B}(\boldsymbol{x}) = \mathrm{e}^{2\pi \mathrm{i} \Omega_{A/B}(\boldsymbol{x})} \det_{a,b} \frac{\partial^2 \widetilde{W}_{\mathrm{eff}}}{\partial u_a \partial u_b}$$

Mirror symmetry for A-, B-twisted indices

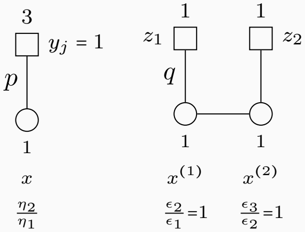
- For two mirror theories $\mathcal{T} = T^{\boldsymbol{\sigma}}_{\boldsymbol{\rho}}[SU(n)]$ and theory $\mathcal{T}^\vee = T^{\boldsymbol{\rho}}_{\boldsymbol{\sigma}}[SU(n)]$

$$I^{\mathcal{T}}_{g,A}(q, z_i) = I^{\mathcal{T}^\vee}_{g,B}(q^\vee, z_i^\vee), \quad \forall g$$

- In particular

$$\mathcal{H}^{\mathcal{T}}_A(\hat{\boldsymbol{x}}; q, z_i)\Big|_{\hat{\boldsymbol{x}} \in \mathcal{S}_{BE}} = \mathcal{H}^{\mathcal{T}^\vee}_B(\hat{\boldsymbol{x}}^\vee; q^\vee, z_i^\vee)\Big|_{\hat{\boldsymbol{x}}^\vee \in \mathcal{S}^\vee_{BE}}$$

Example of mirror symmetry



- Both systems have three solutions $\boldsymbol{x}_{1,2,3}$ to Q-system/BAEs

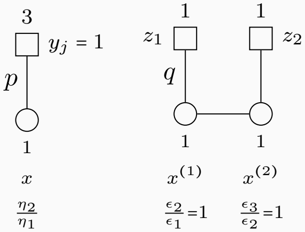
$$\mathcal{H}_A^{\text{quiver}}(\boldsymbol{x}_1) = \frac{3(p^2-1)^4 z^{2/3}}{p^2(\sqrt[3]{z}-p)^2(p\sqrt[3]{z}-1)^2}, \ \mathcal{H}_A^{\text{quiver}}(\boldsymbol{x}_{2,3}) = \ldots$$

$$\mathcal{H}_B^{\text{QED}}(\boldsymbol{x}_1) = \frac{3(q^2-1)^4 \epsilon^{2/3}}{q^2(\sqrt[3]{\epsilon}-q)^2(q\sqrt[3]{\epsilon}-1)^2}, \ \mathcal{H}_B^{\text{QED}}(\boldsymbol{x}_{2,3}) = \ldots.$$

- Mirror symmetry

$$\mathcal{H}_A^{\text{quiver}}(\boldsymbol{x}_a) = \mathcal{H}_B^{\text{QED}}(\boldsymbol{x}_a)\Big|_{\substack{\epsilon\to z\\ q\to\frac{1}{p}}}, \quad a = 1,2,3$$

Example of mirror symmetry



- Both systems have three solutions $\boldsymbol{x}_{1,2,3}$ to Q-system/BAEs

$$\mathcal{H}_B^{\text{quiver}}(\boldsymbol{x}_1) = -\frac{3(\sqrt[3]{z}-p)(p\sqrt[3]{z}-1)}{p\sqrt[3]{z}},\ \mathcal{H}_B^{\text{quiver}}(\boldsymbol{x}_{2,3}) = \ldots$$

$$\mathcal{H}_A^{\text{QED}}(\boldsymbol{x}_1) = -\frac{3(\sqrt[3]{\epsilon}-q)(q\sqrt[3]{\epsilon}-1)}{q\sqrt[3]{\epsilon}},\ \mathcal{H}_A^{\text{QED}}(\boldsymbol{x}_{2,3}) = \ldots.$$

- Mirror symmetry

$$\mathcal{H}_B^{\text{quiver}}(\boldsymbol{x}_a) = \mathcal{H}_A^{\text{QED}}(\boldsymbol{x}_a)\Big|_{\substack{\epsilon\to z\\ q\to\frac{1}{p}}}, \quad a = 1,2,3$$

Implication for QISs

- Quantum integrable systems associated to mirror dual 3d $N = 2^*$ theories are very different.
- Bispectral duality: the spectra of the two QISs are one-to-one correspondent.
  [Gaiotto-Koroteev]
- $\mathcal{H}_A^T(\hat{\boldsymbol{x}}_a), \mathcal{H}_B^{T^\vee}(\hat{\boldsymbol{x}}_a^\vee)$ may be identified with certain conserved charges in QISs, which are then also identified.
- Our construction leads to quick determination of bispectral dual integrable systems: Q-systems with partitions exchanged.

Conclusion and outlook

Conclusion

- Rational Q-systems are more suited than BAEs to describe Bethe/gauge correspondence.
- For gauge theorists: We constructed rational Q-systems for quickly solving SUSY vacua of 3d $\mathcal{N} = 2^*$ A-type quiver on S^1.
- For integrability experts: We give easy criteria for bispectral dual integrable systems based on rational Q-systems.

Outlook

- Efficient calculation of A-,B-twisted indices using algebraic methods.
- Generalisation to orthosymplectic quivers and open spin chains.

Thank you for your attention!